# Blind Voronoi Game

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## Abstract

In the classical Voronoi game, two players compete over a user space by placing their facilities in a certain order, each trying to maximize the number of users served by their facilities. We introduce a new variant of the game, so called *blind Voronoi game*, in which the distribution of users in the underlying space is initially unknown to the players. As the game proceeds, players obtain a partial information about the distribution, which is limited to the distance of closest users to the facilities placed so far. We investigate mixed strategies for the players in this blind setting. In particular, we show that in the two-round blind Voronoi game on a line segment, there is a mixed strategy for the second player that guarantees him at least $1/3$ of users in expectation. In two dimensions, we show that our strategy guarantees an expected payoff of $1/5$ for the second player, when users are arbitrary distributed in the plane. We generalize our strategy to any $d$-dimensions and to any number of $k$ rounds, for $k, d \geq 1$.

## 1 Introduction

The Voronoi game was introduced by Ahn *et al.* [1] as a competitive facility location problem. The game consists of two players P1 and P2, and a set of users in an underlying space. Th players compete over the users by alternatively placing their facilities in the space, each trying to maximize the number of users served by their facilities, assuming that each user seeks service from the closest facility.

The Voronoi game has been the subject of active research over the past decade, and various variants of the problem have been studied in the literature, such as Voronoi game on line segments [3], in the plane [10], on graphs [2, 11, 12], and in simple polygons [6]. The problem has been also studied both in one round [7, 9], where P1 places all his facilities before P2 starts placing his facilities, and in the $k$-round version, where players alternate by placing one facility at a time for $k$ rounds [4]. The problem has been also studied in the discrete and continues spaces. In the former, users are considered as discrete points in the space (e.g., in [3, 4, 5]), while in the latter, users are distributed continuously over the space (e.g., in [1, 7, 10]).

In this paper, we introduce a new variant of the Voronoi game which we call the *blind Voronoi game*. The blind version differs from the previous variants of the game in that the distribution of users is initially unknown to the players. During the game, players obtain a partial information about the distribution, which is limited to the distance of closest users to the facilities placed so far. The blind version of the game is suitable in real situations of the facility location problem, in which no prior information about the location or distribution of the users is available to the players. In other words, players are "blind" to the location and distribution of the users in the space.

We study the *k-round discrete blind Voronoi game*, in which two players P1 and P2 compete over a user space by placing one facility at a time, for a total of $k$ rounds. The user distribution is initially unknown to the players, and the only information available to each player during the game is the distance of closest users to the current facilities. The *payoff* of each player is defined as the fraction of users served by his facilities. The goal of P2 is to maximize his expected payoff in the worst case, where the expectation is taken over the random choices of the player, and the worst case is taken over all possible user distributions. On the other hand, P1's goal is to minimize P2's expected payoff. We note that maximizing payoff for P1 is not a proper goal, as there is always a distribution of users in which the expected number of users for P1 is zero: just consider a distribution very dense at the last facility of P2. Moreover, we note that a pure strategy for the second player will not work, as the player has no information about the location of users, and hence, the adversary can easily adjust a distribution for which the player's payoff is zero. Therefore, in order to guarantee a payoff greater than zero, the second player needs to use a mixed strategy, i.e., use randomization.

We investigate the $k$-round discrete blind Voronoi game in $d$-dimensional Euclidean space. In one-dimensional two-round game, where users are discretely distributed on a line segment, we show that the second player has a mixed strategy that guarantees him an expected payoff of $1/3$, regardless of the initial distribution of the users. When users are discretely distributed in the plane, our mixed strategy guarantees an expected payoff of $1/5$ for the second player in the two-round

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game. We generalize our strategy to any $d$-dimensions and to any number of $k$ rounds, $d, k \geq 1$, yielding an expected payoff of $1/(kd+1)$ for the second player in the worst case.

## 2 Preliminaries

Let $\ell$ be a line segment, and $\pi$ be a distribution of users on $\ell$. Given two points $u, x \in \ell$, we say that $u$ is *served* (or is *covered*) by $x$, if $u$ is closer to $x$ than any other facility on $\ell$. Given a point $x \in \ell$ and a distribution $\pi$ on $\ell$, we denote by $N_\pi(x)$ the number of users in the distribution served by $x$. We assume that no two users or facilities can share the same location. We call two facilities *adjacent* if no user lies on the line segment between them. The *payoff* of each player is defined as the fraction of users served by his facilities. We denote the minimum expected payoff of P2 at the end of the game by $\Pi_2$. The goal of P2 is to maximize $\Pi_2$, while P1's goal is to minimize $\Pi_2$.

A *candidate set* $C$ is a set of points along with a probability function $p$ that assigns to each point $x \in C$ a probability $p(x)$ such that $\sum_{x \in C} p(x) = 1$. We denote by $G(C)$ the minimum expected number of users served by $C$, i.e., $G(C) = \min_\pi \left\{\sum_{x \in C} p(x) N_\pi(x)\right\}$, where the minimum is taken over all possible distributions $\pi$ of the users. Each point in a candidate set is called a *candidate point*. A candidate set $C$ with a maximum possible $G(C)$ is called an *optimal candidate set*.

## 3 Blind Voronoi game on a line segment

In this section, we consider the two-round blind Voronoi game on a line segment. Let $\ell$ be the underlying line segment. We solve the game in a top-down approach, analogous to the backward induction in extensive games. Namely, we first suppose that all moves are done, except the last move of P2. After resolving the last move of P2 , we suppose that the first moves of P1 and P2 are finished, and resolve the second move of P1. Similarly, we resolve the first move of P2, assuming that P1 has placed his first facility, and finally we consider the first move of P1.

We start by the last move of P2. Depending on the order of facilities on the line and their adjacency, several cases can arise as depicted in Figure 1. In this figure, $f$ and $s$ denote the facilities of the first and the second player, respectively. Moreover, the empty circles show candidate points for the second facility of P2. The cases are distinguished as follows. We assume, w.l.o.g., that the leftmost facility on the line segment is $f$. (The other case is symmetric.) Thus the first three facilities of the players have two possible permutations: either $ffs$ or $fsf$. Cases (a) to (d) of Figure 1 correspond to different arrangements for $ffs$, based on whether two facilities

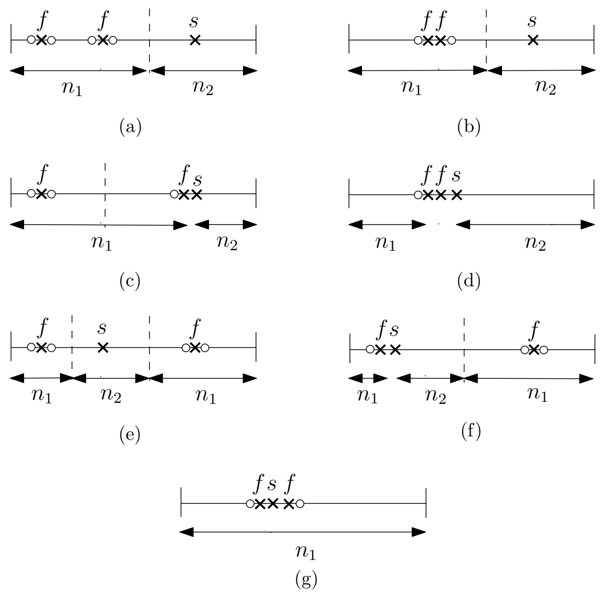


Figure 1: Seven cases for the last move of P2

are adjacent or not. (Adjacent facilities are shown close to each other with no empty circle in between.) Cases (e) to (g) of the figure demonstrate different arrangements for $fsf$. Any other configuration is analogous or symmetric to one of these cases.

As illustrated in the figure, all candidate points for the second facility of P2 are chosen adjacent to P1's facilities. The next lemma shows that this is indeed the best choice for the candidate sets.

**Lemma 1** *Any optimal candidate set for the last move of P2 consists of candidate points adjacent to P1's facilities, on those sides which are free from adjacent facilities. The probability of choosing each of these candidate points in an optimal candidate set must be equal.*

**Proof.** Let $C$ be an optimal candidate set for P2's last move. Let $\rho(y)$ denote the probability that a point $y \in \ell$ is covered by $C$, i.e., the sum of probabilities of candidate points that cover $y$. Note that

$$\begin{aligned} G(C) &= \min_\pi \left\{\sum_{x \in C} p(x) N_\pi(x)\right\} \\ &= \min_\pi \left\{\sum_{u \in \pi} \rho(u)\right\} \\ &\geq n \cdot \min_{y \in \ell} \{\rho(y)\}, \end{aligned}$$

where $\pi$ is taken over all possible distributions of users on $\ell$, and $y$ is taken over all points in $\ell$ where users may be located. Since there is a distribution $\pi$ where all users are dense at a point $y$ that minimizes $\rho(y)$, we have $G(C) = n \cdot \min_{y \in \ell} \{\rho(y)\}$. This means that we must cover every point $y \in \ell$ with some positive probability, otherwise $G(C) = 0$. The only way to cover a point $y$ adjacent to a P1 facility is with an adjacent candidate, and therefore, $\rho(y) = p(y)$ for such points. Moreover, the set of adjacent candidates covers all of $\ell$. Thus, the minimum of $\rho(y)$ is attained at an adjacent point.

It follows that the optimal strategy is to maximize the minimum of $\rho(y) = p(y)$ at adjacent points. This is achieved by selecting uniformly from just the adjacent points. If one adjacent point is more likely than another, its probability can be reduced and the minimum raised. If a non-adjacent point is in the candidate set $C$, it can be removed and its probability distributed evenly among the adjacent points, also raising the minimum of $\rho(y)$. □

**Lemma 2** *For cases (a) to (g) of Figure 1, the value of* $\Pi_2$ *is* $\frac{1}{4}$, $\frac{1}{2}$, $\frac{1}{3}$, $1$, $\frac{1}{4}$, $\frac{1}{3}$, *and* $\frac{1}{2}$, *respectively.*

**Proof.** Fix one of the cases in Figure 1. Let $C$ be the set of candidate points in that case, and $k$ be the number of candidate points in $C$. Let $I$ be the portion of line segment $\ell$ not already covered by P2, and $n_1$ be the number of users lying in $I$ (see Figure 1). The candidate points in $C$ are chosen in such a way that they jointly cover the whole area of $I$. As the probability of choosing each candidate point is equal by Lemma 1, the expected number of users served by the candidate points in $C$ is $n_1/k$. Considering that the number of users already served by P2 is $n_2 = n - n_1 \geq 0$, the expected number of users served after the last move of P2 is $n_2 + (n - n_2)/k \geq n/k$. Since $n_2$ can be zero in a distribution, the above inequality reduces to an equality at its minimum. Therefore, $\Pi_2 = \frac{1}{k}$ in the corresponding case. Since the size of the optimal candidate sets in cases (a) to (g) are 4, 2, 3, 1, 4, 3, 2 respectively, the statement of the lemma follows. □

By Lemma 2, the minimum value of $\Pi_2$ achievable by the second player is 1/4. However, this minimum value corresponds to two cases that the second player can avoid by choosing a proper strategy for placing his first facility. We show this in the next lemma.

**Lemma 3** *The best move for the first facility of P2 is to place his facility adjacent to the first facility of P1.*

**Proof.** If the first move of P2 is not adjacent to the first move of P1 (see Figure 2a), then P1 can place his last facility in the second round in such a way that either case (a) or case (e) of Figure 1 occurs, for which we already know that $\Pi_2$ is 1/4 by Lemma 2. On the other hand, if P2 places his first facility adjacent to the first facility of P1, he can guarantee a value of $\Pi_2 \geq 1/3$ in all cases derived from his move. □

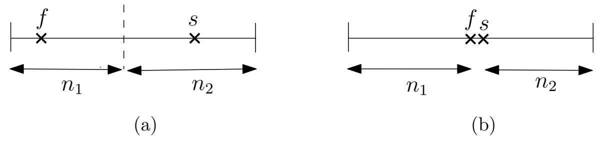


Figure 2: Two cases for the first move of P2

Lemmas 1–3 together yield the following theorem.

**Theorem 4** *There is a strategy for the second player in the two-round blind Voronoi game on a line segment that guarantees* $\Pi_2 \geq \frac{1}{3}$.

**Remark.** It is easy to see that the first player has always a strategy to force $\Pi_2$ to be at most $1/3$. The first move of P1 is arbitrary, as there is still no other facility, and there is no information about the users. If P2 places his first facility adjacent to the first facility of P1, which is indeed the best strategy for P2 by Lemma 3, then P1 in his second move, can put his second facility far from the first two facilities, in order to either case (c) or case (f) of Figure 1 arises, in both of which $\Pi_2$ is $1/3$.

## 4 Blind Voronoi game in $\mathbb{R}^d$

In this section, we generalize our result for the blind Voronoi game in one dimension to any fixed dimension $d \geq 2$, and for any number of rounds $k \geq 2$. In the following, we denote by $r(f)$ the distance of a facility $f$ to its nearest user.

Recall that the main idea behind our strategy in one dimension was to cover all points on the line where users may be located by a set of candidate points. We generalize this idea to $d$ dimensions as follows. Let $F$ be a set of facilities placed by the first player in $\mathbb{R}^d$. We call a set $C$ of points in $\mathbb{R}^d$ a *proper candidate set* with respect to $F$, if in the Voronoi diagram of $C \cup F$, the unoin of Voronoi cells of $C$ cover all regions in $\mathbb{R}^d$ that may contain users. This ensures P2 to cover every point that a user may be located with some positive probability, when choosing at random from the candidate set.

To present the generalized idea, we start by explaining our strategy in two dimensions. Suppose that P1 has placed his first facility at a point $f$ in the plane. Consider a circle $B$ centered at $f$ with radius $r(f)/2$. We choose three points evenly spaced on the boundary of $B$ as our candidate set $C$. (See Figure 3.) Note that the Voronoi cell of $f$ in the Voronoi diagram of $C \cup \{f\}$ is contained in $B$, and hence, is empty of any user. Therefore, all users are covered by the union of the Voronoi cells of $C$, and hence, $C$ is a proper candidate set. Now, if P2 chooses uniformly at random from $C$, he receives one third of the users in expectation in the worst case.

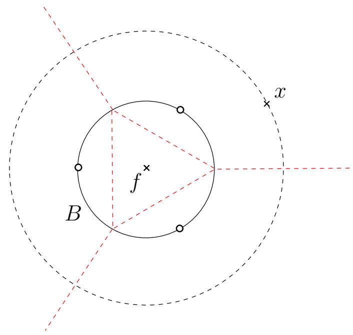


Figure 3: A facility $f$ and its closest user $x$. Three candidate points are evenly spaced on the boundary of a circle $B$ centered at $f$ of radius $\|x - f\|/2$.

Now, suppose that P1 places his second facility on a point $f'$. Again, we choose three points evenly spaced on a circle centered at $f'$ with radius $r(f')/2$, and add these three points to $C$. Now, $C$ has six candidate points, one of which is already chosen by P2 in the first round. (See Figure 4.) Moreover, $C$ is a proper candidate set with respect to $\{f, f'\}$, as the Voronoi cells of $f$ and $f'$ are both empty of any user. Now, P2 chooses from the remaining 5 candidate points of $C$ uniformly at random, and hence, receives at least 1/5 of the users in expectation. This yields a strategy for the second player in the two-round blind Voronoi game in the plane that guarantees $\Pi_2 \geq \frac{1}{5}$.

To extend our strategy to higher dimensions, we first define some notions. Given a point $p \in \mathbb{R}^d$ and a real value $r > 0$, we denote by $B(p, r)$ a $d$-dimensional ball of radius $r$ centered at $p$. The distance of a point $p$ to a point set $S$ is defined as $\min_{q \in S} \|p - q\|$. A straightforward extension of our two-dimensional idea to higher dimensions would be as follows. Let $f$ be a facility of P1, and let $r = r(f)/2$. We choose $d + 1$ points evenly spaced on the boundary of $B(f, r)$ as the candidate points. However, in $d \geq 5$ dimensions, the Voronoi cell of $f$ is no longer contained in $B(f, r(f))$, and hence, there may be users outside $B(f, r(f))$ covered by $f$. To overcome this issue, we need to choose the radius $r$ small

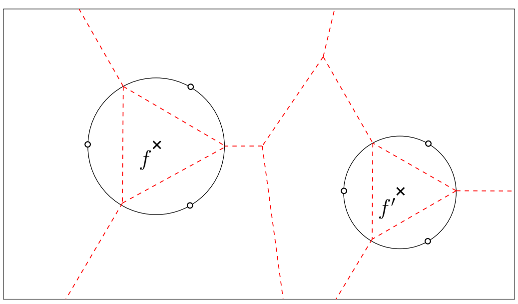


Figure 4: The Voronoi diagram of $C \cup \{f, f'\}$

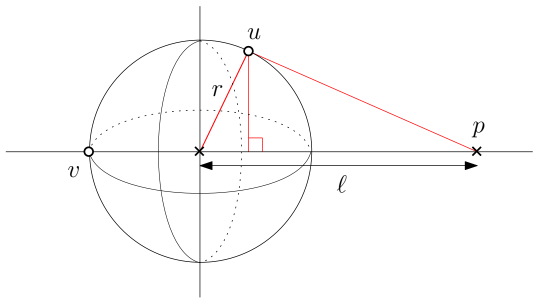


Figure 5: Point $p$ at distance $\ell$ to the center of a regular simplex.

enough to make sure that the Voronoi cell of $f$ is empty of any user. The following lemma is crucial for finding such a proper radius.

**Lemma 5** *Let $S$ be a regular $d$-dimensional simplex whose circumsphere has radius $r$, and let $p$ be a point at distance $\ell$ to the center of $S$. Then the distance of $p$ to the vertex set of $S$ is at most*

$$\sqrt{r^2 + \ell^2 - \frac{2r\ell}{d}}.$$

**Proof.** Fix a vertex $v$ of $S$. Suppose, w.l.o.g., that $S$ is centered at the origin, and that $v = (-r, 0, 0, \ldots, 0)$. As the angle subtended by any two vertices of $S$ through the origin is $\arccos(-1/d)$ [8], the $x$-coordinate of any other vertex of $S$ is $r/d$. Now, consider a point $p$ in the halfpence $x \geq 0$ at distance $\ell$ to the origin. Due to symmetry of $S$, the maximum distance of $p$ to the vertex set of $S$ is attained when $p$ lies on the line through $v$ and the origin, i.e., $p = (\ell, 0, 0, \ldots, 0)$. Now, fix an arbitrary vertex $u$ of $S \setminus \{v\}$. We can assume w.l.o.g. (by rotating around the $x$-axis) that $u$ lies in the $xy$-plane. As $u$ has distance $r$ to the origin, the $y$-coordinate of $u$ is $\sqrt{r^2 - (r/d)^2}$. (See Figure 5.) Therefore, the distance of $p$ to $u$ (and to any other vertex of $S \setminus \{v\}$) is

$$\sqrt{\left(\ell - \frac{r}{d}\right)^2 + r^2 - \left(\frac{r}{d}\right)^2} = \sqrt{r^2 + \ell^2 - \frac{2r\ell}{d}}.$$

□

Lemma 5 is interesting on its own. For example, it implies that any point on the circumsphere of a regular simplex $S$ has distance at most $\sqrt{(2 - 2/d)}r$ to the vertex set of $S$, where $r$ is the radius of the circumsphere of $S$.

We are now ready to provide a general strategy for the blind Voronoi game in any fixed dimensions. In the following we denote by $\mathrm{VD}(P)$ the Voronoi diagram of a point set $P$.

**Theorem 6** *For all integers $k, d \geq 1$, there is a strategy for the second player in the $k$-round blind Voronoi game in $\mathbb{R}^d$ that guarantees $\Pi_2 \geq \frac{1}{kd+1}$.*

**Proof.** Suppose we are in the $k$-th round, $k \geq 1$, and let $F$ be the set of $k$ facilities placed by $P1$. For each facility $f \in F$, we define $C_f$ as a set of $d+1$ points evenly spaced on the boundary of a ball of radius $r(f)/d$ centered at $f$. We take $C = \bigcup_{f\in F} C_f$ as the candidate set for P2. We claim that $C$ is a proper candidate set. To prove this, we show that the Voronoi cell of every point of $F$ in $\mathrm{VD}(C \cup F)$ is empty of any user. Fix a facility $f \in F$, and let $r = r(f)/d$. Consider an arbitrary user $u$, and let $\ell$ be the distance of $u$ to $f$. Note that $\ell \geq r(f) = rd$. Now, by Lemma 5, the distance of $u$ to the point set $C_f$ is at most $\sqrt{\ell^2 + r(r - \frac{2\ell}{d})}$, which is strictly less than $\ell$ for all values $0 < r < \frac{2\ell}{d}$. The latter inequality holds, since $r \leq \ell/d$ by our choice of $r$, and therefore, $u$ is closer to a point in $C_f$ than to $f$. Hence, $u$ cannot lie in the Voronoi cell of $f$ in $\mathrm{VD}(C \cup F)$, which completes the proof of the claim.

Now, the generalized strategy for the second player is as follows. During the first $k-1$ rounds, P2 places $k-1$ facilities arbitrarily on $k-1$ candidate points of $C$. In the $k$-th round, P2 places his $k$-th facility uniformly at random on one of the remaining candidate points of $C$. Let $S$ be the final set of facilities placed by P2. Since $S \subseteq C$, for any facility $s \in S$, the Voronoi cell of $s$ in $\mathrm{VD}(C \cup F)$ is completely contained in the Voronoi cell of $s$ in $\mathrm{VD}(S \cup F)$. In other words, any user lying in the Voronoi cell of $s$ in $\mathrm{VD}(C \cup F)$ is covered by $s$ at the end of the game. Let $R \subset S$ be the $k-1$ facilities placed by P2 before the last round, and let $n_1$ be the number of users lying in the Voronoi cells of $R$ in $\mathrm{VD}(C \cup F)$. As the Voronoi cells of $F$ in $\mathrm{VD}(C \cup F)$ are empty, the remaining $n - n_1$ users are covered by the Voronoi cells of the candidate points in $C \setminus R$. Since the probability of choosing each candidate point in $C \setminus R$ is equal in the last round, the expected number of users lying in the Voronoi cell of the selected facility in the last round is $(n - n_1)/(kd + 1)$, where $kd + 1$ is the size of $C \setminus R$. Therefore, the expected number of users lying in the Voronoi cells of $S$ in $\mathrm{VD}(C \cup F)$ is $n_1 + (n - n_1)/(kd+1) \geq n/(kd + 1)$, which completes the proof. □

## 5 Conclusion

In this paper, we introduced a new variant of the Voronoi game, so called blind Voronoi game, in which the distribution of users is initially unknown to the players. We provided a mixed strategy for the second player in the two-round blind Voronoi game that guarantees an expected payoff of 1/3, when users are distributed on a line, and an expected payoff of $1/(2d + 1)$, when users are arbitrary distributed in $\mathbb{R}^d$. Our strategy can be applied to other variants of the problem, such as the $k$-round $m$-facility game, where each player places $m$ facilities at each of the $k$ rounds. Other variants of the problem remain open for further research, such as blind Voronoi game on graphs and blind Voronoi game in simple polygons.

**Acknowledgments.** The authors would like to thank the anonymous reviewers for their valuable comments.

## References

[1] H.-K. Ahn, S.-W. Cheng, O. Cheong, M. Golin, and R. van Oostrum. Competitive facility location: the Voronoi game. *Theoret. Comput. Sci.*, 310(1):457–467, 2004.

[2] S. Bandyapadhyay, A. Banik, S. Das, and H. Sarkar. Voronoi game on graphs. *Theoret. Comput. Sci.*, 562:270–282, 2015.

[3] A. Banik, B. B. Bhattacharya, and S. Das. Optimal strategies for the one-round discrete Voronoi game on a line. *J. Comb. Optim.*, 26(4):655–669, 2013.

[4] A. Banik, B. B. Bhattacharya, S. Das, and S. Das. Two-round discrete Voronoi game along a line. In *Proc. 3rd Internat. Workshop Frontiers Algorithmics*, pages 210–220, 2013.

[5] A. Banik, B. B. Bhattacharya, S. Das, and S. Das. The 1-dimensional discrete Voronoi game. *Operations Research Letters*, 47(2):115–121, 2019.

[6] A. Banik, S. Das, A. Maheshwari, and M. Smid. The discrete Voronoi game in a simple polygon. *Theoretical Computer Science*, 2019.

[7] O. Cheong, S. Har-Peled, N. Linial, and J. Matousek. The one-round Voronoi game. In *Proc. 18th Annu. ACM Sympos. Comput. Geom.*, pages 97–101, 2002.

[8] H. S. M. Coxeter. *Regular polytopes.* Dover, New York, 1973.

[9] M. de Berg, S. Kisfaludi-Bak, and M. Mehr. On one-round discrete Voronoi games. In *Proc. 30th Annu. Internat. Sympos. Algorithms Comput.*, pages 37:1–37:17, 2019.

[10] S. P. Fekete and H. Meijer. The one-round Voronoi game replayed. *Comput. Geom. Theory Appl.*, 30(2):81–94, 2005.

[11] M. Mavronicolas, B. Monien, V. G. Papadopoulou, and F. Schoppmann. Voronoi games on cycle graphs. In *Proc. 33rd Internat. Sympos. Math. Found. Comput. Sci.*, pages 503–514, 2008.

[12] X. Sun, Y. Sun, Z. Xia, and J. Zhang. The one-round multi-player discrete Voronoi game on grids and trees. In *International Computing and Combinatorics Conference*, pages 529–540. Springer, 2019.