HOMOGENIZATION FOR PARTIAL DIFFERENTIAL EQUATIONS

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I would like to dedicate this modest expository paper to Professor Siavash Shahshahani on his 60th birthday. Siavash has always been a superb teacher and a great friend. My deepest thanks for his endless encouragement and support.

Homogenization.

The Hamilton-Jacobi equation

$$u_t + H(x, u_x) = 0 \tag{1.1}$$

and its viscous cousin

$$u_t + H(x, u_x) = \alpha \Delta u, \qquad \alpha > 0 \ , \tag{1.2}$$

are often used to model the formation of crystals. When there is impurity or the lack of experimental data, we may assume that H is random. If such randomness is stochastically *stationary* and *ergodic*, then in macroscopic coordinates the PDE (1) or (2) simplifies to a *homogenized* Hamilton-Jacobi equation. Indeed, if x and t are macroscopic variables and $u^\epsilon(x,t) = \epsilon u\big(\frac{x}{\epsilon}, \frac{t}{\epsilon}\big)$, then u^ϵ satisfies

$$u^\epsilon_t + H\left(\frac{x}{\epsilon}, u^\epsilon_x\right) = \epsilon \alpha \Delta u^\epsilon \ . \tag{1.3}$$

When H is stationary, the function $H\big(\frac{x}{\epsilon}, p\big)$ is highly oscillatory in x-variable for small ϵ. The huge fluctuation in H results in the convergence of u^ϵ to a function $\bar{u}$ that now solves a Hamilton-Jacobi equation of the form

$$\bar{u}_t + \bar{H}_\alpha(\bar{u}_x) = 0 \ , \tag{1.4}$$

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where $\bar{H}_\alpha$ is known as the *effective Hamiltonian* of the equation (1.2).

The simplest nontrivial randomness we may consider for H is when ω is selected uniformly from the unit cube $[0,1]^d$ and

$$H(x,p,\omega) = H_0(x+\omega,p)$$

for a fixed Hamiltonian $H_0(x,p)$ that is x-periodic of period one; $H_0(x+e_j,p) = H(x,p)$ for each j where e_j denotes the unit vector in the j-th direction. In this case, the role of the randomness is rather artificial and we may simply study

$$u_t^\epsilon + H_0\left(\frac{x}{\epsilon}, u_x^\epsilon\right) = \epsilon\alpha\Delta u^\epsilon \tag{1.5}$$

for a fixed (nonrandom) Hamiltonian H_0 that is periodic in x-variable. The convergence of u^ϵ to $\bar{u}$ in the periodic case was established by Lions et al [LPV] when $\alpha = 0$. Later Evans [E] treats the case $\alpha > 0$. In both [LPV] and [E], the proof of convergence follows from the solvability of the auxiliary equation

$$H_0(x, p + w_x) = \lambda + \alpha\Delta w \tag{1.6}$$

where $p \in \mathbb{R}^d$, λ is a constant and $w : \mathbb{R}^d \to \mathbb{R}$ is a periodic function. It turns out that for every $p \in \mathbb{R}^d$, there exists a unique constant λ for which (1.6) has a periodic solution w. In fact the unique constant λ is nothing other than $\bar{H}_\alpha(p)$. To see this, observe that if w solves (1.6), then

$$u(x,t) = w(x) + p\cdot x - t\lambda \tag{1.7}$$

solves (1.5). Hence

$$u^\epsilon(x,t) = \epsilon w\left(\frac{x}{\epsilon}\right) + p\cdot x - t\lambda \tag{1.8}$$

converges to

$$\bar{u}(x,t) = p\cdot x - t\lambda\ . \tag{1.9}$$

But such a function $\bar{u}$ solves (1.4) if and only if $\lambda = \bar{H}_\alpha(p)$.

We now turn our attention to the nonperiodic case. Consider a Hamiltonian $H(x,p) = H(x,p,\omega)$ that is stationary and ergodic with respect to x. As a concrete example, assume

$$H(x,p,\omega) = \tfrac{1}{2}|p|^2 + \omega(x)\ , \tag{1.10}$$

where ω is chosen randomly from the space of continuous functions $\Omega = C(\mathbb{R}^d, \mathbb{R})$. Here we have a probability measure P on Ω such that

$$\int f(\tau_x\omega)P(d\omega) = \int f(\omega)P(d\omega) \tag{1.11}$$

for every x, where

$$(\tau_x\omega)(y) = \omega(x+y) \tag{1.12}$$

is the shift operator. We also assume that P is ergodic;

$$\lim_{\ell\to+\infty} \frac{1}{(2\ell)^d} \int_{-\ell}^{\ell} \cdots \int_{-\ell}^{\ell} f(\tau_x\omega)dx = \int f\, dP\ . \tag{1.13}$$

As for (1.6), we now need to solve

$$H(x, v_x, \omega) = \lambda + \alpha\Delta v \tag{1.14}$$

where v satisfies

$$v(x) = v(x,\omega) = x\cdot p + o(|x|) \tag{1.15}$$

as $|x| \to +\infty$. Note that this is consistent with what we had in the periodic case because if we set $v(x) = w(x) + x\cdot p$ with w a solution of (1.6), then $v(x) = x\cdot p + O(1)$. When $\alpha = 0$ and H is given by (1.10), then it is not hard to see that (1.14) cannot be solved for every $\omega \in \Omega$ and every $p \in \mathbb{R}^d$. If H is convex in p, then we can use control theory to represent u^ϵ by a variational formula:

$$u^\epsilon(x,t,\omega) = \inf_y \left\{ g(y) + \epsilon S\left(\frac{x}{\epsilon}, \frac{t}{\epsilon}, \frac{y}{\epsilon}, \omega\right)\right\}\ , \tag{1.16}$$

$$S(x,t,y,\omega) = \inf\left\{ \int_0^t L(\xi,\dot{\xi},\omega)d\theta \mid \xi(0)=y,\ \xi(t)=x,\ \xi \text{ Lipschitz}\right\}\ , \tag{1.17}$$

where $L(x,q,\omega)$ is the convex conjugate of $H(x,p,\omega)$ in the p-variable.

In Rezakhanlou-Tarver [RT] and Souganidis [So], (1.16) was used to reduce the convergence of u^ϵ to the convergence of $S^\epsilon(x,y,t,\omega) = S(\frac{x}{\epsilon}, \frac{t}{\epsilon}, \frac{y}{\epsilon}, \omega)$. Using a *subadditive ergodic theorem*, we obtain

$$\lim_{\epsilon\to 0} S^\epsilon(x,y,t,\omega) \ = \ t\bar{L}\left(\frac{x-y}{t}\right) \tag{1.18}$$

for some deterministic convex function $\bar{L}$. In fact (1.16) and (1.18) can be used to show that if

$$u^\epsilon(x,0,\omega) = g(x)\ ,$$

then

$$\lim_{\epsilon\to 0} u^\epsilon(x,t,\omega) = \inf_y \left\{ g(y) + \bar{L}\left(\frac{x-y}{t}\right)\right\} \tag{1.19}$$

with probability one. By the celebrated Hopf-Lax-Oleinik formula, the right-hand side of (1.19) is the unique solution of (1.4) with the initial condition $\bar{u}(x,0) = g(x)$, where $\bar{H}_0$ is the convex conjugate of $\bar{L}$.

We next study (1.2) when $\alpha > 0$ and H is stationary and ergodic in the x-variable. Without loss of generality, assume $\alpha = \frac{1}{2}$. It turns out that if H is of the form (1.10) then the convergence of u^ϵ can be recast as a *large deviation principle* for a Brownian motion in random media. More precisely, let $\beta(t)$ denote a Brownian motion and let E^y denote the expectation with respect to β. Then by the Feymann-Kac formula,

$$u^\epsilon(x,t,\omega) = -\epsilon \log E \left[\exp \left(\epsilon^{-1} g \left(\left(\frac{t}{\epsilon} \right) \right) + \int_0^{\frac{t}{\epsilon}} \omega(\beta(\theta)) d\theta \right) \right] . \tag{1.20}$$

In the context of probability theory, the convergence of u^ϵ to $\bar{u}$ is equivalent to a large deviation principle for a Brownian motion β in a random media ω with a rate function $I(u) = \bar{L}(u) - \inf \bar{L}$. In this context, the convergence of u^ϵ was established by Sznitman. See Sznitman [Sz] where such a large deviation principle is thoroughly addressed.

If H is not of the form of (1.10) the large deviation interpretation and formula (1.20) is no longer available and the question of the convergence of u^ϵ remains open.

> **Problem 1.** When $\alpha > 0$, show that $u^\epsilon \to \bar{u}$ for a $\bar{u}$ that solves a Hamilton-Jacobi equation.

Central Limit Theorem.

As we mentioned in the previous section, the graph of the function $u(x,t,\omega)$ may be used to model the boundary of a crystal that is evolving with time. The limit $\bar{u}(x,t)$ gives us a *macroscopic* description of the boundary surface. Microscopically though, the boundary surface is rough and fluctuates about the macroscopic solution. A central limit theorem is established in some cases when $\alpha = 0$. The method of Rezakhanlou [R] applies to those cases for which (1.14) has a nice solution. More precisely, imagine that there exists a continuous (random) function $v(x,p)$ such that

$$H(x, v_x) = \bar{H}(p) \ , \tag{2.1}$$

and the process

$$\frac{1}{\sqrt{\epsilon}} \big(\epsilon v \big(\frac{x}{\epsilon}, p \big) - x \cdot p \big) \tag{2.2}$$

converges to a continuous process $B(x,p)$. In other words,

$$v^\epsilon(x,p) = x \cdot p + \sqrt{\epsilon} \ B(x,p) + o(\sqrt{\epsilon}) \ . \tag{2.3}$$

It is shown in [R] that if (2.3) is true, then

$$u^\epsilon(x,t) = \bar{u}(x,t) + \sqrt{\epsilon}\ Z(x,t) + o(\sqrt{\epsilon})\ , \tag{2.4}$$

where the random process $Z(x,t)$ is given by

$$Z(x,t) = \inf_{y\in I(x,t)} \{B(x,p(x,y,t)) - B(y,p(x,y,t))\} \tag{2.5}$$

where

$$p(x,y,t) = \nabla \bar{L}\left(\frac{x-y}{t}\right)$$

and $I(x,t)$ is the set of y at which the infimum in (1.19) is attained.

There are several classes of examples for which the condition (2.3) can be readily verified.

Example 2.1 Suppose $\bar{H}:\mathbb{R}\to\mathbb{R}$ is a given convex function and $H(x,p,\omega) = \bar{H}\left(\frac{p}{\omega(x)}\right)$. Then we may choose

$$v(x,p) = p\int_0^x \omega(y)dy\ .$$

Now (2.3) is a central limit theorem for the process $\int_0^x \omega$. More precisely, if we assume

$$\epsilon \int_0^{\frac{x}{\epsilon}} \omega(y)dy = x + \sqrt{\epsilon}\,\hat{B}(x) + o(\sqrt{\epsilon})\ ,$$

then $B(x,p) = p\hat{B}(x)$ and

$$Z(x,t) = \inf_{y\in I(x,t)} \left\{\bar{L}'\left(\frac{x-y}{t}\right)(\hat{B}(x) - \hat{B}(y))\right\}\ .$$

Example 2.2 Suppose $H(x,p) = \frac{1}{2}p^2 + \omega(x)$ and $d=1$. Let E denote the expectation with respect to the randomness ω and assume that $\sup\omega = 0$ with probability one. Set

$$\varphi(\lambda) = 2E\sqrt{\lambda - \omega(0)}\ .$$

The function φ is increasing and we write φ^{-1} for its inverse. We then define

$$\bar{H}(p) = \begin{cases} -\varphi^{-1}(-p) & \text{if } p \le -\varphi(0)\ , \\ 0 & \text{if } -\varphi(0) < p < \varphi(0)\ , \\ \varphi^{-1}(p) & \text{if } p \ge -\varphi(0)\ . \end{cases}$$

If $p\notin(-\varphi(0),\varphi(0))$, we may choose

$$v(x,p) = \begin{cases} 2\int_0^x \sqrt{\bar{H}(p) - \omega(y)}\,dy & \text{if } p \ge \varphi(0)\ , \\ -2\int_0^x \sqrt{\bar{H}(p) - \omega(y)}\,dy & \text{if } p \le -\varphi(0)\ . \end{cases}$$

It is shown in [R] that if

$$\lim_{n\to\infty} \frac{1}{n^2} \int_{-n}^{n} \frac{dy}{\sqrt{\omega(y)}} = +\infty \tag{2.6}$$

with probability one, then (2.4) is valid provided that we assume (2.3) for $p \notin (-\varphi(0), \varphi(0))$ only.

Problem 2. Establish a central limit theorem when H is given by (1.10) and $d > 1$.

This problem is significantly harder than the one-dimensional case. This is because (1.14) can not be solved explicitly , and even if we can prove the existence of the process $v(x,p)$ in some cases, it is not clear at all that (2.3) is any easier that (2.4).

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