A Note on Elliptic Equations with Nonlinearities that are Sum of a Sublinear and a Superlinear Term

Hossein Torabi Tehrani

Dedicated to Dr. Siavash Shahshahani on his 60th birthday

Introduction

In this note we are mainly concerned with existence and multiplicity results for semilinear elliptic equations of the form

$$(1)\qquad \begin{cases} -\Delta u = a(x)|u|^{q-1}u + g(u) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

in a bounded smooth domain Ω of $\mathbb{R}^n$ $(n \geq 3)$. Here, $a(x)$ is a nonnegative continuous function in Ω, $0 < q < 1$, and we always assume that $g(u)$ behaves like $|u|^{p-1}u$ for some $1 < p$ near zero, in the sense that $g(u) = |u|^{p-1}u + \circ(|u|^p)$ at $u = 0$. Other conditions will vary and consequently be indicated in the appropriate hypotheses.

Note that the nonlinearity $h(u) = a(x)|u|^{q-1}u + g(u)$ is, for $u > 0$, a combination of convex and concave terms. This type of equations has been recently investigated by a number of authors (see references [1], [2], [4]). In particular in [1] Ambrosetti, Brezis and Cerami consider the case $a(x) = \lambda > 0$;

$$(2)\qquad \begin{cases} -\Delta u = \lambda|u|^{q-1}u + g(u) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

and prove:

Theorem 0.1 *([1]) There exists $\Lambda > 0$ such that for $0 < \lambda < \Lambda$, (2) has a positive and a negative solution.*

The proof of this simple result is based on the method of sub- and super-solutions and requires *no restriction on the growth of g at infinity.* On the other hand, using minimax arguments (in particular Mountain Pass Theorem) the authors establish the existence of a second pair of fixed sign solutions for all $0 < \lambda < \Lambda$, under the assumption $g(s) = |s|^{p-1}s + \circ(|s|^p)$ at $s = 0$ and $s = \infty$, $1 < p < n + 2/n - 2$. We recall that the first pair of solutions are in fact local minimizers of I_λ, the associated functional of (2), and are located on *negative* level sets of it. More precisely, if $u_+ > 0$, $u_- < 0$ denote the first pair of solutions and $v_+ > 0$, $v_- < 0$ the second pair, then

$$I_\lambda(u_\pm) < 0, \qquad I_\lambda(v_\pm) > 0,$$

where

$$I_\lambda(u) = \int \frac{1}{2}|\nabla u|^2 - \frac{\lambda}{q+1}|u|^{q+1} - G(u)$$

and $G(s) = \int_0^s g(t)dt$. Later on Ambrosetti, Azorero and Peral [2] improved the results of [1].

Theorem 0.2 *([2]) If $g(s) = |s|^{p-1}s + \circ(|s|^p)$ at $s = 0, s = \infty$, for some $1 < p \leq n+2/n-2$, then there exists $\Lambda^* > 0$ such that, for $0 < \lambda < \Lambda^*$, (2) has a third solution $u_3, u_3 \neq u_\pm$, with $I_\lambda(u_3) < 0$.*

In addition, in the same paper, the existence of another solution v_3, with positive "energy", $I_\lambda(v_3) > 0$, is established under more restrictive assumptions on g (in particular, $p < n + 2/n - 2$). On the other hand when g is odd, Lusternik-Schnirelman theory for Z_2-invariant functionals (see [3], [6]) is used in [1] to prove

Theorem 0.3 *([1]) Let g be odd, $g(s) = |s|^{p-1}s + \circ(|s|^p)$ at $s = 0, s = \infty$.*

(i) If $1 < p \leq n + 2/n - 2$, then there exists $\Lambda^{**} > 0$, such that for all $\lambda \in (0, \Lambda^{**})$ problem (2) has infinitely many solutions with $I_\lambda(u) < 0$.

(ii) If $1 < p < n + 2/n - 2$, then for all $\lambda \in (0, \Lambda^{**})$, (2) has infinitely many solutions with $I_\lambda(u) > 0$.

The proof of all these results, with the exception of Theorem 0.1, are based on variational arguments and require the standard growth restriction $|g(s)| \leq C(1 + |s|^p), p \leq \frac{n+2}{n-2}$. Indeed solutions are obtained as critical points of I_λ on the natural function space $H_0^1(\Omega)$. Such growth restrictions are then needed to ensure that the functional is well defined on $H_0^1(\Omega)$.

Our main goal in this note is to prove a conjecture raised in [2] and show how the presence of the "sublinear" term, $\lambda|u|^{q-1}u$, in fact enables one to use variational tools to obtain multiplicity results like Theorems (0.2), (0.3) above, for solutions with *negative* energy *without any growth restriction on the "superlinear" term g*. The main result of the first section of this paper is the following :

Theorem 0.4 *Let $g(s) = |s|^{p-1}s + \circ(|s|^p)$ at $s = 0$, and assume $g(s)$ is nondecreasing or Lipschitz continuous in a neighborhood of zero.*

(i) There exists $\Lambda^* > 0$ such that for $\lambda \in (0, \Lambda^*)$, (2) has a third solution w with $I_\lambda(w) < 0$. If, furthermore, g is C^1 in a neighborhood of zero, then w changes sign in Ω.

(ii) If, in addition, $g(-s) = -g(s)$ for s near zero, then there exists $\Lambda^* > 0$ such that, for $0 < \lambda < \Lambda^*$, (2) has infinitely many solutions with $I_\lambda(u) < 0$.

The proof of this result is a combination of two ingredients. First we replace g with a function $\tilde{g}$ that is equal to g close to zero and has subcritical growth at infinity (recall that $p = n+2/n-2$ is the critical exponent from the viewpoint of Sobolev embedding). Theorems (0.2) and (0.3) are then applied to get the respective multiplicity results for

$$(3) \qquad \begin{cases} -\Delta u = \lambda|u|^{q-1}u + \tilde{g}(u) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

Then we prove *a priori* L_∞ estimates for solutions of (3) with *negative* energy, from which it finally follows that, for λ sufficiently small, the solutions of (3) with *negative* energy in fact solve the original equation (2).

In section 2 we turn to the general form of equation (1) and allow $a \in C(\Omega)$. Since then, we are unable to find suitable sub- and super-solutions for (1), we will use variational tools and establish the existence of two pairs of positive (respectively, negative) solutions when $\int_\Omega a(x)^{\frac{2}{1-q}}$ is sufficiently small. Finally, we will show how the techniques of section 1 can be modified to prove multiplicity results, similar to those in section 1, for equation (1).

Now a few words about the notation. Throughout the paper we will make use of the following notations. $||\cdot||$ denotes the norm in the space $H_0^1(\Omega)$. $|\cdot|_p, 1 \leq p \leq \infty$, as usual, denotes the norm in the space $L^p(\Omega)$. $C, C_1, C_2, \ldots$ will be used to denote (possibly different) constants whose exact values are immaterial.

1 The case $a(x) = \lambda$

Here we consider the semilinear equation

$$\text{(I)} \qquad \begin{cases} -\Delta u = \lambda |u|^{q-1}u + g(u) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

We shall assume that $g \in C(\mathbb{R}; \mathbb{R})$ satisfies

$$(1.5) \qquad g(s) = |s|^{\alpha-2}s + \circ(|s|^{\alpha-1}) \text{ at } s = 0, \qquad 2 < \alpha.$$

Note that formally, weak solutions of (I) in $H_0^1(\Omega)$ are critical points of the functional

$$I_\lambda(u) = \int \frac{|\nabla u|^2}{2} - \frac{\lambda}{q+1}|u|^{q+1} - G(u) \qquad u \in H_0^1(\Omega)$$

where $G(s) = \int_0^s g(t)dt$. However, our assumption does not allow the conclusion that I_λ is differentiable or even finite on $H_0^1(\Omega)$. So we start by choosing $2 < \beta < 2^* = \frac{2n}{n-2}$, $\beta < \alpha$, and define a new function $\tilde{g}_\lambda$

$$(1.6) \qquad \tilde{g}_\lambda(s) = \begin{cases} |s|^{\beta-2}s & s \le -\lambda \\ \psi_\lambda & -\lambda \le s \le -\frac{\lambda}{2} \\ g(s) & -\frac{\lambda}{2} \le s \le \frac{\lambda}{2} \\ \phi_\lambda & \frac{\lambda}{2} \le s \le \lambda \\ |s|^{\beta-2}s & \lambda \le s \end{cases}$$

Here, increasing functions $\psi_\lambda \in C([-\lambda, -\frac{\lambda}{2}]; \mathbb{R})$ and $\phi_\lambda \in C([\frac{\lambda}{2}, \lambda]; \mathbb{R})$ are chosen in a suitable way so as to make $\tilde{g}_\lambda$ continuous on $\mathbb{R}$. Note that if g is nondecreasing in a neighborhood of zero, then, for λ sufficiently small, $\tilde{g}_\lambda$ will be nondecreasing on $\mathbb{R}$. Similarly, if g is odd close to zero, i.e.

$$(1.7) \qquad g(s) = -g(-s) \quad \text{for } |s| < \delta, \quad \delta > 0$$

then for λ small we can take $\psi_\lambda(-s) = -\phi_\lambda(s)$ which makes $\tilde{g}_\lambda$ a globally defined *odd* function. Next we fix $\lambda > 0$ small, and study the solutions of the semilinear equation

$$(1.8) \qquad \begin{cases} -\Delta u = \lambda |u|^{q-1}u + \tilde{g}_\lambda(u) & \text{in} \quad \Omega \\ u = 0 & \text{on} \quad \partial\Omega \end{cases}$$

with negative "energy". These are the critical points of the functional

$$(1.9) \qquad \tilde{I}_\lambda(u) = \int \frac{|\nabla u|^2}{2} - \frac{\lambda}{q+1}|u|^{q+1} - \tilde{G}_\lambda(u)$$

where $\tilde{G}_\lambda(s) = \int_0^s \tilde{g}_\lambda(t)dt$, with $\tilde{I}_\lambda(u) < 0$.

Proposition 1.1 *Let $u \in H_0^1(\Omega)$ be a solution of (1.8) with $\tilde{I}_\lambda(u) < 0$, then for λ small*

$$(1.10) \qquad |u(x)| \le C\lambda^\zeta \quad \textit{for all } x \in \overline{\Omega}$$

where $0 < C$ and $1 < \zeta$ are independent of λ.

Proof: From the definition of $\tilde{g}_\lambda$ it readily follows that for λ small

$$\begin{cases} 0 \le \beta \tilde{G}_\lambda(s) \le \tilde{g}_\lambda(s)s & \text{for } |s| \ge \lambda \\ 0 \le \beta \tilde{G}_\lambda(s) = \beta \int_0^s \tilde{g}_\lambda(t)dt \le 2\beta \frac{|s|^\alpha}{\alpha} \le 2\frac{\beta}{\alpha}\lambda^\beta & \text{for } -\lambda \le s \le \lambda \end{cases}$$

Thus

$$0 \le \beta \tilde{G}_\lambda(s) \le \tilde{g}_\lambda(s)s + 2\frac{\beta}{\alpha}\lambda^\beta \qquad \forall s \in \mathbb{R} \tag{1.11}$$

Since u is a solution of (1.8) and $\tilde{I}_\lambda(u) < 0$, we have

$$\int |\nabla u|^2 - \lambda |u|^{q+1} - \tilde{g}_\lambda(u)u = 0 \tag{1.12}$$

$$\int \frac{1}{2}|\nabla u|^2 - \frac{\lambda}{q+1}|u|^{q+1} - \tilde{G}_\lambda(u) < 0 \tag{1.13}$$

Hence, using (1.11) we get

$$(\frac{\beta}{2} - 1)\int \tilde{G}_\lambda(u) \le \lambda(\frac{1}{q+1} - \frac{1}{2})\int |u|^{q+1} + C\lambda^\beta |\Omega|$$

Consequently, taking (1.13) into account and using Sobolev inequality we have

$$\begin{aligned} \int |\nabla u|^2 &\le C_1 \lambda \int |u|^{q+1} + C_2\lambda^\beta \\ &\le C_1\lambda(\int |\nabla u|^2)^{\frac{q+1}{2}} + C_2 \lambda^\beta \end{aligned} \tag{1.14}$$

Next we define $f : \mathbb{R}^+ \to \mathbb{R}$; $f(t) = t - C_1\lambda t^{\frac{q+1}{2}}$. Setting $\int |\nabla u|^2 = a$, equation (1.14) writes as $f(a) \le C_2\lambda^\beta$. This implies that $a = \int |\nabla u|^2 \le C_3\lambda^r, r > 1$. In fact f is a C^2 convex function and since $q < 1$, $f(t) = 0$ has the unique solution $t_0 = (C_1\lambda)^{\frac{2}{1-q}}$. Now if $f(t_0 + h) = C_2\lambda^\beta$, then $f(t_0) + f'(t_0)h \le f(t_0 + h) = C_2\lambda^\beta$, which yields $h \le C_2\lambda^\beta / f'(t_0)$. Thus $t_0 + h \le \frac{2C_2}{1-q}\lambda^\beta + (C_1\lambda)^{\frac{2}{1-q}}$. So going back to equation (1.14) we infer

$$\int |\nabla u|^2 \le C_1\lambda^{\frac{2}{1-q}} + C_2\lambda^\beta \le C\lambda^r \quad r = \min(\beta, \frac{2}{1-q}) > 2 \tag{1.15}$$

Thus, by Sobolev inequality,

$$|u|_{2^*} \le C(\int |\nabla u|^2)^{\frac{1}{2}} \le C\lambda^{\frac{r}{2}} \tag{1.16}$$

Now since u solves (1.8) and $\tilde{g}_\lambda$ has subcritical growth at infinity, we can use L^p elliptic regularity theory to get L^∞ bound on u. In fact we have

$$-\Delta u = \lambda |u|^{q-1}u + \tilde{g}_\lambda(u) = h(x)$$

so for $\gamma_1 = 2^*/(\beta - 1)$,

$$\begin{aligned} |h|_{\gamma_1} &\le \lambda ||u(x)|^q|_{\gamma_1} + |\tilde{g}_\lambda(u)|_{\gamma_1} \\ &\le \lambda |u|^q_{q\gamma_1} + |u|^{\beta-1}_{2^*} \\ &\le C\lambda |u|^q_{\gamma_1} + C(\lambda)^{\frac{r}{2}(\beta-1)} \\ &\le C\lambda(\lambda)^{\frac{r}{2}q} + C(\lambda)^{\frac{r}{2}(\beta-1)} \\ &\le C\lambda^{\zeta_1} \qquad \zeta_1 = \min(1 + \frac{r}{2}q, \frac{r}{2}(\beta - 1)) > 1 \end{aligned}$$

Thus, by Calderon-Zygmund inequality,

$$\begin{aligned} |u|_{W^{2,\gamma_1}} &\leq C(|u|_{\gamma_1} + |h|_{\gamma_1}) \\ &\leq C\lambda^{\tilde{\zeta_1}} \qquad \tilde{\zeta_1} = \min(\zeta_1, r/2) > 1 \end{aligned}$$

After a finite number of iterations we get:

$$|u|_\infty \leq C\lambda^\zeta \qquad \text{for some } \zeta > 1.$$

This completes the proof. □

A direct consequence of this result is that for λ sufficiently small, any solution of (1.8) with negative energy is in fact a solution of (I). Now using this device we can improve the results of [1] and [2].

Theorem 1.1 *Let g be a continuous function satisfying (1.5), i.e.*

$$g(s) = |s|^{\alpha-2}s + \circ(|s|^{\alpha-1}) \quad \text{at } s = 0, \quad \alpha > 2$$

Assume further that g is either nondecreasing or Lipschitz continuous in a neighborhood of zero. Then

(1) There exists $\lambda^* > 0$ such that, for $0 < \lambda < \lambda^*$, (I) has three solutions $u_1 > 0, u_2 < 0$ and u_3, with $I_\lambda(u_i) < 0, i = 1,2,3$. If, in addition, g is C^1 near zero, then u_3 changes sign in Ω.

(2) If g satisfies (1.7), that is

$$g(-s) = -g(s) \qquad \text{for } |s| < \delta, \quad \delta > 0,$$

then there exists $\lambda^{**} > 0$, such that for $0 < \lambda < \lambda^{**}$, (I) has infinitely many solutions with $I_\lambda(u) < 0$.

Proof : 1. The existence of a positive and a negative solution is essentially proved in [1]. Here we shall present a slightly different argument which makes it possible to get the first two solutions under weaker conditions on g. In fact if $e > 0$ is the solution of $-\Delta u = 1$ in Ω with zero boundary condition, then by (1.5) it is clear that, for λ small, $\overline{u}(x) = \frac{\lambda}{4|e|_\infty}e(x)$ is a super-solution, and $\underline{u}(x) = \epsilon\phi_1(x)$, for ϵ sufficiently small, is a sub-solution for (I), where ϕ_1 is the first eigenfunction of $-\Delta$ in Ω with Dirichlet boundary condition. We take ϵ so small that $\epsilon\phi_1 < \overline{u}(x)$. Now we can apply Theorem 2.4 of [6] and obtain u_1, a solution of (I) as the solution of the following minimization problem:

$$\min_{u\in M} I_\lambda(u) = \min_{u\in M} \int \frac{1}{2}|\nabla u|^2 - \frac{\lambda}{q+1}|u|^{q+1} - G(u)$$

where

$$M = \{u \in H_0^1 : \underline{u}(x) \leq u(x) \leq \overline{u}(x), x \in \Omega\}$$

Since g satisfies (1.5),

$$I_\lambda(\epsilon\phi_1) \leq \epsilon^2 \int \frac{1}{2}|\nabla\phi_1|^2 - \frac{\lambda}{q+1}\epsilon^{q+1} \int \phi_1^{q+1} + C_1\epsilon^\alpha + \circ(\epsilon^\alpha) < 0 \text{ for } \epsilon \text{ small},$$

so $I_\lambda(u_1) < 0$ and $0 < u_1(x) \leq \frac{\lambda}{4}$ for $x \in \Omega$. Now if g is either increasing or Lipschitz continuous near zero, we can apply strong maximum principle and conclude that:

$$\text{(1.17)} \qquad \begin{cases} \underline{u}(x) < u_1(x) < \overline{u}(x) & \text{in } \Omega \\ \frac{\partial}{\partial n}(u_1 - \underline{u}) < 0 & \text{on } \partial\Omega \\ \frac{\partial}{\partial n}(\overline{u} - u_1) < 0 & \text{on } \partial\Omega \end{cases}$$

Thus, u_1 is a local minimizer of I_λ in C^1 topology and since $0 < u_1(x) \le \frac{\lambda}{4}$, a local minimizer of $\tilde{I}_\lambda$ in C^1 topology as well. Now in view of Theorem 8 of [5], u_1 is a local minimizer of $\tilde{I}_\lambda$ in H_0^1 topology. Similarly, working with $-\overline{u}$ (respectively $-\underline{u}$) as a sub- (respectively super-) solution, we obtain a negative solution $u_2(x)$, $\tilde{I}_\lambda(u_2) < 0, -\frac{\lambda}{4} \le u_2(x) < 0$, which is another local minimizer of $\tilde{I}_\lambda$ in H_0^1 topology. Now Theorem 2.1 part(i) of [2] applies and we find a third solution u_3 of (1.8) with $\tilde{I}_\lambda(u_3) < 0$, which by proposition 1, for λ small, is in fact a third solution of (I). If g is C^1 near zero, we may take $\tilde{g}_\lambda \in C^1(\mathbb{R}, \mathbb{R})$. Now we recall Theorem 2.2 of [1] (note that the proof given there in fact works for arbitrary C^1 functions):

There exists $A > 0$ such that, for λ small, problem (I) has at most one positive (negative) solution u_+ (u_-) such that $|u_+(u_-)| \le A$.

Since $u_1 > 0, u_2 < 0$ and $\tilde{I}_\lambda(u_i) < 0$ which, for λ small, implies $|u_i(x)| \le \lambda/2$ for $x \in \Omega$, the above result shows that u_3 necessarily changes sign in Ω.

2. We take $\lambda < \delta$ and replace g by the *odd* function $\tilde{g}_\lambda$. Theorem 2.5 part 1 of [1] applies and we get infinitely many solutions of (1.8) with $\tilde{I}_\lambda(u) < 0$. An application of proposition 1 completes the proof. □

2 The general case

In this section we are concerned with the following equation

$$\text{(II)} \qquad \begin{cases} -\Delta u = a(x)|u|^{q-1}u + g(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where now $a(x)$ is a nonnegative continuous function. Since $a(x)$ is not a constant, we are unable to find sub- and super-solutions for (II), nevertheless we still manage to show the existence of fixed sign solutions of (II) by using variational methods. In our first result we shall assume that the superlinear term satisfies the standard growth restriction, although we will remove this condition later. For $u \in H_0^1(\Omega)$ we set

$$I(u) = \int \frac{1}{2}|\nabla u|^2 - a(x)\frac{|u|^{q+1}}{q+1} - G(u)$$

with $G(s) = \int_0^s g(t)dt$. Let $a^+ = \max\{a, 0\}$.

Theorem 2.1 *Let g be a continuous function satisfying (1.5), that is*

$$g(s) = |s|^{\alpha-2}s + \circ(|s|^{\alpha-1}) \text{ at } s = 0, \quad \alpha > 2$$

Furthermore let

$$\text{(2.19)} \qquad |g(s)| \le C(1 + |s|^p) \qquad 1 < p < n + 2/n - 2$$

$$\text{(2.20)} \qquad g(s)s \ge 0 \qquad \text{for all } s$$

Then there exists $\delta > 0$ such that for $|a|_{\frac{2}{1-q}} < \delta$

(1) Problem (II) has two solutions, $w_1 > 0, w_2 < 0$, with $I(w) < 0$.

(2) If, in addition, g satisfies

$$\text{(2.21)} \qquad 0 \le \beta G(u) \le g(u)u \qquad \text{for } |u| \text{ large}, \ \beta > 2$$

then (II) has another pair of fixed sign solutions $v_1 > 0, v_2 < 0$ with $I(v_i) > 0$, $i = 1, 2$.

(3) If g is odd, then (II) has infinitely many solutions with $I(u) < 0$.

Proof : For $u \in H_0^1(\Omega)$ we define:

$$I^+(u) = \int \frac{1}{2}|\nabla u|^2 - \int a(x)\frac{|u^+|^{q+1}}{q+1} - G(u^+)$$

$$I^-(u) = \int \frac{1}{2}|\nabla u|^2 - \int a(x)\frac{|u^-|^{q+1}}{q+1} - G(-u^-)$$

Nonnegative (respectively, nonpositive) solutions of (II) correspond to critical points of I^+ (respectively, I^-). Using (1.5), (2.19) and Holder inequality we have

$$I^{\pm}(u) \geq 1/2||u||^2 - C|a|_{\frac{2}{1-q}}||u||^{q+1} - \epsilon||u||^2 - C(\epsilon)||u||^{p+1}$$

So there exists $\delta, r, a > 0$ such that, for $|a|_{\frac{2}{1-q}} < \delta$, we have

$$I^{\pm}(u) \geq a > 0 \qquad \text{for } ||u|| = r \tag{2.22}$$

1. Since g is continuous and has subcritical growth (see (2.19)), I^+ and I^- are weakly lower semi-continuous functionals. On the other hand if $0 \leq \phi_1 \in C_0^\infty(\Omega)$ and $0 \geq \phi_2 \in C_0^\infty(\Omega)$ then clearly

$$I^+(\epsilon\phi_1) < 0, \qquad I^-(\epsilon\phi_2) < 0 \qquad \text{for } 0 < \epsilon \text{ small} \tag{2.23}$$

So the following minimization problems:

$$\min_{||u||\leq r} I^+(u) \qquad \min_{||u||\leq r} I^-(u)$$

are solvable and by (2.22) and (2.23)

$$I^+(w_1) = \min_{||u||\leq r} I^+(u) < 0, \qquad ||w_1|| < r$$

$$I^-(w_2) = \min_{||u||\leq r} I^-(u) < 0, \qquad ||w_2|| < r$$

Thus, w_1 and w_2 are, respectively, nonnegative and nonpositive solutions of (II). Now since $-\Delta w_1 = a(x)|w_1|^{q-1}w_1 + g(w_1) \geq 0$, maximum principle implies that $w_1 > 0$. Similarly, we have $w_2 < 0$.

2. We now use Mountain Pass Theorem (MPT) to prove the existence of v_1 and v_2. In fact it is clear that by (2.21), I^+ satisfies (PS) condition and we can find a positive function z^+ such that $I^+(z^+) < 0, ||z^+|| > r$. Now since w_1 is a local minimizer of I^+ we can apply MPT to conclude that

$$\alpha =: \inf_{\gamma\in\Gamma}\max_{0\leq t\leq 1} I^+(\gamma(t)) \geq a > 0$$

is a critical value of I^+, where

$$\Gamma = \{\gamma \in C([0,1]; H_0^1) : \gamma(0) = w_1, \gamma(1) = z^+, \gamma(t) \geq 0, 0 \leq t \leq 1\}$$

We take v_1 to be the corresponding critical point. Similarly we show the existence of v_2, $I(v_2) > 0$, working with I^-.

3. If g is odd, then the functional I is even and we can follow the proof of Theorem 2.5-1 in [1] with minor changes. In fact set

$$\Sigma = \{A \subset H_0^1 : 0 \notin A, u \in A \Rightarrow -u \in A\}$$

and denote by $\gamma(A)$ the Z_2-genus of A. Define

$$A_{n,r} = \{A \in \Sigma; A \text{ compact }, A \subset B_r, \gamma(A) \geq r\}$$

where r is defined in (2.22). Then

$$b_{n,r} = \inf_{A \in A_{n,r}} \max_{u \in A} I(u)$$

are critical values of I. We need to show that $b_{n,r} < 0$ for all $n \in N$. To see this for $n \in N$ given, take n mutually disjoint balls $B_1, B_2, ...B_n$ with $B_i \subset \Omega, 1 \leq i \leq n$, and n functions $\psi_1, \psi_2, ..., \psi_n$, $0 \leq \psi_i \in C_0^\infty(B_i)$, $1 \leq i \leq n$. Let $H_n = \text{Span}\{\psi_1, \psi_2, ..., \psi_n\}$ then clearly $S_{n,\epsilon} = \partial(H_n \cap B_\epsilon) \in A_{n,r}$ and for ϵ sufficiently small, $I(u) < -\nu < 0$ for $u \in S_{n,\epsilon}$. This completes the proof of the theorem. □

Next we show how by adopting the techniques used in section 1 we can prove the existence of solutions with negative energy for (II) *without growth restriction on g*. Since the argument is similar to part 1 we will be brief here. As in section 1 we assume that g satisfies (1.5) and replace g with $\tilde{g}_\lambda$ where now $\lambda = |a|_{\frac{N}{1-q}}$. For solutions of

$$\left\{ \begin{array}{ll} -\Delta u = a(x)|u|^{q-1}u + \tilde{g}_\lambda(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{array} \right. \tag{2.24}$$

with $\tilde{I}_\lambda(u) = \int \frac{1}{2}|\nabla u|^2 - a(x)\frac{|u|^{q+1}}{q+1} - \tilde{G}_\lambda(u) < 0$, we have the same *a priori* estimate as in proposition 1.

Proposition 2.2 *Let $u \in H_0^1(\Omega)$ be a solution of (2.24) with $\tilde{I}_\lambda(u) < 0$, then for λ small,*

$$|u(x)| \leq C\lambda^\zeta \quad \textit{for all } x \in \overline{\Omega}$$

where $0 < C$ and $1 < \zeta$ are independent of λ.

Proof : Following the proof of Proposition 1, we get

$$\begin{aligned} \int |\nabla u|^2 &\leq C_1 \int a(x)|u|^{q+1} + C_2\lambda^\beta \\ &\leq C_1 |a|_{\frac{2}{1-q}} (\int |\nabla u|^2)^{\frac{q+1}{2}} + C_2\lambda^\beta && (2.25) \\ &\leq C_1\lambda(\int |\nabla u|^2)^{\frac{q+1}{2}} + C_2\lambda^\beta \quad \text{since } \frac{2}{1-q} < \frac{N}{1-q} && (2.26) \end{aligned}$$

This implies:

$$\int |\nabla u|^2 \leq C\lambda^r \qquad r > 2 \tag{2.27}$$

But u is a solution of

$$-\Delta u = a(x)|u|^{q-1}u + \tilde{g}_\lambda(u) = h(x)$$

So, for $\gamma_1 = \frac{2^*}{\beta - 1}$, we have

$$\begin{aligned} |h|_{\gamma_1} &\leq (\int |a(x)|^{\gamma_1}|u(x)|^{q\gamma_1})^{\frac{1}{\gamma_1}} + |u|_{2^*}^{\beta-1} \\ &\leq (\int |a(x)|^{\frac{2\gamma_1}{1-q}})^{\frac{1-q}{2\gamma_1}} \cdot (\int |u|^{\frac{2q\gamma_1}{1+q}})^{\frac{q+1}{2\gamma_1}} + C\lambda^{r/2(\beta-1)} \\ &\leq C\lambda|u|^q_{\frac{2q\gamma_1}{q+1}} + C(\lambda)^{r/2(\beta-1)} \quad \text{since } \frac{2\gamma_1}{1-q} \leq \frac{N}{1-q} && (2.28) \\ &\leq C\lambda|u|^q_{\gamma_1} + C(\lambda)^{r/2(\beta-1)} \\ &\leq C\lambda^{\zeta_1} \qquad \zeta_1 = \min(1 + r/2q, r/2(\beta-1)) > 1 \end{aligned}$$

and finally

$$\begin{aligned} |u|_{\tilde{\gamma}_1} \leq |u|_{W^{2,\gamma_1}} &\leq C(|u|_{\gamma_1} + |h|_{\gamma_1}) \\ &\leq C\lambda^{\tilde{\zeta}_1} \qquad\qquad \tilde{\zeta}_1 > 1 \end{aligned}$$

where $\frac{1}{\tilde{\gamma}_1} = \frac{1}{\gamma_1} - \frac{2}{N}$. Next we take $\gamma_2 = \frac{\tilde{\gamma}_1}{\beta - 1}$ and repeat the above procedure. Note that we only need to do this as long as $\gamma_k \leq \frac{N}{2}$, so estimate (2.28) is valid at each step, since $\frac{2\gamma_k}{1-q} \leq \frac{N}{1-q}$. □

With this at hand, we can now remove the growth restriction on g in Theorem 2 by assuming a more stringent condition on $a(x)$.

Theorem 2.3 *Let g satisfy (1.5), that is*

$$g(s) = |s|^{\alpha-2}s + \circ(|s|^{\alpha-1}) \text{ at } s = 0, \quad \alpha > 2,$$

and

$$g(s)s \geq 0 \qquad \text{for } |s| \leq l, \qquad l > 0.$$

Then there exists $\delta > 0$, such that for $|a|_{\frac{N}{1-q}} < \delta$,

(1) Problem (II) has two solutions, $w_1 > 0, w_2 < 0$, with $I(w) < 0$.

(2) If g is odd in a neighborhood of zero, i.e.

$$g(-s) = -g(s) \qquad \text{for } |s| < \nu, \quad \nu > 0$$

then (II) has infinitely many solutions with $I(u) < 0$.

Proof : Follow the proof of Theorem 1. The suitable form of L_∞ estimate is now provided by Proposition 2. □

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