

On Geometric Representations of Galois Group

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Dedicated to professor Shahshahani on the occasion of his 60th birthday

Abstract

One can think of an ℓ -adic Galois representation attached to an elliptic curve over \mathbb{Q} , as an action of the Galois group over the ℓ -adic completion of the fundamental group of that elliptic curve. We use this idea to attach Galois representations to curves over \mathbb{Q} of higher genus. Then, we suggest some techniques of Galois representation theory, fit to study these new representations.

1 Introduction

Given an elliptic curve E over \mathbb{Q} and fixing a prime ℓ , Tate associated a representation

$$\rho_\ell : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow GL_2(\mathbb{Z}_\ell)$$

by taking the inverse limit over n of the groups of ℓ^n -torsion points of E as Galois modules. On the other hand, one can think of the ℓ^n -torsion points of E as the Galois group of the function field extension associated to the multiplication map $\times \ell^n : E \rightarrow E$. Since every Galois etale covering of an elliptic curve is again an elliptic curve, we can take a limit over all Galois etale covers $E' \rightarrow E$ of degree ℓ^n for some n , and obtain the same Galois module structure on $\mathbb{Z}_\ell \times \mathbb{Z}_\ell$ as part of the algebraic fundamental group of E .

For an arbitrary curve X the algebraic fundamental group $\pi_1^{alg}(X)$ is defined by Grothendieck as $\lim Gal(K'/K)$ where K is the function field of X and K' runs over all Galois extensions of K such that the corresponding curve X' is etale over X . For example, $\pi_1^{alg}(\mathbb{P}^1) = 1$ and for every elliptic curve E we have

$$\pi_1^{alg}(E) = \prod \mathbb{Z}_\ell \times \mathbb{Z}_\ell.$$

Grothendieck proved that, for a curve X of genus g , the algebraic fundamental group of X is isomorphic to the completion of the ordinary topological fundamental group of X over \mathbb{C} [Gr]. If X is defined over \mathbb{Q} we can induce an action of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on $\pi_1^{alg}(X)$.

One can define the ℓ -adic algebraic fundamental group $\pi_{1,\ell}^{alg}(X)$ as the limit

$$\lim Gal(K'/K)$$

where K is the function field of X and K' runs over all degree ℓ^n Galois extensions of K such that the corresponding curve X' is etale over X . The group $\pi_{1,\ell}^{alg}(X)$ is isomorphic to the ℓ -adic completion of $\pi_1^{top}(X)$. If X is defined over \mathbb{Q} one can associate an ℓ -adic Galois representation to X which is a direct summand of the above mentioned representation

$$\rho_\ell : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow Aut(\pi_{1,\ell}^{alg}(X)).$$

The Frattini subgroup $\Phi(G)$ of the pro- ℓ group $G = \pi_{1,\ell}^{alg}(X)$ is fixed by each automorphism [Ri-Za]. Therefore we get a representation

$$\rho_\ell : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow Aut(G/\Phi(G)).$$

Since $\Phi(G) = \overline{G^p[G, G]}$ we can think of $G/\Phi(G)$ as a finite dimensional vector space over the field \mathbb{F}_p with p elements [Ri-Za]. This way we have recovered the mod- p Galois representation associated to X via Tate module.

The classical approach to definition of an ℓ -adic Galois representation is to consider the action of Galois group on the Tate module associated to the Jacobian variety $Jac(X)$ as above. This representation can be reconstructed from the one we introduced above by abelianization. But we loose lots of information during this process. We believe the structure of fundamental group can help us to understand the Galois action better.

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2 The fundamental group revisited

The fundamental group plays a fundamental role in geometry. Hempel in 1976 proved that Poincaré conjecture holds iff there exists a unique epimorphism

$$\pi_1^{top}(X) \rightarrow F(g) \times F(g)$$

up to automorphisms of $\pi_1^{top}(X)$ and automorphisms of the components $F(g)$ which are free groups with g generators [He]. Existence of such an epimorphism is evident from the presentation

$$\pi_1^{top}(X) = \langle a_1, \dots, a_g, b_1, \dots, b_g \mid \prod_{i=1}^g [a_i, b_i] = 1 \rangle.$$

Having this presentation, one can show that the fundamental group is a torsion-free group with trivial center. We also have access on a necessary and sufficient condition for a one-relator group to be isomorphic to the fundamental group [Co-Zi].

For a free group F , we say that two surjective maps $\varphi_1, \varphi_2 : \pi_1^{top}(X) \rightarrow F$ are equivalent if there exist an automorphism of $\pi_1^{top}(X)$ and an automorphism of F which take φ_1 to φ_2 , i.e. there exists $\alpha \in Aut(\pi_1^{top}(X))$ and $\beta \in Aut(F)$ such that the following diagram is commutative

$$\begin{array}{ccc} \varphi_1 : \pi_1^{top}(X) & \longrightarrow & F \\ \alpha \downarrow & & \downarrow \beta \\ \varphi_2 : \pi_1^{top}(X) & \longrightarrow & F \end{array}$$

We say that φ_1 and φ_2 are strictly equivalent if there exist an automorphism α of $\pi_1^{top}(X)$ such that $\varphi_2 = \varphi_1 \circ \alpha$. Zieschang in 1964 proved that if rank of F is less than or equal to g there exists only one equivalence class of surjective maps $\pi_1^{top}(X) \rightarrow F$ [Zi]. With the same assumption, Kurchanov and Grigorchuk in 1989 proved that there exists only one strict equivalence class of such surjective maps [Ku-Gr]. Existence of such a surjection is an easy consequence of the presentation of the fundamental group. Nielsen in 1927 proved that every automorphism of $\pi_1^{top}(X)$ is induced by an automorphism of the corresponding free group which preserves the relation R [Ni].

Suppose H_0 and H_1 are subgroups of $\langle a_1, \dots, a_g, b_1, \dots, b_g \mid \prod_{i=1}^g [a_i, b_i] = 1 \rangle$ which are generated by a_1, \dots, a_g and b_1, \dots, b_g respectively. The subgroups L^+ and L^- defined as such

$$L^- = \{ \varphi \in Aut(\pi_1^{top}(X)) : \varphi(H_0) = H_0 \}$$

$$L^+ = \{ \varphi \in Aut(\pi_1^{top}(X)) : \varphi(H_1) = H_1 \}$$

are conjugate in $Aut(\pi_1^{top}(X))$. The group $Aut(\pi_1^{top}(X))$ is generated by L^- and L^+ .

Every automorphism of $\pi_1^{top}(X)$ induces an automorphism of the abelian group $\pi_1^{top}(X)/[\pi_1^{top}(X), \pi_1^{top}(X)]$ whose matrix coordinatizes a representation

$$\rho : Aut(\pi_1^{top}(X)) \longrightarrow Sp_{2g}(\mathbb{Z})$$

Here $Sp_{2g}(\mathbb{Z})$ is the group of matrices S of order $2g$ such that $SJS^t = \pm J$ where

$$J = \begin{pmatrix} 0 & I_g \\ -I_g & 0 \end{pmatrix}.$$

The representation ρ takes elements of L^- to lower triangular matrices

$$\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}$$

and elements of L^+ go to upper triangular matrices

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}.$$

3 The completion of fundamental group

Providing information about the completed fundamental group is the key to a better understanding of the Galois representation. Let G denote the pro- p completion of the fundamental group. Taking pro- p completions from the surjective maps $\pi_1^{top}(X) \rightarrow F(g) \times F(g)$ and $\pi_1^{top}(X) \rightarrow F(2g)$ one gets surjections from G to the pro- p completions $F(g)_p \times F(g)_p$ and $F(2g)_p$. Also, every automorphism of $\pi_1^{top}(X)$ induces an automorphism of G . One can ask if the analogue of the Poincaré conjecture is true here. More precisely, one can ask if there is only one strict equivalence class of surjections

$$G \longrightarrow F(g)_p \times F(g)_p$$

The representation $\rho : Aut(\pi_1^{top}(X)) \longrightarrow Sp_{2g}(\mathbb{Z})$ induces a map

$$\hat{\rho} : Aut(G) \longrightarrow Sp_{2g}(\mathbb{Z})_p$$

which after combining with the natural surjection

$$Sp_{2g}(\mathbb{Z})_p \longrightarrow Sp_{2g}(\mathbb{Z}_p)$$

induces a representation

$$\rho_X : Aut(G) \longrightarrow Sp_{2g}(\mathbb{Z}_p).$$

If we combine ρ with the Galois action on $\pi_1^{alg}(X)$ one has associated a p -adic Galois representation to the algebraic curve X :

$$\rho_p : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow Sp_{2g}(\mathbb{Z}_p).$$

This is the same as the Galois representation associated to the Tate module of the Jacobian variety of X .

4 The Galois action on $\pi_1^{alg}(X)$

The outer automorphism group of $\pi_1^{alg}(X)$ as a quotient of $Aut(\pi_1^{alg}(X))$ accepts a representation from the Galois group

$$\rho_p : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow Aut(\pi_1^{alg}(X)) \longrightarrow Out(\pi_1^{alg}(X))$$

Mochizuki proved that if X and X' are curves over \mathbb{Q} of genus greater than or equal to 2, the map

$$Isom_{\mathbb{Q}}(X, X') \longrightarrow Out_{Gal(\overline{\mathbb{Q}}/\mathbb{Q})}(Out(\pi_1^{alg}(X)), Out(\pi_1^{alg}(X')))$$

is a one-to-one correspondence, where $Out_{Gal(\overline{\mathbb{Q}}/\mathbb{Q})}$ denotes Galois equivariant isomorphisms between the two groups [Mo]. In particular, the Galois representation on $Out(\pi_1^{alg}(X))$ completely determines the curve X over \mathbb{Q} .

The representation $\hat{\rho}$ breaks through $Out(\pi_1^{alg}(X))$ because inner automorphisms are neutral when reduced to abelianization of a group

$$\hat{\rho} : Aut(\pi_1^{alg}(X)) \longrightarrow Out(\pi_1^{alg}(X)) \longrightarrow Sp_{2g}(\mathbb{Z})_p.$$

Thus for each prime p one can associate a p -adic representation to $Out(\pi_1^{alg}(X))$ by considering the pro- p part of the outer automorphism group

$$\rho_X : Out(\pi_1^{alg}(X)) \longrightarrow Sp_{2g}(\mathbb{Z}_p).$$

and therefore the Galois representation

$$\rho_p : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow Sp_{2g}(\mathbb{Z}_p)$$

could be associated to the Galois action on $Out(\pi_1^{alg}(X))$. This shows that the language of outer automorphisms is an appropriate one for studying the arithmetic of curves over \mathbb{Q} . To study the geometric Galois representation

$$\rho : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow Out(\pi_1^{alg}(X))$$

the first step, is to study $Out(\pi_1^{alg}(X))$.

5 The completion of mapping class group

The mapping class group $MC(X)$ of X is defined as the factor of the group of homeomorphisms of X as a Riemann surface by the subgroup of elements isotopic to identity. The mapping class group is isomorphic to the group of outer automorphisms of the topological fundamental group

$$MC(X) \cong Out(\pi_1^{top}(X)).$$

There has been many efforts to introduce a finite presentation for this group. The ones introduced by Birman in 1974 for the case of genus 2 look particularly simple. The generators $\sigma_1, \dots, \sigma_5$ together with the following relations generate $MC(X_2)$.

$$\begin{aligned} \sigma_i \sigma_j &= \sigma_j \sigma_i, & \text{for } |i - j| \geq 2, 1 \leq i, j \leq 5 \\ \sigma_i \sigma_{i+1} \sigma_i &= \sigma_{i+1} \sigma_i \sigma_{i+1}, & 1 \leq i \leq 4 \\ (\sigma_1 \sigma_2 \dots \sigma_5)^6 &= 1 \\ (\sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5^2 \sigma_4 \sigma_3 \sigma_2 \sigma_1)^2 &= 1 \\ \sigma_1 \sigma_2 \sigma_3 \sigma_4 \sigma_5^2 \sigma_4 \sigma_3 \sigma_2 \sigma_1 \perp \sigma_i, & 1 \leq i \leq 5 \end{aligned}$$

where $x \perp y$ means that x and y commute. For $g \geq 3$ genus, Dehn 1938 [De], Lickorish 1965 [Li], Hatcher and Thurston 1980 [Ha-Th], Wajnryb 1983 [Wa], gave presentations of the mapping class group. In all these presentations, the number of generators increases with g . However, Suzuki in 1977 showed that one can manage

with four generators [Su]. We give automorphisms in $Aut(\pi_1^{top}(X))$ whose image in $MC(X) = Aut(\pi_1^{top}(X))/Inn(\pi_1^{top}(X))$ generate the mapping class group.

$$\begin{aligned} \alpha_0 &: \begin{pmatrix} a_1 \rightarrow b_1^{-1}, a_j \rightarrow a_j, j \neq 1 \\ b_1 \rightarrow b_1^{-1}a_1b_1, b_j \rightarrow b_j, j \neq 1 \end{pmatrix} \\ \alpha_1 &: \begin{pmatrix} a_i \rightarrow a_{i+1}, 1 \leq i \leq g-1, a_g \rightarrow a_1 \\ b_i \rightarrow b_{i+1}, 1 \leq i \leq g-1, b_g \rightarrow b_1 \end{pmatrix} \\ \alpha_2 &: \begin{pmatrix} a_i \rightarrow a_i, 1 \leq i \leq g \\ b_1 \rightarrow b_1^{-1}, b_j \rightarrow b_j, 2 \leq j \leq g \end{pmatrix} \\ \alpha_3 &: \begin{pmatrix} a_2 \rightarrow b_2a_2(b_1^{-1}a_1b_1)(a_2^{-1}b_2^{-1}a_2) \\ a_j \rightarrow a_j, j \neq 2 \\ b_1 \rightarrow b_1(a_2^{-1}b_2^{-1}a_2) \\ b_j \rightarrow b_j, 2 \leq j \leq g \end{pmatrix} \end{aligned}$$

We define the groups $MC^-(X)$ and $MC^+(X)$ as the images of L^- and L^+ under the canonical homomorphism $Aut(\pi_1^{top}(X)) \rightarrow MC(X)$. The generators for $MC^-(X)$ and $MC^+(X)$ are also introduced by Suzuki in 1977. The automorphisms α_1, α_2 , and α_3 generate $MC^-(X)$. The generators of $MC^+(X)$ are the following. Here s_i denotes the word $b_i^{-1}a_i^{-1}b_ia_i$ for $1 \leq i \leq g$.

$$\begin{aligned} \alpha_4 &: \begin{pmatrix} a_1 \rightarrow b_1^{-1}a_1^{-1}b_1, a_j \rightarrow a_j, 2 \leq j \leq g \\ b_1 \rightarrow b_1^{-1}s_1^{-1}, b_j \rightarrow b_j, 2 \leq j \leq g \end{pmatrix} \\ \alpha_5 &: \begin{pmatrix} a_1 \rightarrow s_1^{-1}a_2s_1, a_2 \rightarrow a_1, a_j \rightarrow a_j, 3 \leq j \leq g \\ b_1 \rightarrow s_1^{-1}b_2s_1, b_2 \rightarrow b_1, b_j \rightarrow b_j, 3 \leq j \leq g \end{pmatrix} \\ \alpha_6 &: \begin{pmatrix} a_i \rightarrow a_i, 1 \leq i \leq g \\ b_1 \rightarrow a_1b_1a_2^{-1}s_2(b_1^{-1}a_1^{-1}b_1) \\ b_2 \rightarrow b_2a_2(b_1^{-1}a_1^{-1}b_1)a_2^{-1} \\ b_j \rightarrow b_j, 3 \leq j \leq g \end{pmatrix} \end{aligned}$$

By studying the mapping class group, we have considered generators of $Out(\pi_1^{top}(X))$. In order to understand the algebraic geometric analogue $Out(\pi_1^{alg}(X))$ we shall study $Aut(\pi_1^{alg}(X))$ in more detail.

It is well known that, for a profinite group G which admits a fundamental system of open neighborhoods of the identity consisting of characteristic subgroups, there exists a topological isomorphism,

$$Aut(G) \cong \varinjlim Aut(G/U)$$

where U runs over open characteristic subgroups of G . We have an injection $\pi_1^{top}(X) \rightarrow \pi_1^{alg}(X)$. An element of $Aut(\pi_1^{top}(X))$ fixes every characteristic open

subgroup U and induces a compatible system of elements in $Aut(A/U)$ for different U and therefore an element of $\lim Aut(G/U)$. We have constructed an injection

$$Aut(\pi_1^{top}(X)) \longrightarrow Aut(\pi_1^{alg}(X)).$$

Inner automorphisms of $\pi_1^{top}(X)$ induce inner automorphisms of the completion $\pi_1^{alg}(X)$. Thus we get a second injection,

$$Out(\pi_1^{top}(X)) \longrightarrow Out(\pi_1^{alg}(X)).$$

If we prove that $Aut(\pi_1^{alg}(X))$ is the profinite completion of $Aut(\pi_1^{top}(X))$, we have shown that $Out(\pi_1^{alg}(X))$ is the completion in the profinite topology of $Out(\pi_1^{top}(X)) \cong MC(X)$. It is enough to show that $\pi_1^{alg}(X)$ has a fundamental system of open characteristic subgroups which are completions of open subgroups of $\pi_1^{top}(X)$ in the profinite topology. We know that every automorphism of $\pi_1^{top}(X)$ is induced by an automorphism of $F(a_1, \dots, a_g, b_1, \dots, b_g)$. So it is enough to show that every free group has a fundamental system of open characteristic subgroups. But this is proved to be true for a finitely generated free group. Therefore we have a representation of Galois group landing on the profinite completion of $MC(X)$

$$\rho : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow Out(\pi_1^{alg}(X)) \cong \widehat{MC(X)}.$$

6 The geometric structure of Galois action

Uchida in 1976 [Uc] and Ikeda in 1977 [Ik] proved that every automorphism of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ is inner. Therefore the equivalence class of Galois representations

$$\rho : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow Out(\pi_1^{alg}(X))$$

has only one element. This is unlike the case of p -adic Galois representations associated to elliptic curves.

Translating the Galois representation from the language of automorphisms to the language of mapping class group, provides us a chance to geometrically define invariants of the Galois representation. For example, one shall be able to give a purely geometric definition of the conductor of a Galois representation.

On the other hand, there are particular arithmetic invariants like the weight of a modular form, which show off in the p -adic Galois representation associated to a modular form. In the geometric picture we have introduced, one shall be able to give purely geometric interpretation of these arithmetic invariants.

7 Serre conjecture after Wiles

Associating Galois representations to modular forms raised the question that if a given Galois representations is modular. Failure of characterization of these representations, led Serre in 1987 [Se] to the following conjecture

Conjecture 7.1 (Serre) *Every 2-dimensional odd irreducible mod p Galois representation is modular.*

In fact, Serre conjecture was stated much more detailed than the above formulation. But the above version is the part which has remained unproven today. Serre predicted explicit weight and level for the modular form which is related to the given mod p Galois representation.

In search for the right modular form, assuming the above conjecture, it was proved that

Theorem 7.2 *Let $l \geq 3$ be a prime not dividing the conductor of ρ mod p and let ρ come from $\Gamma_1(M)$ where $M = Nl^\alpha$ and $(N, l) = 1$, then ρ can be induced from a modular form on $\Gamma_1(N)$.*

The above result is the fruit of the efforts of many mathematicians. In particular, one of the steps was due to Ribet who proved [Ri]

Theorem 7.3 (Ribet) *If ρ comes from a modular form on $\Gamma_1(N)$ and of weight k where $N > 3$, for a prime $l > 3$ for which $2 \leq k \leq l + 1$ one can induce ρ from a modular form of weight 2 on $\Gamma_1(Nl)$.*

On the other hand Shimura associated an elliptic curve defined over some number field to such a modular form. So, if Serre conjecture is true, ρ comes from the p -adic Galois representation associated to an elliptic curve.

Now suppose we have found an elliptic curve E over \mathbb{Q} whose Galois representation induces ρ . Then by the Shimura-Taniyama-Weil conjecture which was proved by Wiles [Wi] followed by Taylor, Diamond, and Conrad, E is modular. If ρ is a representation to $GL_2(\mathbb{F}_p)$ we could hope that ρ comes from an elliptic curve over \mathbb{Q} . If this be true we have found an equivalent formulation of Serre's conjecture, namely

Conjecture 7.4 *Every 2-dimensional odd irreducible mod p Galois representation to $GL_2(\mathbb{F}_p)$ comes from an elliptic curve over \mathbb{Q} .*

Nevertheless the above conjecture is interesting in its own right. This conjecture claims that every mod p representation is geometric.

Going back to the case of a smooth curve X of genus ≥ 2 , we could associate a mod p Galois representation by mod p reduction of

$$\rho_p : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow Sp_{2g}(\mathbb{Z}_p),$$

or by inducing a representation on $\pi_1^{alg}(X)/\Phi(\pi_1^{alg}(X))$

$$\rho : Gal(\overline{\mathbb{Q}}/\mathbb{Q}) \rightarrow Aut(\pi_1^{alg}(X)/\Phi(\pi_1^{alg}(X))),$$

where $\Phi(\pi_1^{alg}(X))$ denotes the Frattini subgroup of $\pi_1^{alg}(X)$ and taking the mod p component of this representation. We could also associate a mod p representation

by considering the Jacobian variety of X and considering the action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the p -adic Tate module. One can ask again if every representation like this comes from a curve X defined over \mathbb{Q} . More precisely

Conjecture 7.5 *Every odd irreducible mod p Galois representation to $Sp_{2g}(\mathbb{F}_p)$ comes from a smooth curve defined over \mathbb{Q} .*

We shall mention that, not every mod p representation associated to a curve of genus ≥ 2 is irreducible. For example, $\text{Jac}(X)$ could be isogenous to a direct sum of elliptic curves. But if we do not assume irreducibility, ρ could be a sum of characters, which evidently is not induced from a smooth curve.

References

- [Co-Zi] D. J. Collins, H. Zieschang; *Combinatorial Group Theory and Fundamental Groups* In Algebra VII by A.N. Parshin and I.R. Shafarevich; Encyclopedia of mathematical sciences, Vol. 58, Springer-Verlag 1993.
- [De] M. Dehn; *Die Gruppen der Abbildungsklassen*, Acta Math. 69, 135-206. Zbl.19,253.
- [Gr] A. Grothendieck; *SGA1*, Lecture notes in math. 224, Springer-Verlag 1971.
- [Ha-Th] Hatcher A., Thurston W.; *A presentation for the mapping class group of a closed orientable surface*, Topology 19, 221-237. Zbl.447.57005.
- [He] Hempel J.; *3-manifolds*, Ann. Math. Stud. 86. Zbl.345.57001.
- [Ik] Ikeda M.; *Completeness of the absolute Galois group of the rational number field*, J. Reine Angew. Math., 291, 1-22.
- [Ku-Gr] Kurchanov P.F., Grigorchuck R.I.; *Classification of epimorphisms from the fundamental groups of a surface to free groups*, Int. Conf. in memory of A.I. Mal'tsev, Novosibirsk 1989, p. 72 (Russian). Zbl.725.20001.
- [Li] Likorish W.B.R.; *On the homeomorphisms of a non-orientable surface*, Proc. Cam. Philos. Soc. 61, 61-65. Zbl.131.208.
- [Mo] Mochizuki S.; Preprint.
- [Ni] Nielsen J.; *Om flytningsgrupper i den hyperbolske plan*, Mat. Tidsskrift B. 65-75, Jbuch 53, 543.
- [Ri-Za] Ribes L., Zalesski P.A.; *Profinite Groups* Springer 2000.
- [Ri] K. Ribet; *Report on mod- p Galois representations* in Motives, Vol. 1. Editors J.P. Serre et al.

- [Se] J. P. Serre; *Sur les representations modulaires de degre 2 de $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$* , Duke Math. J. 54 (1987), 179-230.
- [Su] S. Suzuki; *On homeomorphisms of a 3-dimensional handle body*, Can. J. Math. 29, No.1, 111-124. Zbl.339-57001.
- [Uc] K. Uchida; *Isomorphisms of Galois groups*, J. Math. Soc. Japan, 28, 617-620.
- [Wa] B. Wajnryb; *A simple presentation for the mapping class group of an orientable surface*, Isr. J. Math. 45, No. 2, 3, 157-174. Zbl.533.57022.
- [Wi] A. Wiles; *Modular elliptic curves and Fermat's last theorem*, Ann. Math. Studies 141 (1995), 443-551.
- [Zi] H. Zieschang; *Alternierende Produkt in freien Gruppen*, Abh. Math. Semin. Univ. Ham. 27, 13-31. Zbl.135,418.