On the Biplanar and k-Planar Crossing Numbers*

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Abstract

The biplanar crossing number of a graph G is the minimum number of crossings over all possible drawings of the edges of G in two disjoint planes. We present new bounds on the biplanar crossing number of complete graphs and complete bipartite graphs. In particular, we prove that the biplanar crossing number of complete bipartite graphs can be approximated to within a factor better than 3, improving upon the best previously known factor of 4.03. For complete graphs, we establish an approximation factor of 3.17, improving the best previous factor of 4.34. We provide similar improved bounds for the k-planar crossing number of complete graphs and complete bipartite graphs, for any positive integer k. We also investigate the relation between (ordinary) crossing number and biplanar crossing number of general graphs in more depth. In particular, we prove that any graph with crossing number at most 11 is biplanar.

Keywords Crossing number; Biplanar embedding; Counting method; Asymptotic approximation factor.

1 Introduction

An embedding (or drawing) of a graph G in the Euclidean plane is a mapping of the vertices of G to distinct points in the plane and a mapping of edges to smooth curves between their corresponding vertices, such that no edge passes through any vertex other than its endpoints. A planar embedding of a graph is a drawing of the graph in the plane such that edges intersect only at their endpoints. A graph admitting such a drawing is called *planar*. A *biplanar embedding* of a graph $G = (V, E)$ is a decomposition of the graph into two graphs $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$ such that $E = E_1 \cup E_2$ and $E_1 \cap E_2 = \emptyset$, together with planar embeddings of G_1 and G_2. In this case, we call G *biplanar*.

Biplanar embeddings are central to the study of graph thickness [14, 21], with applications to VLSI design [15]. In such applications, one often seeks to distribute the edges of a non-planar graph across multiple layers, such that each layer contains a planar subgraph. For instance, in VLSI circuits, these layers correspond to distinct levels of metallization, or the two sides of a printed circuit board. While planarity can be recognized in linear time, biplanarity testing is NP-complete [13], making it substantially more challenging.

Let $cr(G)$ be the minimum number of edge crossings over all drawings of G in the plane, and let $cr_k(G)$ be the minimum of $cr(G_1) + \cdots + cr(G_k)$ over all possible decompositions of G into k subgraphs $G_1, \ldots, G_k$. We call $cr(G)$ the *crossing number* of G, and $cr_k(G)$ the *k-planar crossing number* of G. Throughout this paper, we only consider *simple drawings* for each subgraph G_i, in which no two edges intersect more than once, and no three edges intersect at a point (such drawings

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are sometimes called nice drawings). Moreover, we denote by n the number of vertices, and by m the number of edges of a graph.

Determining the crossing number of complete graphs and complete bipartite graphs has been the subject of extensive research over the past decades. In 1955, Zarankiewicz [23] conjectured that the crossing number $cr(K_{p,q})$ of the complete bipartite graph $K_{p,q}$ is equal to

$$Z(p,q) := \left\lfloor \frac{p}{2} \right\rfloor \left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{q}{2} \right\rfloor \left\lfloor \frac{q-1}{2} \right\rfloor .$$

He also established a drawing with that many crossings. In 1960, Guy [9] conjectured that the crossing number $cr(K_n)$ of the complete graph K_n is equal to

$$Z(n) := \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor .$$

Both conjectures have remained open after more than six decades. For the biplanar case, even formulating such conjectures seems to be hard. As noted in [5], techniques like embedding method and the bisection width method which are useful for bounding ordinary crossing numbers do not seem applicable to the biplanar case.

In 1971, Owens [15] described a biplanar embedding of K_n with almost $\frac{7}{24}Z(n)$ crossings. The construction was later improved by Durocher et al. [8], but the upper bound remained asymptotically the same. In 2006, Czabarka et al. [5] presented a biplanar embedding for $K_{p,q}$ with about $\frac{2}{9}Z(p,q)$ crossings. They also proved that $cr_2(K_n) \geq n^4/952$ and $cr_2(K_{p,q}) \geq p(p-1)q(q-1)/290$. Shahrokhi et al. [18] generalized these lower bounds to the k-planar case. Pach et al. [16] proved that for every graph G and any positive integer k, $cr_k(G) \leq \left(\frac{2}{k^2} - \frac{1}{k^3}\right) cr(G)$. This includes as a special case the inequality $cr_2(G) \leq \frac{3}{8}cr(G)$, originally proved by Czabarka et al. [6].

Our results. In this paper, we present several new bounds for approximating the biplanar and k-planar crossing number of complete graphs and complete bipartite graphs. Given a positive integer k and a real constant $\alpha \geq 1$, we say that $cr_k(K_n)$ is *approximated* to within a factor of α, if there is an upper bound $f(n)$ and a lower bound $g(n)$ on the value of $cr_k(K_n)$ such that $\lim_{n\to\infty} \frac{f(n)}{g(n)} \leq \alpha$. Here, α is called an *asymptotic approximation factor* for $cr_k(K_n)$. Similarly, we say that $cr_k(K_{p,q})$ is approximated to within a factor of α, if there is an upper bound $f(p,q)$ and a lower bound $g(p,q)$ on the value of $cr_k(K_{p,q})$ such that $\lim_{p,q\to\infty} \frac{f(p,q)}{g(p,q)}$ exists and is no more than α. The results presented in this paper are summarized below.

- We prove that for all $p, q \geq 30$, $cr_2(K_{p,q}) \geq p(p-1)q(q-1)/213$. This significantly improves the best current lower bound of $cr_2(K_{p,q}) \geq p(p-1)q(q-1)/290$, due to Czabarka et al. [5]. Combined with the upper bound of $cr_2(K_{p,q}) \leq \frac{2}{9}Z(p,q) + o(p^2q^2)$[1] [5], our result implies an asymptotic approximation factor of 2.96 for $cr_2(K_{p,q})$, improving over the best previously known asymptotic factor of 4.03.

- For complete graphs, we show that $cr_2(K_n) \geq \frac{n^4}{694}$, improving the best current lower bound of $cr_2(K_n) \geq \frac{n^4}{952}$ [5]. Combined with the upper bound of $cr_2(K_n) \leq \frac{7}{24}Z(n) + o(n^4)$ due to Owens [15], we achieve an asymptotic approximation factor of 3.17 for $cr_2(K_n)$, improving the best previously known approximation factor of 4.34.

[1] By definition, $f(x,y) = o(g(x,y))$ if $\lim_{x,y\to\infty} \frac{f(x,y)}{g(x,y)} = 0$.

Table 1: Summary of asymptotic approximation factors for the biplanar and k-planar crossing numbers.

		Asymptotic Approx. Factor	
Crossing Number	Graph Class	Previous	New
biplanar	complete bipartite graphs	4.03	2.96
biplanar	complete graphs	4.34	3.17
k-planar ($k \geq 3$)	complete bipartite graphs	13.53	9.15
k-planar ($k \geq 3$)	complete graphs	13.50	7.25

- We investigate the relation between $cr(G)$ and $cr_2(G)$ in general graphs, and pose a new problem of finding the maximum integer $\xi(r)$, for a given integer $r \geq 0$, such that $cr(G) \leq \xi(r)$ implies $cr_2(G) \leq r$, for all graphs G. For the special case of $r = 0$, we show that $\xi(r) \geq 11$. It implies that any graph G that can be drawn in the plane with at most 11 crossings is biplanar.

- We extend our lower bounds for the biplanar crossing number to the k-planar case, for any integer $k \geq 3$. In particular, we show that for sufficiently large n, $cr_k(K_n) \geq n^4/(232k^2)$, improving the best current lower bound of $cr_k(K_n) \geq n^4/(432k^2)$, due to Shahrokhi et al. [18]. Considering the upper bound of $cr_k(K_n) \leq \frac{2}{k^2}Z(n)$ due to Pach et al. [16], we obtain an asymptotic approximation factor of 7.25 for $cr_k(K_n)$, improving over the best previously known factor of 13.53.

- Finally, we prove that for any integer $k \geq 3$, $cr_k(K_{p,q}) \geq p(p-1)q(q-1)/(73.2k^2)$, improving the current lower bound of $cr_k(K_{p,q}) \geq p(p-1)q(q-1)/(108k^2)$ due to Shahrokhi et al. [18]. Combined with the upper bound of $cr_k(K_n) \leq \frac{2}{k^2}Z(p,q)$ [16], we obtain an asymptotic approximation factor of 9.15 for $cr_k(K_{p,q})$, improving the best current factor of 13.5.

Table 1 provides a summary of the asymptotic approximation factors for the biplanar and k-planar crossing numbers of complete graphs and complete bipartite graphs.

2 Key Combinatorial Lemmas

We begin by presenting two key combinatorial lemmas that form the foundation of our main results. The first lemma establishes a general technique for deriving lower bounds on the k-planar crossing number of a graph G, using known lower bounds on the (ordinary) crossing number of G. This lemma applies to broad families of graphs that are closed under edge removal—a property satisfied by many natural graph classes, including simple graphs and bipartite graphs.

Lemma 1. *Let $\mathcal{G}$ be a hereditary class of graphs that is closed under edge removal. Suppose there exist a positive constant α and an arbitrary function $f(x)$ such that for every graph $G \in \mathcal{G}$ with m edges and n vertices,*

$$cr(G) \geq \alpha m - f(n).$$

Then for all $G \in \mathcal{G}$ and all positive integers k,

$$cr_k(G) \geq \alpha m - k \cdot f(n).$$

Proof. Fix a graph $G \in \mathcal{G}$ with m edges and n vertices. Let $G = \bigcup_{i=1}^{k} G_i$ be a decomposition of G into k edge-disjoint subgraphs $G_i = (V, E_i)$, such that the total number of crossings $\sum_{i=1}^{k} cr(G_i)$ is minimized. Since $\mathcal{G}$ is hereditary and closed under edge removal, each G_i also belongs to $\mathcal{G}$. Hence,

$$cr(G_i) \geq \alpha m_i - f(n),$$

by proposition, where $m_i = |E_i|$ for each i. Summing over all i, we obtain

$$\begin{aligned} cr_k(G) = \sum_{i=1}^{k} cr(G_i) &\geq \sum_{i=1}^{k} (\alpha m_i - f(n)) \\ &= \alpha \sum_{i=1}^{k} m_i - k \cdot f(n) = \alpha m - k \cdot f(n). \end{aligned}$$

□

A classical combinatorial technique for establishing lower bounds on the crossing number of graphs is the counting method (see, e.g., [10, 17]). In this paper, we employ the following generalization of the counting method, which allows us to relate the k-planar crossing number of a graph to that of a frequently occurring subgraph.

Lemma 2 (Counting Method)**.** *Let G be a simple graph that contains α copies of a subgraph H. Suppose that in every k-planar drawing of G, each edge crossing is contained in at most β copies of H. Then,*

$$cr_k(G) \geq \left\lceil \frac{\alpha}{\beta} \cdot cr_k(H) \right\rceil.$$

Proof. Let D be a k-planar drawing of G that realizes $cr_k(G)$. Each of the α copies of H in G induces a k-planar drawing in D containing at least $cr_k(H)$ crossings. By assumption, each crossing in D is shared by at most β of these copies. Therefore, the total number of crossings in D must be at least $\alpha \cdot cr_k(H)/\beta$, which yields the lemma statement. Note that a ceiling is put in the right-hand side of the inequality, because $cr_k(G)$ is always an integer. □

3 Lower Bounds for Complete Bipartite Graphs

In this section, we present new lower bounds on the biplanar crossing number of complete bipartite graphs. In particular, we improve upon the following bound by Czabarka et al. [5], which states that for all $p, q \geq 10$,

$$cr_2(K_{p,q}) \geq \frac{p(p-1)q(q-1)}{290}.$$

From Euler's formula, we have $cr(G) \geq m - 3(n-2)$ for simple graphs, and $cr(G) \geq m - 2(n-2)$ for bipartite graphs. Using Lemma 1, we immediately get a lower bound of $cr_2(G) \geq m - 6(n-2)$ for simple graphs, and a lower bound of $cr_2(G) \geq m - 4(n-2)$ for bipartite graphs.

To establish stronger lower bounds, we need to incorporate more powerful ingredients. A graph is called *k-planar* if it can be drawn in the plane such that each edge is crossed at most k times. Such a drawing is referred to as a *k-planar drawing*. It is known that every 1-planar drawing of a 1-planar graph has at most $n-2$ crossings [7]. (Note the difference between k-planar drawing and k-planar crossing numbers.) Removing one edge per crossing yields a planar graph. Therefore, every 1-planar bipartite graph has at most $3n-6$ edges. Karpov [11] proved that for every 1-planar bipartite graph with at least 4 vertices, the inequality $m \leq 3n-8$ holds. In a recent work, Angelini et al. [2] proved that for every 2-planar bipartite graph we have $m \leq 3.5n-7$. We use these results to obtain the following stronger lower bound.

Lemma 3. *For every bipartite graph G with $n \geq 4$,*

$$cr_k(G) \geq 3m - (8.5n - 19)k.$$

Proof. Let G be a bipartite graph with n vertices and m edges. Fix a drawing of G with a minimum number of crossings. If $m > 3.5n - 7$, then by [2], there must be an edge in the drawing with at least three crossings. We repeatedly remove such an edge until we reach a drawing with $\lfloor 3.5n - 7 \rfloor$ edges. Now, by Karpov's result, there must be an edge in the drawing with at least two crossings. We repeatedly remove such an edge until we reach a drawing with $3n - 8$ edges. Let G' be the bipartite graph corresponding to the remaining drawing. We know by Euler's formula that $cr(G') \geq (3n-8) - 2(n-2)$. Therefore,

$$\begin{aligned} cr(G) &\geq 3(m - \lfloor 3.5n - 7 \rfloor) + 2(\lfloor 3.5n - 7 \rfloor - (3n - 8)) + (3n - 8) - 2(n - 2) \\ &\geq 3m - \lfloor 3.5n - 7 \rfloor - (3n - 8) - 2(n - 2) \\ &\geq 3m - 8.5n + 19. \end{aligned}$$

Applying Lemma 1 yields $cr_k(G) \geq 3m - (8.5n - 19)k$. □

For complete bipartite graphs, Lemma 3 implies that $cr_2(K_{p,q}) \geq 3pq - 17(p+q) + 38$, for all $p, q \geq 2$. We use Lemma 3 along with a counting argument to obtain the following improved bound on $cr_2(K_{p,q})$.

Theorem 4. *For all $p, q \geq 30$,*

$$cr_2(K_{p,q}) \geq \frac{p(p-1)q(q-1)}{213}.$$

Proof. Using the counting method (Lemma 2) for $K_{p,p}$ and $K_{p+1,p}$ we have

$$cr_2(K_{p+1,p}) \geq \left\lceil \frac{p+1}{p-1} cr_2(K_{p,p}) \right\rceil.$$

This is because $K_{p+1,p}$ contains $p+1$ copies of $K_{p,p}$, and each crossing realized by two edges, belongs to at most $\binom{p-1}{p-2} = p-1$ of these copies. Using a similar argument for $K_{p+1,p}$ and $K_{p+1,p+1}$, we get

$$cr_2(K_{p+1,p+1}) \geq \left\lceil \frac{p+1}{p-1} \left\lceil \frac{p+1}{p-1} cr_2(K_{p,p}) \right\rceil \right\rceil. \tag{1}$$

By Lemma 3, $cr_2(K_{15,15}) \geq 203$. Plugging into (1), yields $cr_2(K_{16,16}) \geq 266$. Notably, this is the smallest value of p for which the recurrence relation (1) outperforms the direct bound from Lemma 3, which alone would only give $cr_2(K_{16,16}) \geq 262$. Now, we use the recurrence relation (1) iteratively for $p = 16$ through $p = 30$ to get

$$cr_2(K_{30,30}) \geq 3554. \tag{2}$$

Note that in computing the above value, the two rounding operations in relation (1) are applied at each step of the recurrence, rather than only at the end. We can now apply the counting method on $K_{30,30}$ and $K_{p,q}$ to obtain

$$cr_2(K_{p,q}) \geq \frac{\binom{p}{30}\binom{q}{30}}{\binom{p-2}{28}\binom{q-2}{28}} cr_2(K_{30,30}) = \frac{p(p-1)q(q-1)}{30 \times 29 \times 30 \times 29} cr_2(K_{30,30}).$$

Plugging (2) in the above inequality yields the theorem statement. □

Remark. The exact value of the denominator obtained in the above proof is around 212.97. One may continue applying the recurrence relation (1) to obtain better bounds for $K_{p,p}$, when $p > 30$. This leads to a slightly improved constant in the denominator, but it does not seem to reduce the constant below 212. Indeed, the denominator seems to converge to a value around 212.4, for large values of p.

4 Biplanar Crossing Number of Complete Graphs

We now consider the biplanar crossing number of complete graphs. Czabarka et al. [5] used a probabilistic method to prove that for large values of n,

$$cr_2(K_n) \geq \frac{n^4}{952}.$$

We improve this lower bound using the counting method.

Theorem 5. *For all $n \geq 24$,*

$$cr_2(K_n) \geq \frac{n(n-1)(n-2)(n-3)}{698}.$$

Proof. We know from [1] that for every G with $n \geq 3$, $cr(G) \geq 5m - \frac{139}{6}(n-2)$. Applying Lemma 1, we get

$$cr_2(G) \geq 5m - \frac{139}{3}(n-2).$$

This in particular implies $cr_2(K_{25}) \geq 435$. Now, we use the counting method (Lemma 2) on K_{25} and K_n to get

$$cr_2(K_n) \geq \frac{\binom{n}{25} cr_2(K_{25})}{\binom{n-4}{21}} \geq \frac{n(n-1)(n-2)(n-3)}{\frac{25\times 24\times 23\times 22}{435}},$$

which implies the theorem statement. $\square$

We can slightly improve this result, using an iterative counting method similar to what we used in the previous section.

Theorem 6. *For large values of n,*

$$cr_2(K_n) \geq \frac{n^4}{694}.$$

Proof. Using the counting method (Lemma 2) for K_n and K_{n+1} we have

$$cr_2(K_{n+1}) \geq \left\lceil \frac{(n+1)cr_2(K_n)}{n-3} \right\rceil. \tag{3}$$

Starting from $cr_2(K_{25}) \geq 435$, we use the recurrence relation (3) iteratively from $n = 25$ to 50 to obtain $cr_2(K_{50}) \geq 7965$. Now, we use the counting method on K_{50} and K_n to get

$$\begin{aligned} cr_2(K_n) \geq \frac{\binom{n}{50} cr_2(K_{50})}{\binom{n-4}{46}} &\geq \frac{n(n-1)(n-2)(n-3)}{\frac{50\times 49\times 48\times 47}{7965}} \\ &\geq \frac{n(n-1)(n-2)(n-3)}{693.94}, \end{aligned}$$

which implies $cr_2(K_n) \geq \frac{n^4}{694}$ for sufficiently large n. $\square$

5 Crossing Number vs Biplanar Crossing Number

Czabarka et al. [6] defined c^* as the smallest constant such that for every graph G, $cr_2(G) \leq c^* \cdot cr(G)$. They proved that $0.067 \leq c^* \leq \frac{3}{8} = 0.375$. It is known that $cr(K_n) \leq \frac{n^4}{64}$ [22]. By Theorem 6, for sufficiently large n, $cr_2(K_n) \geq \frac{n^4}{694}$. Therefore, our results from Section 4 imply an improved bound of $c^* \geq \frac{64}{694} \approx 0.092$, improving the previous bound of 0.067. To investigate the relationship between $cr(G)$ and $cr_2(G)$ in a more fine-grained form, we introduce the following interesting problem.

Problem 1. *Given a positive integer r, find the largest integer $\xi(r)$ such that for every graph G, $cr(G) \leq \xi(r)$ implies $cr_2(G) \leq r$.*

For the special case of $r = 0$, the problem is to find the largest integer ξ such that drawing a graph with at most ξ crossings in the plane guarantees that the graph is biplanar. As proved by Battle et al. [3] and Tutte [20], K_9 is not biplanar (see [4] for a recent short proof). Moreover, we know that $cr(K_9) = 36$ [12]. Therefore, $\xi(0) < 36$.

The inequality $cr_2(G) \leq \frac{3}{8}cr(G)$, due to Czabarka et al. [6], implies that if $cr(G) \leq 2$, then G is biplanar. Therefore, $\xi(0) \geq 2$. We can strengthen this bound as follows. Recall that by Kuratowski's theorem, every non-planar graph contains a subdivision of $K_{3,3}$ or K_5. Therefore, there is no non-planar graph with less than 9 edges. This leads to the following observation.

Observation 1. *Every graph with at most 8 edges is planar. The only non-planar graph with 9 edges is $K_{3,3}$, and the only non-planar graphs with 10 edges are K_5, $K_{3,3}$ with an extra edge, and $K_{3,3}$ with a subdivided edge.*

From this simple observation, we can conclude that $\xi(0) \geq 4$ as follows. Suppose a graph G is drawn in the plane with at most 4 crossings. The number of edges involved in these four crossings is at most 8. If we remove these 8 edges from the drawing, the remaining drawing has no crossing. Moreover, the subgraph of G that contains only these 8 (or fewer) edges is planar by Observation 1. Therefore, G is the union of two planar graphs, and hence is biplanar.

We will significantly improve this lower bound in the following theorem.

Theorem 7. *Every graph G with $cr(G) \leq 11$ is biplanar. In other words, $\xi(0) \geq 11$.*

Proof. Let G be a graph with crossing number $cr(G) \leq 11$, and fix a drawing D of G in the plane that attains this minimum number of crossings. We iteratively remove from D an edge that is involved in the greatest number of crossings, continuing until no crossings remain. Let D_1 denote the final crossing-free drawing, and let D_2 be the drawing composed by the removed edges. Let G_1 and G_2 be the subgraphs of G corresponding to D_1 and D_2, respectively.

By construction, D_1 has no crossings, and hence, G_1 is planar. If D_2 contains no crossings, then G_2 is planar as well, and G is biplanar. Otherwise, suppose D_2 has at least one crossing. We claim that G_2 contains at most 10 edges in this case. To see this, consider two edges e_1 and e_2 that cross in D_2, and assume without loss of generality that e_1 was removed from D before e_2. The crossing between e_1 and e_2 would have been eliminated when e_1 was removed, and hence, e_2 must have been involved in at least one other crossing at the time of its removal. This shows that at least one edge in D was involved in two or more crossings. Consequently, the first removal step must reduce the total number of crossings by at least two. Each subsequent edge removal reduces the number of crossings by at least one. Since D had at most 11 crossings initially, this implies that no more than 10 edges were removed in total. Therefore, G_2 consists of at most 10 edges.

If G_2 has at most 8 edges, then it is planar by Observation 1, and we are done. Otherwise, suppose G_2 has 9 or 10 edges. Here, each edge in G_2, except possibly the first two, eliminated exactly one crossing in D. By Observation 1, if G_2 is non-planar, then it must be one of K_5, $K_{3,3}$, $K_{3,3}$ with a subdivided edge, or $K_{3,3}$ with an extra edge. We distinguish two cases.

Case 1: G_2 is K_5 or $K_{3,3}$. Let e be the last edge removed during the process. At the time of its removal, e crossed exactly one edge f, which remained in G_1. We now swap e and f: place e into G_1 and move f to G_2. This preserves the planarity of G_1, as e introduces no new crossings. Furthermore, G_2 now differs from K_5 or $K_{3,3}$ by the removal of e, and adding f does not restore the non-planar structure. Indeed, if G_2 were to become isomorphic to K_5 or $K_{3,3}$ again, f would have to connect the same pair of vertices as e, which is impossible since no two parallel edges crossed in D. Thus, G_2 is planar, completing a valid biplanar decomposition.

Case 2: G_2 is a $K_{3,3}$ with either a subdivided edge or an extra edge. In both scenarios, G_2 has exactly 10 edges. Hence, all edges in G_2 beyond the first one were involved in exactly one crossing in D. Let e be such an edge, chosen so that it is neither part of the subdivided edge (in the first scenario) nor the extra edge (in the second). Let f be the unique edge in G_1 that crossed e in D. Since f was involved in only that crossing, it remained in G_1 after e was removed. We now swap e and f: move e into G_1 and f into G_2. As before, G_1 remains planar because e introduces no new crossings. For G_2, removing e breaks the augmented $K_{3,3}$ structure, and f, being neither adjacent to nor parallel with e, cannot restore it. Thus, the modified G_2 is planar.

In all cases, we obtain a partition of the edge set of G into two planar subgraphs G_1 and G_2, which proves that G is biplanar. $\square$

One may consider extending our proof to graphs with 12 crossings, but the argument becomes significantly more intricate as the crossing number increases. For instance, suppose G_2 is a $K_{3,3}$ with two extra edges. When attempting to swap an edge $e \in G_2$ with its unique crossing edge $f \in G_1$ (as in our proof), it may happen that the addition of f to G_2 creates a new non-planar subgraph, for example, a different copy of $K_{3,3}$ formed by f and the two extra edges. In that case, one must backtrack further and remove not just f, but also edges added in earlier steps. This cascading dependency across multiple steps would make the analysis more complicated.

6 k-Planar Crossing Number of K_n and $K_{p,q}$

In this section, we provide improved lower bounds on the k-planar crossing number of complete bipartite and complete graphs. Shahrokhi et al. [18] proved that for any positive integer k, and sufficiently large integers p, q, and n:

$$cr_k(K_{p,q}) \geq \frac{p(p-1)q(q-1)}{108k^2},$$

and

$$cr_k(K_n) \geq \frac{n(n-1)(n-2)(n-3)}{432k^2}.$$

We improve these results using the ideas developed in Sections 3 and 4.

Theorem 8. *For all $p, q \geq 8k+2$,*

$$cr_k(K_{p,q}) \geq \frac{p(p-1)q(q-1)}{\frac{512}{7}k^2}.$$

Proof. We apply the counting method (Lemma 2) on $K_{8k+2,8k+2}$ and $K_{p,q}$. By Lemma 3, for every bipartite graph G, $cr_k(G) \geq 3m - (8.5n - 19)k$. This yields

$$cr_k(K_{8k+2,8k+2}) \geq 56k^2 + 81k + 12.$$

Hence,

$$cr_k(K_{p,q}) \geq \frac{\binom{p}{8k+2}\binom{q}{8k+2} cr_k(K_{8k+2,8k+2})}{\binom{p-2}{8k}\binom{q-2}{8k}} = \frac{p(p-1)q(q-1)cr_k(K_{8k+2,8k+2})}{(8k+2)(8k+1)(8k+2)(8k+1)}$$
$$\geq \frac{p(p-1)q(q-1)}{\frac{(8k+2)^2(8k+1)^2}{56k^2+81k+12}} \geq \frac{p(p-1)q(q-1)}{\frac{512}{7}k^2},$$

which completes the proof. □

Theorem 9. *For all $n \geq 14k-3$,*

$$cr_k(K_n) \geq \frac{n(n-1)(n-2)(n-3)}{232k^2}.$$

Proof. We use the counting method (Lemma 2) for K_{14k-3} and K_n. Recall that for every G with $n \geq 3$, $cr(G) \geq 5m - \frac{139}{6}(n-2)$ [1]. Therefore, $cr_k(G) \geq 5m - \frac{139}{6}(n-2)k$ by Lemma 1. Thus,

$$cr_k(K_{14k-3}) \geq \frac{497}{3}k^2 - \frac{775}{6}k + 30.$$

Therefore,

$$cr_k(K_n) \geq \frac{\binom{n}{14k-3} cr_k(K_{14k-3})}{\binom{n-4}{14k-7}} = \frac{n(n-1)(n-2)(n-3)cr_k(K_{14k-3})}{(14k-3)(14k-4)(14k-5)(14k-6)},$$

which implies the theorem. □

7 Conclusion

In this paper, we presented several improved bounds on the biplanar and k-planar crossing number of complete graphs and complete bipartite graphs. An obvious open problem is whether the bounds presented in this paper can be further improved. Obtaining similar bounds on the k-planar crossing number of other graph classes is an intriguing open problem. We also posed an open problem of finding the largest positive integer $\xi(r)$ such that $cr(G) \leq \xi(r)$ implies $cr_2(G) \leq r$. In particular, we proved that $11 \leq \xi(0) \leq 35$. This definition can be easily generalized to the k-planar case: given positive integers k and r, find the largest integer $\xi_k(r)$ such that $cr(G) \leq \xi_k(r)$ implies $cr_k(G) \leq r$. Determining the value of $\xi_k(r)$ is an intriguing problem, even for the special case of $r = 0$.

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