Shahshahani's Work in Dynamical Systems

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§1. Introduction. Siavash Shahshahani began his mathematical career as a graduate student in Berkeley in the mid 1960's. It was the golden era of the theory of smooth dynamical systems. Only a few years earlier, Smale had laid the modern foundations of this beautiful theory. His pioneering work prompted an army of dynamicists, global analysts and topologists to study the dynamics of flows and diffeomorphisms from a global point of view. Dynamical systems soon became the talk of the town and Shahshahani, who has always had good taste in mathematics, seized the opportunity and chose to work under the supervision of Smale.

Among the central themes of dynamics at the time were *structural stability* and *genericity* in a given family of dynamical systems. The problem of structural stability is to identify those elements in the family whose qualitative features remain intact after a small but arbitrary perturbation. The problem of genericity, on the other hand, is to pinpoint the dynamical properties shared by typical (in one sense or another) elements in the family. Shahshahani's first work [**Sh1**] addressed these issues for the family of second order ordinary differential equations on a manifold; these equations are global generalizations of Newton's equation of motion in classical mechanics. A particularly interesting subclass consists of the equations obtained from a conservative system by adding a dissipation force. In [**Sh2**], he obtained genericity results and proved a version of Morse inequalities for such dissipative systems. His subsequent work included symplectic structures on integral manifolds [**Sh3**], structural stability of the generalized van der Pol equation [**Sh4**], bounds on the number of periodic solutions of the Abel equation [**Sh6**], as well as a major contribution to mathematical biology [**Sh5**] for which I refer to Edalat's paper in this volume. In recent years, he has mainly been interested in holomorphic foliations on complex manifolds and iterations of rational maps on the Riemann sphere.

§2. Three examples. In what follows, I will try to briefly describe three of Shahshahani's results, hoping to show the flavor of his work in dynamical systems.

• **Generic properties of 2nd order ODE's.** Second order ordinary differential equations (ODE's) arise naturally as equations of motion in classical mechanics. A second order ODE $\ddot{q} = f(q, \dot{q})$ on the real line $\mathbb{R}$ can be thought of as a first order ODE on the tangent bundle $T\mathbb{R} \cong \mathbb{R}^2$ by introducing the velocity variable $v = \dot{q}$. The first order ODE is then represented by the vector field

$$v\frac{\partial}{\partial q} + f(q, v)\frac{\partial}{\partial v}$$

on $\mathbb{R}^2$. One can generalize this idea to a smooth n-manifold M as follows. Let $q = (q^1, \ldots, q^n)$ and $(q, v) = (q^1, \ldots, q^n, v^1, \ldots, v^n)$ be local coordinates on M and

the tangent bundle TM, respectively. A *second order ODE* on M is a vector field X on TM which has the local form

$$X(q, v) = \sum_{i=1}^{n} v^i \frac{\partial}{\partial q^i} + \sum_{i=1}^{n} f_i(q, v) \frac{\partial}{\partial v^i}$$

for some smooth functions f_i. More intrinsically, second order ODE's on M can be defined as vector fields $X : TM \to T^2 M$ which satisfy the condition $D\pi \circ X = \mathrm{id}_{TM}$, where $\pi : TM \to M$ is the canonical projection.

Now let M be a compact smooth manifold and $\mathcal{S}(M)$ be the space of all smooth second order ODE's on M equipped with the Whitney topology. The question arises as to what simple dynamical properties a typical vector field in $\mathcal{S}(M)$ possesses. To make sense of this, let us say that a subset $\mathcal{G} \subset \mathcal{S}(M)$ is *generic* if it contains a countable intersection of open dense sets in $\mathcal{S}(M)$. In particular, a generic set is dense since it is not hard to show that $\mathcal{S}(M)$ is a Baire space.

Shahshahani answered the above question in the following theorem. Recall that a singularity p of a vector field X with the flow $\{\varphi_t\}$ is *hyperbolic* if all the eigenvalues of $D\varphi_1(p)$ are off the unit circle. The *stable* (resp. *unstable*) *manifold* of p is the set of all q such that $\varphi_t(q) \to p$ as $t \to +\infty$ (resp. $t \to -\infty$). Similarly, let γ be a periodic orbit of X, $p \in \gamma$, and f be the Poincaré first return map defined on some local transversal to γ at p, with $f(p) = p$. Then γ is called hyperbolic if all the eigenvalues of $Df(p)$ are off the unit circle. The stable (resp. unstable) manifold of γ is the set of all q such that $\varphi_t(q) \to \gamma$ as $t \to +\infty$ (resp. $t \to -\infty$).

Theorem [Sh1]. *There exists a generic set $\mathcal{G} \subset \mathcal{S}(M)$ such that if $X \in \mathcal{G}$, then*

(i) *all the singularities and periodic orbits of X are hyperbolic;*

(ii) *the stable and unstable manifolds of the singularities and periodic orbits of X intersect transversally;*

(iii) *if $\dim M > 1$, no periodic orbit of X meets the zero section of TM.*

Since the singularities of a second order ODE must belong to the zero section, the theorem implies that generically there are only finitely many singular points. But there may well be a countably infinite number of periodic orbits, even generically, even when M is as simple as a circle [**Sh1**].

The above result is reminiscent of the Kupka-Smale Theorem according to which a generic vector field on a compact manifold has only hyperbolic singularities and periodic orbits, and their stable and unstable manifolds intersect transversally (compare [**K**] and [**Sm2**]). Shahshahani's proof uses the flow-box perturbation techniques of Kupka and Smale, and is based on a fundamental lemma which asserts that if $X \in \mathcal{S}(M)$ is approximated in a flow-box F by a sequence Y_n of vector fields on TM, then X can be approximated in F by a sequence $X_n \in \mathcal{S}(M)$, with each X_n being smoothly conjugate in F to Y_n.

• **Dissipative systems.** In this work, Shahshahani studied a special class of second order ODE's on manifolds which are the global analogues of dissipative systems in classical mechanics. Consider a compact smooth n-manifold M with the tangent bundle TM and the canonical projection $\pi : TM \to M$. Fix a smooth Riemannian metric g on M. Define the *energy function* $E : TM \to \mathbb{R}$ by $E = K + V$. Here the

kinetic energy K is defined by $K(q, v) = g_q(v, v)$, whereas the *potential energy* V is any given smooth function which is constant along the fibers of π. The Hamiltonian energy function E gives rise to a vector field X_E as follows. Denote by ω^* the canonical symplectic form on the cotangent bundle T^*M. Recall that ω^* depends only on the smooth structure of M and if $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ is a local coordinate system on T^*M, then

$$\omega^* = \sum_{i=1}^{n} dp_i \wedge dq^i.$$

The pull-back ω of ω^* under the isomorphism $TM \xrightarrow{\cong} T^*M$ given by the metric g is a symplectic form on TM, which of course depends on g. The vector field X_E on TM is then determined by

$$dE = \omega(\,\cdot\,, X_E).$$

It is easy to check that $D\pi \circ X_E = \mathrm{id}_{TM}$ so that X_E is a second order ODE.

Dissipative systems are obtained by adding to such an X_E a dissipation force. By definition, a vector field Δ on TM is called a *dissipation force* if (i) Δ is "vertical" in the sense that $D\pi \circ \Delta = 0$; (ii) $\widehat{g}(\Delta, \nabla_{\widehat{g}} K) < 0$ away from the zero section of TM. Here $\widehat{g}$ is the induced Riemannian metric on TM with respect to which $\nabla_{\widehat{g}} K$ is vertical and $\nabla_{\widehat{g}} V$ is horizontal. Roughly speaking, condition (i) means that dissipation forces depend only on velocity, whereas (ii) means that they act against the kinetic energy to slow down the system. Vector fields of the form $X_E + \Delta$ are called *dissipative systems*; they are clearly second order ODE's.

In [**Sh2**], Shahshahani determined the dynamical structure of a generic dissipative system. To state his result, recall that the *non-wandering set* Ω_X of a vector field X with the flow $\{\varphi^t\}$ is the set of all points q such that for every neighborhood U of q there exists arbitrarily large t for which $\varphi^t(U) \cap U \neq \emptyset$. The vector field X is said to be Ω-*stable* if its orbit structure on the invariant set Ω_X persists under small perturbations. More specifically, if for every vector field Y sufficiently close to X there is an orbit-preserving homeomorphism $h : \Omega_X \to \Omega_Y$.

Theorem [Sh2]. *Fix a Hamiltonian vector field X_E with a finite number of singularities, all non-degenerate. Then there exists an open and dense subset $\mathcal{D}$ of all dissipation forces on TM such that if $\Delta \in \mathcal{D}$ and $X = X_E + \Delta$, then*

(i) *X is Ω-stable and Ω_X consists of a finite number of hyperbolic singularities;*
(ii) *the tangent bundle TM is the union of stable manifolds of the singularities of X;*
(iii) *at every singularity of X the dimension of the stable manifold is at least as large as the dimension of the unstable manifold.*

In the special case of a vector field X_E of the form $\ddot{x} = f(x)$ on $M = S^1$ with the standard Riemannian structure, a sharper version of the above theorem can be proved (compare Theorem 2 in [**Sh2**]).

Another result obtained by Shahshahani was "Morse inequalities" for dissipative systems. Given a vector field X on a manifold M, Morse inequalities compare the Betti numbers of M to the number of stable manifolds of X of a given dimension. Such inequalities were originally obtained by Morse in the case of gradient vector

fields (see for example [**M**]). They were generalized by Smale to the vector fields now commonly known as "Morse-Smale" [**Sm1**].

Shahshahani's version of Morse inequalities can be stated as follows. With M and a generic $X = X_E + \Delta$ as above, let β_i be the i-th Betti number of M and M_i be the number of stable manifolds of X of dimension i. Observe that by the above theorem, $M_i = 0$ for $i = 0, \ldots, n - 1$.

Theorem [Sh2]. *For each $0 \le k \le n$, we have the inequality*

$$\sum_{i=0}^{k} (-1)^{k+i} M_{n+i} \ge \sum_{i=0}^{k} (-1)^{k+i} \beta_i.$$

Moreover, the case $k = n$ is an equality:

$$\sum_{i=0}^{n} (-1)^i M_{n+i} = \sum_{i=0}^{n} (-1)^i \beta_i = \chi(M).$$

• **Periodic solutions of the Abel equation.** The second half of Hilbert's 16th problem asks for a bound $N = N(d)$ on the number of limit cycles of a polynomial vector field $P(x, y) \, \partial/\partial x + Q(x, y) \, \partial/\partial y$ in the plane in terms of $d = \max\{\deg P, \deg Q\}$. Despite numerous attempts and partial results, the problem has not yet been settled in full generality. Even the fact that a polynomial vector field in the plane has finitely many limit cycles was proved as recently as 1987 by Ilyashenko.

An easier problem of the same nature is to estimate the number of periodic solutions of the Abel differential equation

$$\dot{x} = x^n + a_{n-1}(t)x^{n-1} + \cdots + a_1(t)x + a_0(t),$$

where $x \in \mathbb{R}$, $t \in [0, 1]$, and the a_i are smooth functions on $[0, 1]$. Here, by a periodic solution $x = x(t)$ is meant one which satisfies $x(0) = x(1)$. Shahshahani addressed this problem in the case $n \le 3$. Using an elementary but ingenious perturbation argument, he proved the following

Theorem [Sh6]. *The Abel equation has at most n periodic solutions if $n \le 3$.*

Here is the idea of his proof for the case $n = 3$ (the cases $n = 1, 2$ are easy to deal with). He first observes that simple (=multiplicity 1) periodic solutions persist under a small perturbation of the equation. More generally, he shows that at most k periodic solutions can bifurcate off a periodic solution of multiplicity k. He then uses this to prove that an equation with a periodic solution of multiplicity > 1 has precisely 3 periodic solutions. Finally, for an arbitrary equation he constructs a path connecting it to an equation with at most 3 simple periodic solutions and applies a continuity argument to deduce the result.

It comes as no surprise that his method could not yield similar estimates in higher degrees. In fact, for $n \ge 4$ the Abel equation can have an arbitrary number of periodic solutions (compare [**L**]). Even more dramatic is the fact that when $n \ge 4$ the return maps $x(0) \mapsto x(1)$ of these equations are dense in the space of all orientation-preserving homeomorphisms [**P**]. Very recently, Ilyashenko has found an upper bound for the number of periodic solutions of the equation in terms of n and the

size of the a_i [**I**]. More specifically, he proves that if $n \geq 4$ and $\sup_{t \in [0,1]} |a_i(t)| < C$ for every $0 \leq i \leq n-1$, then the equation has at most $N = N(n,C)$ periodic solutions, where

$$N \leq 8 \exp \left\{ (3C+2) \, \exp \left(\frac{3}{2}(2C+3)^n \right) \right\}.$$

This double exponential bound seems to be far from optimal, but it is the only estimate available at the present time.

§3. Epilogue. Let me conclude with a few non-mathematical words. When Shahshahani returned to Iran in the mid 1970's, he discovered a new generation of talented students eager to learn fresh ideas in mathematics beyond the standard and (then) old-fashioned university program. He responded by redesigning the curriculum, teaching exciting new courses, and coordinating interesting seminars. Through him, many students learned for the first time about differential and algebraic topology, dynamical systems, mathematical biology, and other beautiful subjects. His love and knowledge of mathematics and all-around intellectual character made him an enormously charismatic figure. In the eyes of the students, he personified what being a professional mathematician is all about.

Shahshahani's profound impact on the mathematics of Iran is an undeniable fact. Those of us who have had the privilege of working with him would happily testify to that. But do not take our word for it: the next generations of Iranian mathematicians, for whom Shahshahani has made so many personal sacrifices, will convince you.

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