

Some Results on the Spinor L-function for the Group $\mathrm{GSp}(4)$

Ramin Takloo-Bighash

To Siavash Shahshahani on the occasion of his 60th birthday

ABSTRACT. We collect some known and new results on the local Euler factors of the spinor L-function of the similitude symplectic group of order four using the integral of Novodvorsky.

CONTENTS

Introduction	105
Notation	107
1. The work of Jacquet and Langlands	108
2. The Spinor L-function for $\mathrm{GSp}(4)$	112
2.1. The integral	113
2.2. Non-Archimedean Theory	115
2.3. Archimedean Theory	117
2.4. A possible application	120
2.5. Open problems	121
References	121

INTRODUCTION

In his article “Problems in the theory of automorphic forms”, Robert Langlands set forward a series of conjectures that were to revolutionize the theory and practice of modular forms. Langlands’ approach to the theory was via his theory of L-functions. In this theory, the most important L-series were the ones that had an Euler product, an infinite product of simple functions, one for each prime ideal of the base field, including the primes “at infinity.” One also expects a functional equation similar to the one satisfied by the Riemann Zeta function. In Langlands’ theory, one starts with a global field F , a reductive algebraic group G defined over F , an irreducible automorphic cuspidal representation π of $G(\mathbb{A}_F)$, and a finite dimensional algebraic representation r of the dual of G , denoted by ${}^L G$. Then $\pi = \otimes_v \pi_v$, where for each place v , π_v is a representation of $G(F_v)$. Also, there is a finite set of places, including the places at infinity, such that for $v \notin S$, the representation π_v is unramified. Next, by a process, essentially due to Satake, one associates to π_v ,

$v \notin S$, a conjugacy class $\mathcal{C}_v(\pi)$ in ${}^L G$. We define

$$L_v(s, \pi, r) = \frac{1}{\det(I - r(\mathcal{C}_v(\pi))\mathbb{N}v^{-s})}.$$

For example, if $F = \mathbb{Q}$, $G = \mathbb{G}_m$, π the trivial representation, and r the trivial representation, then for $v = p$ a prime number, we have

$$L_p(s, \pi, r) = \frac{1}{1 - p^{-s}},$$

the p -Euler factor of the Riemann zeta function!

Langlands' first conjecture, and in some sense the most basic one, asserts that it is possible to extend the definition of L_v to $v \in S$, in such a way that if we set

$$L(s, \pi, r) = \prod_v L_v(s, \pi, r),$$

then L is "nice." Here "nice" means that L is meromorphic in the entire complex plane with only a finite number of poles and satisfies the functional equation

$$L(s, \pi, r) = \epsilon(s, \pi, r, \psi)L(1 - s, \pi, \tilde{r}).$$

Here \tilde{r} is the contragredient of the finite dimensional complex representation r .

In this generality, the conjecture is still open. There are, however, many instances where it has been verified. For example, if $G = \mathrm{GL}(n)$ and r the standard representation of ${}^L G = \mathrm{GL}_n(\mathbb{C})$, then the classical results of Godement, Tate, and Jacquet, which predate the paper of Langlands, affirm the conjecture. In the works of Godement-Jacquet, and Tate, the L-function is presented as an integral on some adelic space, which in turn is an infinite product, the factors of which are indexed by the places of the global field. For appropriate choices of the local data, each of these local integrals gives the corresponding local Euler factor. In modern terminology, this is the method of *Integral Representations* or the *Rankin-Selberg Method*. We also note the powerful method of Langlands and Shahidi which has yielded many interesting examples where the conjecture holds.

In this article, we concentrate on a special Rankin-Selberg Integral for the symplectic similitude group of order four $GSp(4)$ over a totally real field. The group $GSp(4)$ is important, both from a historical point of view and from the point of view of applications. This group provides the natural group theoretic framework for Siegel modular forms of genus two, and for that reason has immediate connection to various fundamental problems in arithmetic geometry and number theory. Also in a systematic development of the theory, it is necessary to have worked out various non-trivial lower dimensional examples in details to provide the required empirical data for the revelation of a universal vision.

This paper is organized as follows. Section One gives an overview of the zeta integral machine through an example from the classical work of Jacquet and Langlands [14]. Our overview is heavily based on the exposition of [10]. The reader who is familiar with classical modular forms is encouraged to consult Gelbart's wonderful monograph [9] in order to understand the relevance of the material presented here to the classical theory. Section two contains the theory of Novodvorsky's integral

and the L-function represented by it. In this section, we have included the functional equation, the sketch of unramified computations from [5], a review of the computations at the bad primes from [20], and some thoughts on the archimedean calculations. The archimedean theory described in this section is the main contribution, if any, of this work. Here, we have introduced a global method in order to perform the local computation at the real place for the generic (limit of) discrete series. Our idea is to utilize the global theta correspondence for the dual reductive pair $(\mathrm{GO}(2, 2), \mathrm{GSp}(4))$. Recently, Furusawa, Roberts, and Ralf Schmidt have applied similar global methods involving the theta correspondence to other local problems. We also note the work in progress of Miller and Schmidt on the real unramified principal series. In this section, we have also included a possible arithmetic application of our results to the work in progress of Furusawa and Shalika. Our exposition of their research closely follows an unpublished manuscript by Furusawa.

In the preparation of this work, we have largely benefited from conversations with Jeffrey Adams, Masaaki Furusawa, Brooks Roberts, Freydoon Shahidi and most importantly my graduate advisor Joseph A. Shalika under whose guidance the non-archimedean computations of section two were prepared.

It is our utmost pleasure to dedicate this article to Professor Siavash Shahshahani on the occasion of his 60th birthday. This author owes a great deal of the development of his mathematical and intellectual character, including his passion for analysis, to what he heard and learned in Professor Shahshahani’s classes and lectures. We believe that because of Professor Shahshahani’s teachings the mathematical community of Iran is one long step closer to the realization of the dream outlined in his article in the *Memoirs of the 25th Mathematical Conference of Iran*.

Notation. In this paper, the group $\mathrm{GSp}(4)$ over an arbitrary field K is the group of all matrices $g \in \mathrm{GL}_4(K)$ that satisfy the following equation for some scalar $\nu(g) \in K$:

$${}^t g J g = \nu(g) J,$$

where $J = \begin{pmatrix} & & & 1 \\ & & 1 & \\ & -1 & & \\ -1 & & & \end{pmatrix}$. It is a standard fact that $G = \mathrm{GSp}(4)$ is a reductive group. The map $(F^\times)^3 \rightarrow G$, given by

$$(a, b, \lambda) \mapsto \mathrm{diag}(a, b, \lambda b^{-1}, \lambda a^{-1}),$$

gives a parameterization of a maximal torus T in G . Let χ_1, χ_2 and χ_3 be quasi-characters of F^\times . We define the character $\chi_1 \otimes \chi_2 \otimes \chi_3$ of T by

$$(\chi_1 \otimes \chi_2 \otimes \chi_3)(\mathrm{diag}(a, b, \lambda b^{-1}, \lambda a^{-1})) = \chi_1(a) \chi_2(b) \chi_3(\lambda).$$

We have three standard parabolic subgroups: The Borel subgroup B , The Siegel subgroup P , and the Klingen subgroup Q with the following Levi decompositions:

$$B = \left\{ \begin{pmatrix} a & & & \\ & b & & \\ & & b^{-1}\lambda & \\ & & & a^{-1}\lambda \end{pmatrix} \begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & & r & s \\ & 1 & t & r \\ & & 1 & \\ & & & 1 \end{pmatrix} \right\},$$

$$P = \left\{ \begin{pmatrix} g & & & \\ & \alpha^\tau g^{-1} & & \\ & & & \\ & & & \end{pmatrix} \begin{pmatrix} 1 & r & s \\ & 1 & t & r \\ & & 1 & \\ & & & 1 \end{pmatrix} \mid g \in GL(2) \right\}.$$

Here ${}^\tau g$ is the transposed matrix with respect to the second diagonal, and finally

$$Q = \left\{ \begin{pmatrix} \alpha & & & \\ & g & & \\ & & & \\ & & & \alpha^{-1} \det g \end{pmatrix} \begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & r & s \\ & 1 & r \\ & & 1 & \\ & & & 1 \end{pmatrix} \mid g \in GSp(2) \right\}.$$

Over a local field, we will use the notation $\chi_1 \times \chi_2 \rtimes \chi_3$ for the parabolically induced representation from the minimal parabolic subgroup, by the character $\chi_1 \otimes \chi_2 \otimes \chi_3$. If π is a smooth representation of $GL(2)$, and χ a quasi-character of F^\times , then $\pi \rtimes \chi$ (respectively $\chi \rtimes \pi$) is the parabolically induced representation from the Levi subgroup of the Siegel (resp. Klingen) parabolic subgroup. We define a character of the unipotent radical $N(B)$ of the Borel subgroup by the following:

$$\theta \left(\begin{pmatrix} 1 & x & & \\ & 1 & & \\ & & 1 & -x \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & r & s \\ & 1 & t & r \\ & & 1 & \\ & & & 1 \end{pmatrix} \right) = \psi(x + t).$$

We call an irreducible representation (Π, V_Π) of $GSp(4)$ over a local field generic, if there is a functional λ_Π on V_Π such that

$$\lambda_\Pi(\Pi(n)v) = \theta(n)v,$$

for all $v \in V_\Pi$ and $n \in N(B)$. If such a functional exists, it is unique up to a constant [19]. Freydoon Shahidi has given canonical constructions of these functionals in [17] for representations induced from generic representations. We define Whittaker functions on $G \times V_\Pi$ by

$$W(\Pi, v, g) = \lambda_\Pi(\Pi(g)v).$$

When there is no danger of confusion, after fixing v and suppressing Π , we write $W(g)$ instead of $W(\Pi, v, g)$. For a character Ψ of the unipotent radical of the Borel subgroup, we denote by $\pi_{N, \Psi}$ the twisted by Ψ Jacquet module of the representation π . For any representation π , we will denote by ω_π the central character of π . We will also use Shahidi's notation for intertwining operators and local coefficients from [17].

1. THE WORK OF JACQUET AND LANGLANDS

In this section, we examine the group $GL(2)$. This section serves as motivation for Section Two which contains the main results of the paper. Our exposition is heavily based on [10], to the point of copying, especially pages 5-19. For the sake of familiarity and simplicity, we work over \mathbb{Q} .

Let $G = GL(2)$. Suppose χ is a unitary character of \mathbb{A}^\times . By a χ -cusp form φ on $GL(2)$, we mean an $L^2(Z_{\mathbb{A}}G(\mathbb{Q}) \backslash G(\mathbb{A}))$ function satisfying

$$\varphi \left(\begin{pmatrix} a & \\ & a \end{pmatrix} g \right) = \chi(a)\varphi(g),$$

and

$$\int_{\mathbb{Q}\backslash\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) dx = 0,$$

for almost all $g \in G(\mathbb{A})$. It is clear that if φ is a χ -cusp form, and $g \in G(\mathbb{A})$, then the function $g.\varphi$ on $G(\mathbb{A})$ defined by

$$g.\varphi(h) = \varphi(hg),$$

is again a χ -cusp form. This defines a representation of $G(\mathbb{A})$ on the vector space of χ -cusp forms $L_0^2(\chi)$. It is a fundamental fact that $L_0^2(\chi)$ is a discrete direct sum of irreducible subspaces, each of which appears with multiplicity one. An irreducible representation π of $GL_2(\mathbb{A})$ which is realized as an irreducible subspace H_π of $L_0^2(\chi)$ is called a *cuspidal automorphic representation*.

Suppose π is an irreducible cuspidal automorphic representation of $GL_2(\mathbb{A})$, and $\varphi \in H_\pi$. We introduce a *global zeta integral*

$$(1) \quad \mathcal{Z}(\varphi, s) = \int_{\mathbb{Q}^\times \backslash \mathbb{A}^\times} \varphi\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right) |a|_{\mathbb{A}}^{s-\frac{1}{2}} d^\times a.$$

If φ is “nice enough”, $\mathcal{Z}(\varphi, s)$ defines an entire function in \mathbb{C} . Also, the zeta function $\mathcal{Z}(\varphi, s)$ satisfies a functional equation:

$$\mathcal{Z}(\varphi, s) = \tilde{\mathcal{Z}}(\varphi^w, 1-s),$$

where $w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$, and $\varphi^w(g) = \varphi(gw)$. Also

$$(2) \quad \tilde{\mathcal{Z}}(\varphi, s) = \int_{\mathbb{Q}^\times \backslash \mathbb{A}^\times} \varphi\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right) |a|^{s-\frac{1}{2}} \chi^{-1}(a) d^\times a,$$

with χ the central character of π .

The problem is to relate the function $\mathcal{Z}(\varphi, s)$ to an automorphic L-function $L(s, \pi, r)$ for some representation r of ${}^L G = GL_2(\mathbb{C})$. For this purpose, we start by the Fourier expansion of φ

$$(3) \quad \varphi(g) = \sum_{\xi \in \mathbb{Q}^\times} W_\varphi^\psi\left(\begin{pmatrix} \xi & \\ & 1 \end{pmatrix} g\right).$$

Here

$$(4) \quad W_\varphi^\psi(g) = \int_{\mathbb{Q}\backslash\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) \overline{\psi(x)} dx,$$

where ψ is a non-trivial character of $\mathbb{Q}\backslash\mathbb{A}$. It follows from (1) and (3) that

$$\mathcal{Z}(\varphi, s) = \int_{\mathbb{A}^\times} W_\varphi^\psi\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right) |a|^{s-\frac{1}{2}} d^\times a,$$

for $\Re s$ large enough.

Now we recall some of the properties of the *Whittaker functions* W_φ^ψ . From now on we suppress ψ . We assume that φ is right K-finite. In this situation, W_φ is rapidly decreasing at infinity and satisfies

$$(5) \quad W_\varphi\left(\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} g\right) = \psi(x) W_\varphi(g),$$

for all $x \in \mathbb{A}$. The space of all such W_φ provides the ψ -Whittaker model of π . It is known that such a model is unique ([14], or [19]), and it is equal to the restricted tensor product of local Whittaker models $\mathcal{W}(\pi_p, \psi_p)$, where $\pi = \otimes_p \pi_p$ and $\psi = \prod_p \psi_p$. In particular, we can assume that

$$(6) \quad W_\varphi(g) = \prod_p W_p(g_p),$$

where each $W_p \in \mathcal{W}(\pi_p, \psi_p)$ and for almost all finite p , W_p is *unramified*, i.e. $W_p(k) = 1$ for $k \in K_p = GL_2(\mathbb{Z}_p)$.

Finally, we obtain for $\Re s$ large

$$(7) \quad \mathcal{Z}(\varphi, s) = \prod_p \mathcal{Z}(W_p, s),$$

where

$$(8) \quad \mathcal{Z}(W_p, s) = \int_{\mathbb{Q}_p^\times} W_p\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right) |a|^{s-\frac{1}{2}} d^\times a.$$

First, we collect some of the properties of the local zeta functions $\mathcal{Z}(W, s)$. The fundamental fact for $p < \infty$ is the following: There are a finite number of finite functions c_1, \dots, c_N on \mathbb{Q}_p^\times , depending only on π_p , such that for every $W \in \mathcal{W}(\pi_p, \psi_p)$, there are Schwartz-Bruhat functions Φ_1, \dots, Φ_N on \mathbb{Q}_p satisfying

$$(9) \quad W\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right) = \sum_{i=1}^N c_i(a) \Phi_i(a).$$

Here, a finite function is a function whose space of right translates by \mathbb{Q}_p^\times is finite dimensional; finite functions on \mathbb{Q}_p^\times are thus characters, integer powers of the valuation function, or products and linear combinations thereof. Taking the asymptotic expansion just mentioned for granted, we obtain from Tate's thesis that the integral defining $\mathcal{Z}(W, s)$ converges for $\Re s$ large (independent of s), and in the domain of convergence is equal to a rational function of p^{-s} . In particular, the integral has a meromorphic continuation to all of \mathbb{C} . Furthermore, the family of rational functions $\{\mathcal{Z}(W, s) \mid W \in \mathcal{W}(\pi_p, \psi_p)\}$ admits a common denominator, i.e. a polynomial P such that $P(p^{-s})\mathcal{Z}(W, s) \in \mathbb{C}[p^{-s}, p^s]$, for all W . Also, there exists a W^* in $\mathcal{W}(\pi_p, \psi_p)$ with the property that $\mathcal{Z}(W^*, s) = 1$. The analogous result in the archimedean situation is that there is a W^* such that $\mathcal{Z}(W^*, s)$ has no poles or zeroes.

As in Tate's thesis [22], we also need to establish the functional equation and perform the local unramified computations. The functional equation asserts that there exists a meromorphic function $\gamma(\pi_p, \psi_p, s)$ (rational function in p^{-s} when $p < \infty$) such that

$$(10) \quad \tilde{\mathcal{Z}}(W^w, 1-s) = \gamma(\pi_p, \psi_p, s) \mathcal{Z}(W, s).$$

Here $W^w(g) = W(gw)$ and

$$\tilde{\mathcal{Z}}(W, s) = \int_{\mathbb{Q}_p^\times} W\left(\begin{pmatrix} a & \\ & 1 \end{pmatrix}\right) |a|^{s-\frac{1}{2}} \chi_p^{-1}(a) d^\times a,$$

with χ_p the central character of π_p . The (non-trivial) proof of this equation follows from the fact that the integrals \mathcal{Z} and $\tilde{\mathcal{Z}}$ define functionals (depending on s) satisfying a certain invariance property. Next, one proves that the space of such functionals is at most one dimensional, implying that \mathcal{Z} and $\tilde{\mathcal{Z}}$ must be proportional. The factor γ is simply the factor of proportionality.

We recall π_p is called unramified when

$$\pi_p = \mathrm{Ind}(\mu_1 \otimes \mu_2),$$

with μ_1 and μ_2 unramified characters of \mathbb{Q}_p^\times . We also suppose that W^0 is the unique K_p -invariant function in $\mathcal{W}(\pi_p, \psi_p)$. We also suppose that ψ_p is unramified. Then a direct calculation, using the results of [6] for example, shows that

$$\mathcal{Z}(W^0, s) = \frac{1}{(1 - \mu_1(\varpi_p)p^{-s})(1 - \mu_2(\varpi_p)p^{-s})},$$

where ϖ_p is the local uniformizer at p . Next the conjugacy class in ${}^L G = \mathrm{GL}_2(\mathbb{C})$ canonically associated with π_p is $t_p = \begin{pmatrix} \mu_1(\varpi_p) & \\ & \mu_2(\varpi_p) \end{pmatrix}$. In particular, we have

$$(11) \quad \mathcal{Z}(W^0, s) = L_p(s, \pi, r),$$

with r the standard two dimensional representation of $\mathrm{GL}_2(\mathbb{C})$.

After this preparation, we can prove the conjecture of Langlands for $L(s, \pi, r)$ with r as above. For simplicity, we write $L(s, \pi)$ instead of $L(s, \pi, r)$. We start by extending the definition of $L_v(s, \pi)$ to the ramified and archimedean places. We observe that in equation (11), the right hand side is indeed the greatest common denominator of the family of rational functions $\{\mathcal{Z}(W, s)\}$. Since we have already noted that such a g.c.d. exists, even when the given representation is not unramified, we set

$$(12) \quad L_v(s, \pi) = \mathrm{g.c.d.} \{ \mathcal{Z}(W, s) \},$$

when $v < \infty$. Also, when $v = \infty$, we can choose an appropriate product of Tate's archimedean L-functions, denoted by $L_\infty(s, \pi, r)$, such that the ratio

$$(13) \quad \frac{\mathcal{Z}(W, s)}{L_\infty(s, \pi)}$$

is an entire function for all $W \in \mathcal{W}(\pi_v, \psi_v)$, and it is a nowhere vanishing function for some choice of W .

With this extension, we now proceed to outline the proof. Let S be a set of places, including the place at infinity, such that for $v \notin S$, all the data is unramified. We set

$$(14) \quad L_S(s, \pi) = \prod_{v \notin S} L_v(s, \pi).$$

For each "ramified" non-archimedean place p , we choose W_p such that $\mathcal{Z}(W_p, s) = 1$. Also for the archimedean v , we choose W_v such that $\mathcal{Z}(W_v, s)$ is a non-vanishing entire function $e^{g(s)}$. If we set $W = \prod_v W_v$, with $W_p = W_p^0$ for $p \in S$, we have

$$(15) \quad \mathcal{Z}(\varphi, s) = e^{g(s)} L_S(s, \pi),$$

implying the holomorphicity of L_S . This immediately implies the continuation of L to a meromorphic function with only a finite number of poles.

We finally turn to the functional equation of the completed L-function. Choosing W_p so that $\mathcal{Z}(W_p, s) = L_p(s, \pi)$, we have

$$\begin{aligned}
L(s, \pi) &= \mathcal{Z}(\varphi, s) \\
&= \tilde{\mathcal{Z}}(\varphi^w, 1-s) \\
&= \left(\prod_{p \in S} \tilde{\mathcal{Z}}(W_p^w, 1-s) \right) L_S(1-s, \tilde{\pi}) \\
&= \left(\prod_{p \in S} \frac{\tilde{\mathcal{Z}}(W_p^w, 1-s)}{L_p(1-s, \tilde{\pi})} \right) L(1-s, \tilde{\pi}) \\
&= \left(\prod_{p \in S} \frac{\gamma(\pi_p, \psi_p, s) \mathcal{Z}(W_p, s)}{L_p(1-s, \tilde{\pi})} \right) L(1-s, \tilde{\pi}) \\
&= \left(\prod_{p \in S} \frac{\gamma(\pi_p, \psi_p, s) L_p(s, \pi)}{L_p(1-s, \tilde{\pi})} \right) L(1-s, \tilde{\pi}) \\
&= \left(\prod_{p \in S} \epsilon(s, \pi_p, \psi_p) \right) L(1-s, \tilde{\pi}),
\end{aligned}$$

where

$$\epsilon(s, \pi_p, \psi_p) = \frac{\gamma(\pi_p, \psi_p, s) L_p(s, \pi)}{L_p(1-s, \tilde{\pi})}.$$

Hence, if we set

$$\epsilon(s, \pi) = \prod_{p \in S} \epsilon(s, \pi_p, \psi_p),$$

we have the functional equation

$$(16) \quad L(s, \pi) = \epsilon(s, \pi) L(1-s, \tilde{\pi}),$$

as anticipated by Langlands. One last note is that the function $\epsilon(s, \pi)$ is a monomial function of s . In particular, it has no poles or zeroes.

Remark 1.1. In equation (9), if $c_i = \mu_i \cdot v^{r_i}$, with μ_i a quasi-character, we have

$$L_p(s, \pi) = \prod_{i=1}^N L(s, \mu_i)^{r_i}.$$

The L-functions appearing on the right hand side are Tate's local L-factors for the quasi-characters μ_i . This implies that in order to give an explicit calculation of the local L-factors, we need to determine the finite functions c_i . In [14], this is established by a case by case analysis of representation types for π_p , i.e. principal series vs. special representations vs. supercuspidals.

2. THE SPINOR L-FUNCTION FOR $\mathrm{GSp}(4)$

In this section, we examine the integral representation given by Novodvorsky [15] for $G = \mathrm{GSp}(4)$. The details of the material in the following paragraphs appear in [5], [20], and [21].

2.1. **The integral.** Let φ be a cusp form on $\mathrm{GSp}(4, \mathbb{A})$, belonging to the space of an irreducible cuspidal automorphic representation π . Consider the integral

$$(17) \quad \mathcal{Z}(\varphi, s) = \int_{F^\times \backslash \mathbb{A}^\times} \int_{(F \backslash \mathbb{A})^3} \varphi \left(\begin{pmatrix} 1 & x_2 & & x_4 \\ & 1 & & \\ & z & 1 & -x_2 \\ & & & 1 \end{pmatrix} \begin{pmatrix} y & & & \\ & y & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \right) \frac{\overline{\psi(x_2)} |y|^{s-\frac{1}{2}} dz dx_2 dx_4 d^\times y.$$

If $w = \begin{pmatrix} & & 1 & \\ -1 & & & \\ & & & 1 \\ & -1 & & \end{pmatrix}$, then

$$(18) \quad \mathcal{Z}(\varphi, s) = \mathcal{Z}(\varphi^w, 1-s),$$

where as before $\varphi^w(g) = \varphi(gw)$. Also a usual unfolding process, as described in [5], shows that

$$(19) \quad \mathcal{Z}(\varphi, s) = \int_{\mathbb{A}^\times} \int_{\mathbb{A}} W_\varphi \left(\begin{pmatrix} y & & & \\ & y & & \\ & x & 1 & \\ & & & 1 \end{pmatrix} \right) |y|^{s-\frac{3}{2}} dx d^\times y.$$

Here the Whittaker function W_φ is given by

$$(20) \quad W_\varphi(g) = \int_{(F \backslash \mathbb{A})^4} \varphi \left(\begin{pmatrix} 1 & x_2 & & \\ & 1 & & \\ & & 1 & -x_2 \\ & & & 1 \end{pmatrix} \begin{pmatrix} 1 & x_3 & x_4 \\ & 1 & x_3 \\ & & 1 & \\ & & & 1 \end{pmatrix} g \right) \frac{\overline{\psi(x_1 + x_2)}}{dx_1 dx_2 dx_3 dx_4}.$$

Equation (19) implies that, in order for $\mathcal{Z}(\varphi, s)$ to be non-zero, we need to assume that W_φ is not identically equal to zero. A representation satisfying this condition is called “generic.” Every representation of $\mathrm{GL}(2)$ is generic. On other groups, however, there may exist non-generic cuspidal representations. In fact, those representations of $\mathrm{GSp}(4)$ which correspond to holomorphic cuspidal Siegel modular forms are not generic.

From this point on, we assume that all the representations of $\mathrm{GSp}(4)$, local or global, which appear in the text are generic.

If φ is chosen correctly, the Whittaker function may be assumed to decompose locally as $W(g) = \prod_v W_v(g_v)$, a product of local Whittaker functions. Hence, for $\Re s$ large, we obtain

$$(21) \quad \mathcal{Z}(\varphi, s) = \prod_v \mathcal{Z}(W_v, s),$$

where

$$(22) \quad \mathcal{Z}(W_v, s) = \int_{F_v^\times} \int_{F_v} W_v \left(\begin{pmatrix} y & & & \\ & y & & \\ & x & 1 & \\ & & & 1 \end{pmatrix} \right) |y|^{s-\frac{3}{2}} dx d^\times y.$$

As usual, we have a functional equation: There exists a meromorphic function $\gamma(\pi_v, \psi_v, s)$ (rational function in $\mathbb{N}v^{-s}$ when $v < \infty$) such that

$$(23) \quad \mathcal{Z}(W_v, s) = \gamma(\pi_v, \psi_v, s) \tilde{\mathcal{Z}}(W_v^w, 1 - s),$$

with w as above,

$$\tilde{\mathcal{Z}}(W_v, s) = \int_{F_v^\times} \int_{F_v} W_v \left(\begin{pmatrix} y & & & \\ & y & & \\ & x & 1 & \\ & & & 1 \end{pmatrix} \right) \chi_v^{-1}(y) |y|^{s-\frac{3}{2}} dx d^\times y,$$

and χ_v the central character of π_v .

We also consider the unramified calculations. Suppose v is any non-archimedean place of F such that W_v is right invariant by $GS\!p(4, \mathcal{O}_v)$ and such that the largest fractional ideal on which ψ_v is trivial is \mathcal{O} . Then the Casselman-Shalika formula [6] allows us to calculate the last integral (cf. [5]). The result is the following:

$$(24) \quad \mathcal{Z}(W_v, s) = L(s, \pi_v, \text{Spin}).$$

Let us explain the notation. The connected L -group ${}^L G^0$ is $GS\!p(\mathbb{C})$ (cf. [4]). Let ${}^L T$ be the maximal torus of elements of the form

$$t(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \begin{pmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \alpha_3 & \\ & & & \alpha_4 \end{pmatrix},$$

where $\alpha_1 \alpha_4 = \alpha_2 \alpha_3$. The fundamental dominant weights of the torus are λ_1 and λ_2 , where

$$\lambda_1 t(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \alpha_1,$$

and

$$\lambda_2 t(\alpha_1, \alpha_2, \alpha_3, \alpha_4) = \alpha_1 \alpha_3^{-1}.$$

The dimensions of the representation spaces associated with these dominant weights are four and five, respectively. In our notation, Spin is the representation of $GS\!p(4, \mathbb{C})$ associated with the dominant weight λ_1 , i.e. the standard representation of $GS\!p(4, \mathbb{C})$ on \mathbb{C}^4 . The L-function $L(s, \pi, \text{Spin})$ is called the Spinor, or simply the Spin, L-function of $GS\!p(4)$.

Next step is to use the integral introduced above to extend the definition of the Spinor L-function to ramified non-archimedean and archimedean places. Our definition at these places would be analogous to equations (12) and (13) above.

2.2. Non-Archimedean Theory. In this paragraph, we sketch the computation of the local non-archimedean Euler factors of the Spin L-function given by the integral representation of the previous paragraph. In order for this to make sense, we need the following lemma:

Lemma 2.1 (Theorem 2.1 of [20]). *Suppose Π is a generic representation of $\mathrm{GSp}(4)$ over a non-archimedean local field K , q order of the residue field. For each $W \in \mathcal{W}(\Pi, \psi)$, the function $\mathcal{Z}(W, s)$ is a rational function of q^{-s} , and the ideal $\{\mathcal{Z}(W, s)\}$ is principal.*

Sketch of proof. For $W \in \mathcal{W}(\Pi, \psi)$, we set

$$(25) \quad Z(W, s) = \int_K W \left(\begin{pmatrix} y & & & \\ & y & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \right) |y|^{s-\frac{3}{2}} d^\times y.$$

The first step of the proof is to show that the vector space $\{Z(W, s)\}$ is the same as $\{\mathcal{Z}(W, s)\}$ (cf. Proposition 3.2 of [20]). Next, we use the asymptotic expansions of the Whittaker functions along the torus to prove the existence of the g.c.d. for the ideal $\{Z(W, s)\}$. Indeed, Proposition 3.5 of [20] (originally a theorem in [6]) states that there is a finite set of finite functions S_Π , depending only on Π , with the following property: for any $W \in \mathcal{W}(\Pi, \psi)$, and $c \in S_\Pi$, there is a Schwartz-Bruhat function $\Phi_{c,W}$ on K such that

$$W \left(\begin{pmatrix} y & & & \\ & y & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \right) = \sum_{c \in S_\Pi} \Phi_{c,W}(y) c(y) |y|^{\frac{3}{2}}.$$

The lemma is now immediate. \square

We have the following theorem:

Theorem 2.2. *Suppose Π is a generic representation of the group $\mathrm{GSp}(4)$ over a non-archimedean local field K . Then*

- (1) *If Π is supercuspidal, or is a sub-quotient of a representation induced from a supercuspidal representation of the Klingen parabolic subgroup, then $L(s, \pi, \mathrm{Spin}) = 1$.*
- (2) *If π is a supercuspidal representation of $\mathrm{GL}(2)$ and χ a quasi-character of K^\times , and $\Pi = \pi \rtimes \chi$ is irreducible, we have*

$$L(s, \Pi, \mathrm{Spin}) = L(s, \chi).L(s, \chi.\omega_\pi).$$

- (3) *If χ_1, χ_2 , and χ_3 are quasi-characters of K^\times , and $\Pi = \chi_1 \times \chi_2 \rtimes \chi_3$ is irreducible, we have*

$$L(s, \Pi, \mathrm{Spin}) = L(s, \chi_3).L(s, \chi_1\chi_3).L(s, \chi_2\chi_3).L(s, \chi_1\chi_2\chi_3).$$

- (4) *When Π is not irreducible, one can prove similar statements for the generic subquotients of $\Pi = \pi \rtimes \chi$ (resp. $\Pi = \chi_1 \times \chi_2 \rtimes \chi_3$) according to the classification theorems of Sally-Tadic [16] and Shahidi [18] (cf. theorems 4.1 and 5.1 of [20]).*

Remark 2.3. Sally and Tadic [16] and Shahidi [18] have completed the classification of representations supported in the Borel and Siegel parabolic subgroups. In particular, they have determined for which representations the parabolic induction is reducible. From their result, one can immediately establish a classification for all the generic representations supported in the Borel or Siegel parabolic subgroups.

Sketch of proof. By the proof of the lemma, we need to determine the asymptotic expansion of the Whittaker functions in each case. The argument consists of several steps:

Step 1. Bound the size of S_Π . Fix $c \in S_\Pi$, and define a functional Λ_c on $\mathcal{W}(\Pi, \psi)$ by

$$(26) \quad \Lambda_c(W) = \Phi_{c,W}(0).$$

If $c, c' \in S_\Pi$, and $c \neq c'$, the two functionals Λ_c and $\Lambda_{c'}$ are linearly independent. Furthermore, the functionals Λ_c belong to the dual of a certain twisted Jacquet module $\Pi_{N, \bar{\theta}}$ (notation from [20], page 1095). Hence $\#S_\Pi = \dim \Pi_{N, \bar{\theta}}$. Then one uses an argument similar to those of [19], distribution theory on p -adic manifolds, to bound the dimension of the Jacquet module. The result (proposition 3.9 of [20]) is that if Π is supercuspidal or supported in the Klingen parabolic subgroup (resp. Siegel parabolic, resp. Borel parabolic), then $\#S_\Pi = 0$ (resp. ≤ 2 , resp. ≤ 4). Note that this already implies the first part of the theorem.

From this point on, we concentrate on the Siegel parabolic subgroup, the Borel subgroup case being similar. We fix some notation. Suppose $\Pi = \pi \rtimes \chi$, with π supercuspidal of $GL(2)$. Let λ_Π (resp. λ_π) be the Whittaker functional of Π (resp. π) from [17]. It follows from the proof of the lemma 2.1 that, for $f \in \Pi$, there is a positive number $\delta(f)$, such that

$$\lambda_\Pi\left(\Pi\left(\begin{pmatrix} y & & & \\ & y & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\right)f\right) = \sum_{c \in S_\Pi} \Lambda_c(f)c(y)|y|^{\frac{3}{2}},$$

for $|y| < \delta(f)$. Here, Λ_c is the obvious functional on the space of Π .

Step 2. Uniformity. For $f \in \text{Ind}(\pi \times \chi|P \cap K, K)$, and $\tau \in \mathbb{C}$, define f_τ on G by

$$f_\tau(pk) = \delta_P(p)^{\tau + \frac{1}{2}} \pi \otimes \chi(p)f(k).$$

It is clear that f_τ is a well-defined function on G , and that it belongs to the space of a certain induced representation Π_τ . The *Uniformity Theorem* (proposition 3.9 of [20]) asserts that one can take $\delta(f_\tau) = \delta(f)$.

Step 3. Regular representations. This is the case where $\omega_\pi \neq 1$. In this situation, we have

$$(27) \quad \lambda_\Pi\left(\Pi\left(\begin{pmatrix} y & & & \\ & y & & \\ & & 1 & \\ & & & 1 \end{pmatrix}\right)f\right) = \lambda_\pi(A(w, \Pi)(f)(e))\chi(y)|y|^{\frac{3}{2}} + C(w\Pi, w^{-1})^{-1}\lambda_\pi(f(e))\chi(y)\omega_\pi(y)|y|^{\frac{3}{2}},$$

for $|y| < \delta(f)$. Here $w = \begin{pmatrix} & & & 1 \\ & & & \\ -1 & & & \\ & & -1 & \\ & & & & & 1 \end{pmatrix}$, $A(w, \Pi)$ is the intertwining integral

of [17], and $C(w\Pi, w^{-1})$ is the local coefficient of [17]. The proof of this identity follows from the the above lemma 2.1, and the *Multiplicity One Theorem* [19]. The idea is to find one term of the asymptotic expansion using the open cell; then apply the long intertwining operator to find the other term.

Note that the identity of *Step 3* also applies to reducible cases. For example, if $f \in \Pi$ is in the kernel of the intertwining operator $A(w, \Pi)$, the first term of the right hand side vanishes.

Step 4. Irregular Representations. The idea is the following: we twist everything in *Step 3* by the complex number τ , so that the resulting representation Π_τ is regular. By *Step 2*, the identity still holds uniformly for all τ . By a theorem of Shahidi [17] (essentially due to Casselman and Shalika [6]), we know that the left hand side of the identity is an entire function of τ . This implies that the poles of the right hand side, coming from the intertwining operator and the local coefficient, must cancel out. Next, we let $\tau \rightarrow 0$. An easy argument (l'Hospital's rule!) shows the appearance of $\chi(y)|y|^{\frac{3}{2}}$ and $\chi(y)|y|^{\frac{3}{2}} \log_q |y|$ in the asymptotic expansion.

This finishes the sketch of proof of the theorem. □

2.3. Archimedean Theory. In this section, we collect some results on the computation of the L-factor at the real place. For reasons that will soon be clear, it will be convenient to work with a different realization of the group $GSp(4)$, and also a different formulation of Novodvorsky's integral. Here, in the definition of the

similitude group from the **Notation**, we take $J = \begin{pmatrix} & & & 1 \\ & & & \\ -1 & & & \\ & & -1 & \\ & & & & & 1 \end{pmatrix}$. Also, if φ

is a cusp form on $GSp(4, \mathbb{A})$, belonging to the space of an irreducible automorphic cuspidal representation π , we set

$$(28) \quad \mathcal{Z}(\varphi, s) = \int_{F^\times \backslash \mathbb{A}^\times} \int_{(F \backslash \mathbb{A})^3} \varphi \left(\begin{pmatrix} 1 & u & v \\ & 1 & w \\ & & 1 \\ & & & 1 \end{pmatrix} \begin{pmatrix} y & & & \\ & 1 & & \\ & & 1 & \\ & & & y \end{pmatrix} \right) \overline{\psi(v)} |y|^{s-\frac{1}{2}} du dv dw d^\times y.$$

This is obtained from the integral of the previous section after a simple change of coordinates. We also note that in this new form, the integral is nothing but a split Bessel model ([7]).

We now recall the definition of the group under consideration [11]. Let V be the vector space M_2 , of the two by two matrices, equipped with the quadratic form \det . Let $(,)$ be the associated non-degenerate inner product, and $H = GO(V, (,))$ be the group of orthogonal similitudes of $V, (,)$. The group $GL(2) \times GL(2)$ has a natural

involution t defined by $t(g_1, g_2) = ({}^t b_2^{-1}, {}^t b_1^{-1})$, where the superscript t stands for the transposition. Let $\tilde{H} = (GL(2) \times GL(2)) \rtimes \langle t \rangle$ be the semi-direct product of $GL(2) \times GL(2)$ with the group of order two generated by t . There is an exact sequence

$$(29) \quad 1 \longrightarrow \mathbb{G}_m \longrightarrow \tilde{H} \longrightarrow H \longrightarrow 1,$$

where the homomorphism $\rho : \tilde{H} \rightarrow H$ is defined by $\rho(g_1, g_2)(v) = g_1 v g_2^{-1}$, and $\rho(t)v = {}^t v$, for all $g_1, g_2 \in GL(2)$ and $v \in V$. Also, $\mathbb{G}_m \rightarrow \tilde{H}$ is the natural map $z \mapsto (z, z) \times 1$. It follows that the image of the subgroup $GL(2) \times GL(2) \subset \tilde{H}$ under ρ is the connected component of the identity of H .

Suppose π_1 and π_2 are two irreducible cuspidal automorphic representations of $GL_2(\mathbb{A})$ satisfying

$$\omega_{\pi_1} \cdot \omega_{\pi_2} = 1, \text{ and } \pi_1 \neq \tilde{\pi}_2.$$

Then for φ_1 and φ_2 cusp forms in the spaces of π_1 and π_2 , respectively, one can think of

$$\varphi(h_1, h_2) = \varphi_1(h_1)\varphi_2(h_2)$$

as a cusp form on the algebraic group $\rho(\tilde{H})$.

Next, let f be a Schwartz-Bruhat function on the space $M_{2 \times 2}(\mathbb{A}) \times M_{2 \times 2}(\mathbb{A})$. By a process described in [11] and [24], one associates to the couple (φ, f) a generic cusp form on the group $GS(4)$, denoted by $\theta(f; \varphi)$. Here, in order to stress the dependence on φ_1 and φ_2 , we will use the notation $\theta_f(\varphi_1, \varphi_2)$. We then set

$$(30) \quad \mathcal{Z}(\varphi_1, \varphi_2, f; s) = \mathcal{Z}(\theta_f(\varphi_1, \varphi_2), s).$$

We have our first result as follows.

Theorem 2.4. *We have*

$$\begin{aligned} \mathcal{Z}(\varphi_1, \varphi_2, f, s) &= \int_{\rho(D)(\mathbb{A}) \backslash \rho(G_1)(\mathbb{A})} f(h_1^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} h_2, h_1^{-1} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} h_2) \\ &\quad \left(\int_{F \backslash \mathbb{A}^\times} \varphi_1 \left(\begin{pmatrix} a & \\ & 1 \end{pmatrix} h_1 \right) |a|^{s-\frac{1}{2}} d^\times a \right) \\ &\quad \left(\int_{F \backslash \mathbb{A}^\times} \varphi_2 \left(\begin{pmatrix} b & \\ & 1 \end{pmatrix} h_2 \right) |b|^{-s+\frac{1}{2}} d^\times b \right) dh_1 dh_2, \end{aligned}$$

with subgroups $\rho(G_1)$ and $\rho(D)$ of $\rho(\tilde{H})$ defined by

$$G_1 = \{(h_1, h_2) \mid \det h_1 = \det h_2\},$$

and

$$D = \left\{ \left(\begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}, \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix} \right) \mid \alpha, \beta \in \mathbb{G}_m \right\}.$$

The proof of this theorem is a standard exercise in the theory of the oscillator representation (cf. for example [12], proof of lemma 5.)

It follows from the functional equation of the Jacquet-Langlands Zeta function and the Fourier-Whittaker expansion of the cusp form φ_2 that

Corollary 2.5. *We have*

$$\begin{aligned} \mathcal{Z}(\varphi_1, \varphi_2, f, s) = & \int_{\rho(D')(\mathbb{A}) \backslash \rho(G_1)(\mathbb{A})} f(h_1^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} w^{-1} h_2 w, h_1^{-1} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} w^{-1} h_2 w) \\ & \left(\int_{\mathbb{A}^\times} W_{\varphi_1} \left(\begin{pmatrix} a & \\ & 1 \end{pmatrix} h_1 \right) |a|^{s-\frac{1}{2}} d^\times a \right) \\ & \left(\int_{\mathbb{A}^\times} W_{\varphi_2} \left(\begin{pmatrix} b & \\ & 1 \end{pmatrix} h_2 w \right) |b|^{s-\frac{1}{2}} \omega_{\pi_2}^{-1}(b) d^\times b \right) dh_1 dh_2, \end{aligned}$$

where $w = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$, and

$$D' = \left\{ \left(\begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}, \begin{pmatrix} \beta & \\ & \alpha \end{pmatrix} \right) \mid \alpha, \beta \in \mathbb{G}_m \right\}.$$

Next, if $f = \otimes_v f_v$, $W_{\varphi_i} = \otimes_v W_v^i$, then we obtain

$$\mathcal{Z}(\varphi_1, \varphi_2, f, s) = \prod_v \mathcal{Z}(W_v^1, W_v^2, f_v, s),$$

with the appropriate local zeta functions. The local zeta function introduced here satisfy local functional equations similar to the ones considered before. Also, if all the data is unramified, one can easily perform the local computation.

We now consider the local archimedean computations. Suppose v is a real archimedean place, and W_1, W_2 in the spaces of π_1 and π_2 , respectively, then we can prove the following result:

Theorem 2.6. *For every Schwartz-Bruhat function f_v as above, the complex function $Z(W_v^1, W_v^2, f_v, s)$ has an analytic continuation to a meromorphic function on the complex plane. Furthermore, the ratio*

$$(31) \quad \frac{Z(W_v^1, W_v^2, f_v, s)}{L(s, \pi_1^v) L(1-s, \pi_2^v)}$$

has an analytic continuation to an entire function. Also, there is a choice of the data in such a way that the resulting entire function is nowhere vanishing.

The idea of the proof is the same as that of the proof of Theorem 5.15 of [14].

The most interesting case in the above analysis is when the representations π_1^v and π_2^v are in the discrete series, and the data W_1^v, W_2^v , and f are K-finite. In this situation, one can explicitly calculate the ratio (31) in the terms of hypergeometric functions and gamma functions. Here, the main idea is to use a two-variable version of the zeta function introduced above. There is a description of the discrete series in terms of the Weil representation (paragraph 1, [14]). The final results are indeed given in terms of Meijer's G-function, in which the variable s appears in the parameters, not the argument. The particular G-function which appears in this work is a finite linear combination of functions of the form

$$\Gamma(s+A)_2 F_3(s+A, B; C, D, E; F)$$

for various values of the parameters A, \dots, F .

2.4. A possible application. In [2], Böcherer has proclaimed the following conjecture:

Conjecture 2.7. *Let Φ be a holomorphic cuspidal Siegel eigenform of degree two and weight k with respect to $Sp_4(\mathbb{Z})$. Let*

$$\Phi(Z) = \sum_{T>0} a(T, \Phi) \exp(2\pi\sqrt{-1}\text{tr}(TZ))$$

be its Fourier expansion. For a fundamental discriminant $-D$, i.e. a discriminant of an imaginary quadratic field $\mathbb{Q}(\sqrt{-D})$, let

$$B_D(\Phi) = \sum_{\{T \mid \det(T) = \frac{D}{4}\} / \sim} \frac{a(T, \Phi)}{\epsilon(T)},$$

where \sim denotes the equivalence relation defined by $T_1 \sim T_2$ when $T_1 = {}^t \gamma T_2 \gamma$ for some $\gamma \in SL_2(\mathbb{Z})$ and $\epsilon(T) = \#\{\gamma \in SL_2(\mathbb{Z}) \mid {}^t \gamma T \gamma = T\}$.

Then there exists a constant C_Φ which depends only on Φ such that

$$L\left(\frac{1}{2}, \pi_\Phi \otimes \chi_D\right) = C_\Phi \cdot D^{-k+1} \cdot |B_D(\Phi)|^2,$$

where π_Φ is the automorphic representation of $GSp_4(\mathbb{A}_\mathbb{Q})$ associated with Φ , χ_D is the quadratic character of $\mathbb{A}_\mathbb{Q}^\times$ associated with $\mathbb{Q}(\sqrt{-D})$ and the left hand side denotes the central critical value of the quadratic twist by χ_D of the degree four Spinor L -function for π_Φ .

Böcherer proved this assertion in the cases of the Klingen Eisenstein series and the Saito-Kurokawa lifting in [2]. Later he and Schulze-Pillot proved this in the case of the Yoshida lifting in [3]. More recently Masaaki Furusawa and Joseph Shalika have started investigating this conjecture from a different angle. Their approach to the problem is to generalize Hervé Jacquet's *relative trace formula* for $GL(2)$ to $GSp(4)$. Jacquet has used his $GL(2)$ relative trace formula in [13] to give another proof for an important theorem of Waldspurger [23]. Böcherer's conjecture provides a natural generalization of this theorem for Siegel modular forms.

From Furusawa-Shalika's conjectural relative trace formula, one should be able to draw a conclusion similar to the one in [13], namely:

Assertion 2.8. *For a globally generic cuspidal automorphic representation of $GSp_4(\mathbb{A}_F)$ with the trivial central character, E a quadratic extension of the base number field F , and χ_E , the quadratic character of \mathbb{A}_F^\times corresponding to E , we have:*

$$(32) \quad L\left(\frac{1}{2}, \pi\right) \cdot L\left(\frac{1}{2}, \pi \otimes \chi_E\right) \neq 0$$

if and only if there exists a triple (D, π_D, Ψ_D) , where D is a central simple quaternion algebra over F containing E , π_D is a cuspidal automorphic representation of G_D , the quaternion similitude unitary group of degree two over D , which corresponds to π in the functorial sense, and Ψ_D is a cusp form in the space of π_D such that

$$(33) \quad \int_{\mathbb{A}_F^\times R_D(F) \backslash R_D(\mathbb{A}_F)} \Psi_D(r) \tau(r) dr \neq 0.$$

Here R_D denotes the Bessel subgroup of G_D and τ is a certain character of $R_D(\mathbb{A}_F)$ (cf. [8] for precise definitions).

Moreover, the detailed analysis should yield an identity that expresses the special value (32) as the square norm of one of the period integrals (33) multiplied by a positive constant C_π which itself depends only on π and not on the quadratic extension E . Also a simple explicit formula for C_π , given essentially as a ratio of Petersson inner products, is expected. It is believed that since the expected formula for C_π involves the Petersson inner product of certain vectors in the Whittaker model of the representation π , the proof of Böcherer's conjecture in general is out of the reach of classical methods.

The special L-values considered above appear in Furusawa-Shalika's relative trace formula as Novodvorsky's integral. For this reason, in order to prove the conjecture of Böcherer, one needs precise information about the local behavior of Novodvorsky's integrals at all the places of the number field F , including the ramified places and the places at infinity.

2.5. Open problems. Below we list two problems directly related to the subject matter of this work.

2.5.1. *Comparison with the Langlands-Shahidi Method.* It is a natural question to determine whether the local L-factors given by Novodvorsky's integral are the same as the ones given by the Langlands-Shahidi method (see page 116 of [10]). The ambient group to consider is the group $GSpin_7$ which has a parabolic subgroup with Levi factor isomorphic to $GSpin_5 \times GL_1$. It is well-known that $GSpin_5$ is isomorphic to $GSp(4)$. Then we can use the results of [1] at least for the cases where the given representations are tempered. We hope to address this problem in a subsequent joint work with Mahdi Asgari.

2.5.2. *Test Vectors.* In [20], we have determined the local L-function given by Novodvorsky's integral. It is easy to see that this local L-factor is actually the value of the integral at a distinguished vector in the space of the representation. For trace formula applications, it is necessary to have some information about this distinguished vector. This is still an open problem which we would like to address in a subsequent paper.

REFERENCES

- [1] M. Asgari, *Local L-functions for split spinor groups*, to appear in Canadian J. of Math.
- [2] S. Böcherer, *Bemerkungen über die Dirichletreihen von Koecher und Maass*, Preprint Math. Gottingensis Heft **68**(1986).
- [3] S. Böcherer and R. Schulze-Pillot, *The Dirichlet series of Koecher and Maass and modular forms of weight $\frac{3}{2}$* , Math. Z. **209**(1992), 273-287.
- [4] A. Borel, *Automorphic L-functions*, Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, pp. 27-61, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979.
- [5] D. Bump, *Rakin-Selberg Method: A Survey*, In *Number Theory, Trace Formulas, and Discrete Groups* (K. E. Bombieri, and D. Goldfeld, eds.), Academic Press, San Diego, CA, 1989, 49-109.
- [6] W. Casselman, and J. A. Shalika, *The unramified principal series of p-adic reductive groups II: The Whittaker function*, Compositio Math. **41** (1980), 207-231.

- [7] M. Furusawa, *On L-functions for $GSp(4) \times GL(2)$ and their special values*, J. Reine Angew. Math. **438** (1993), 187-218.
- [8] M. Furusawa and J. A. Shalika, *The fundamental lemma for the Bessel and Novodvorsky subgroups of $GSp(4)$* , C. R. Acad. Sci. Paris Sér. I Math. **328**(1999). no. 2, 105-110.
- [9] S. Gelbart, *Automorphic forms on adèle groups*, Annals of Mathematics Studies, No. 83. Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1975. x+267 pp.
- [10] S. Gelbart and F. Shahidi, *Analytic Properties of Automorphic L-Functions*, Prospectives in Mathematics, Academic Press, 1988.
- [11] M. Harris and S. S. Kudla, *Arithmetic automorphic forms for the nonholomorphic discrete series of $GSp(2)$* , Duke Math. J. **66**(1) (1992), 59-121.
- [12] M. Harris, D. Soudry, and R. Taylor, *l -adic representations associated to modular forms over imaginary quadratic fields, I. Lifting to $GSp_4(\mathbb{Q})$* , Invent. Math. **112**(1993), 377-411.
- [13] H. Jacquet, *Sur un résultat de Waldspurger*, Ann. Scient. Éc. Norm. Sup., 4 série, t. **19** (1986), 185-229.
- [14] H. Jacquet and R. P. Langlands, *Automorphic forms on $GL(2)$* , Springer Lecture Notes in Mathematics **114**, (1970).
- [15] M. Novodvorsky, *Automorphic L-functions for the symplectic group GSp_4* , In *Automorphic Forms, Representations, and L-Functions*, AMS Proc. Sym. Pure. Math. **33**(2)(1979).
- [16] P. Sally and M. Tadic, *Induced representations and classifications for $GSp(2, F)$ and $Sp(2, F)$* , Mem. Soc. Math. France **52**(1993), 75-133.
- [17] F. Shahidi, *On certain L-functions*, Amer. J. Math. **103** (1981), 297-355.
- [18] —, *A proof of Langlands' conjecture on Plancherel measure; Complementary series for p -adic groups*, Ann. Math. **132**(1990), 273-330.
- [19] J. A. Shalika, *The multiplicity one theorem on $GL(n)$* , Ann. Math. **100** (1974), 171-193.
- [20] R. Takloo-Bighash, *L-functions for the p -adic group $GSp(4)$* , Amer. J. Math. **122**(6) (2000), 1085-1120.
- [21] —, *L-functions for the real group $GSp(4)$* , in preparation.
- [22] J. Tate, *Fourier analysis in number fields, and Hecke's zeta-functions*. 1967 Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965) pp. 305–347 Thompson, Washington, D.C.
- [23] J.-L. Waldspurger, *Sur les valeurs de certaines fonction L automorphes en leur centre de symétrie*, Compositio Math. **54** (1985), 173-242.
- [24] B. Roberts, *Global L-packets for $GSp(2)$ and theta lifts*, Documenta Math. **6**(2001), 247-314.