

Applications of an Estimate for the First Eigenvalue of Δ on Riemannian Manifolds

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Abstract

Let $f : M \rightarrow \bar{M}$ be an isometric immersion between complete Riemannian manifolds. We prove a rigidity type theorem for spheres in \bar{M} under some conditions on the curvatures of manifolds and external diameter of the image of f . In particular, we show that spheres are the only complete hypersurfaces of the Euclidean spaces which are inside a ball of radius $R > 0$ such that the norm of their mean curvature is $\leq 1/R$ and their Ricci curvature is non-negative. Moreover, as a second application, we give a partial affirmative answer to this conjecture that the Gauss image of a complete, oriented and non-compact hypersurface of Euclidean space of non-zero constant mean curvature cannot lie totally inside an open hemisphere. The statement of the main theorem of this paper, Theorem 1.5 (see also Theorem 1.6), is based on the correction of the main result of the paper of Coghlan-Itokawa-Kosecki [CIK]. The proofs are based on an estimate for the first eigenvalue of the Laplacian on Riemannian manifolds.

Introduction

In this paper, we extend the following rigidity type theorem of Markvorsen [M1] to the complete Riemannian manifold M which is not necessarily compact:

- Suppose that $f : M^n \rightarrow \bar{M}^{n+1}$ is an isometric immersion from a connected compact manifold M into a complete manifold \bar{M}^{n+1} whose sectional curvature is bounded from above by some constant b . Suppose that the image of f is contained in a closed normal ball B_r of radius r with $r < \pi/2\sqrt{b}$ if $b > 0$. Suppose that the maximum of the absolute value of the mean curvature of $f(M)$ is equal to $m_b(r)$ where

$$m_b(r) := \begin{cases} \sqrt{b} \cot(r \sqrt{b}) & \text{if } b > 0, \\ \frac{1}{r} & \text{if } b = 0, \\ \sqrt{-b} \coth(r \sqrt{-b}) & \text{if } b < 0. \end{cases}$$

Then M is equal to the boundary of B_r .

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In fact, this paper is based on the paper of Coghlan-Itokawa-Kosecki [CIK] whose main theorem is not correct. The proofs are based on a lower bound for the first eigenvalue of the Dirichlet problem on Riemannian manifolds in terms of non-positive C^2 -functions which satisfy a partial differential inequality. That is a generalization of a result which is due to Barta, see [Ch, p. 70] and also [CY, Cor. 1]. Moreover, the main theorem of this paper, Theorem 1.5, can be interpreted as a generalization of Hopf's theorem to complete Riemannian manifolds which are not necessarily compact. This paper is a continuation of the author's work in [R]. See also [JX].

Moreover, as a second application, we prove the following conjecture under an extra condition on the first eigenvalue of Laplacian (see [HOS] and [L]):

- The Gauss image of a complete, oriented and non-compact hypersurface of a Euclidean space of non-zero constant mean curvature cannot lie totally inside an open hemisphere.

Also, our result generalizes and sharpens the result of [L, Thm. 3], see Corollary 3.3.

1 Lower Bound for the First Eigenvalue

In this section, we find a lower bound for the first eigenvalue of the Dirichlet problem on Riemannian manifolds.

Let N be a Riemannian manifold (of class C^3) and let $\langle \cdot, \cdot \rangle$ denote the Riemannian metric on N . We denote the associated covariant derivative of N by D . For $p \in N$, we denote the distance from p to x by $r(x) = r_p(x)$. The function $r_p(x)$ is smooth on $N \setminus (\{p\} \cup C_p)$, where C_p denotes the cut locus of p . Also, we denote the Hessian of $r(x)$ by $Hess(r)(v, w) := \langle D_v^{\nabla r}, w \rangle$, for all vectors v and w in the tangent bundle of N . We denote the closed ball with center at $q \in N$ and radius $R > 0$ by $B(q, R)$.

Lemma 1.1. *Let M be a complete Riemannian manifold and let Ω be a bounded domain with smooth boundary in M . Let v be a non-positive C^2 -function which is defined on an open neighborhood of $\overline{\Omega}$ and*

$$\Delta v \geq -\delta v \geq 0,$$

where δ is a non-negative number. Let u be a non-negative C^2 -function which is defined on the domain of v and

$$\begin{cases} \Delta u \geq -\lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where λ is a non-negative number. Suppose that $\int_{\Omega} uv \neq 0$, then we have $\delta \leq \lambda$.

Proof. By Green's theorem, we have

$$\int_{\Omega} (u \Delta v - v \Delta u) = \int_{\partial\Omega} u \frac{\partial v}{\partial n} - v \frac{\partial u}{\partial n},$$

where $\frac{\partial u}{\partial n}$ denotes the exterior normal derivative of u on $\partial\Omega$. Since $u \geq 0$ in Ω and $u = 0$ on $\partial\Omega$, we have $\frac{\partial u}{\partial n} \leq 0$. Then we obtain

$$\int_{\Omega} (u \Delta v - v \Delta u) \leq 0,$$

or

$$\int_{\Omega} u \Delta v \leq \int_{\Omega} v \Delta u.$$

Therefore, we have

$$-\delta \int_{\Omega} uv \leq \int_{\Omega} u \Delta v \leq \int_{\Omega} v \Delta u \leq -\lambda \int_{\Omega} uv.$$

This completes the proof of the lemma. \square

Definition 1.2. Let M be a complete Riemannian Manifold. If M is compact, we define $\lambda_1(M) := 0$. If M is non-compact, we define

$$\lambda_1(M) := \lim_{R \rightarrow \infty} \lambda_1(B(a, R)),$$

where $\lambda_1(B(a, R))$ denotes the first eigenvalue of the Dirichlet problem

$$\begin{cases} \Delta h = -\lambda h, & \text{in } B(a, R), \\ h = 0, & \text{on } \partial B(a, R), \end{cases}$$

where $B(a, R)$ denotes the open ball with radius R and center at a . When $\partial B(a, R)$ is not smooth, it is possible to define $\lambda_1(B(a, R))$ as well, see [Ch, p. 21].

Next corollary generalizes (half of) a result which is due to Barta [Ch, p. 70]. See also [CY, Cor. 1] and [FS, Thm. 1].

Corollary 1.3. *Let M be a complete Riemannian manifold. Let v be a non-positive C^2 -function which is defined on the closed ball $\overline{B}(a, r) \subset M$ with smooth boundary and*

$$\Delta v \geq -\delta v \geq 0,$$

where δ is a non-negative number. Let $\lambda_1 := \lambda_1(B(a, r))$ denote the first eigenvalue of the Dirichlet problem on the ball $B(a, r)$ (see Definition 1.2). Suppose that v is not identically zero on $B(a, r)$, then $\delta \leq \lambda_1$.

Proof. By basic properties of the first eigenvalue (see for instance [Ch]), we know that there is a non-negative C^2 -function u which is defined on $\overline{B}(a, r)$ and

$$\begin{cases} \Delta u = -\lambda_1 u, & \text{in } B(a, r), \\ u = 0, & \text{on } \partial B(a, r), \end{cases}$$

By Lemma 1.1, we have $\delta \leq \lambda_1$. This completes the proof of the corollary. \square

Next corollary generalizes Lemma 1 of [P] and [Ch, p. 53].

Corollary 1.4. *Let M be a complete Riemannian manifold and let Ω be a bounded domain with smooth boundary in M . Let w be a C^2 -function which is defined on an open neighborhood of $\overline{\Omega}$ and $\Delta w \geq 1$. Let u be a non-positive C^2 -function which is defined on an open neighborhood of $\overline{\Omega}$ and*

$$\begin{cases} \Delta u \geq -\lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

where λ is a non-negative number. Then we have

$$\frac{1}{\max_{\overline{\Omega}} w - \min_{\overline{\Omega}} w} \leq \lambda.$$

Proof. Define the function v (on the domain of w) as

$$v(x) := w(x) - \max_{\overline{\Omega}} w.$$

Then we have

$$\Delta v \geq 1 = \frac{1}{v} v \geq - \left(\frac{1}{\max_{\overline{\Omega}} w - \min_{\overline{\Omega}} w} \right) v.$$

Therefore, by Lemma 1.1, we have

$$\frac{1}{\max_{\overline{\Omega}} w - \min_{\overline{\Omega}} w} \leq \lambda.$$

This completes the proof of the corollary. \square

Now, we state and prove a theorem on complete Riemannian manifolds which can be viewed as a generalization of this fact that on every compact manifold every subharmonic function is constant (Hopf's theorem). Also, it can be interpreted as a generalization of Theorem 2 of [FS]. See also [F].

Theorem 1.5. *Let M be a complete and connected Riemannian manifold and let v be a non-positive C^2 -function on M . Suppose that*

$$\Delta v \geq -\delta v \geq 0,$$

where $\delta > \lambda_1(M)$ is a non-negative number. Then v is constant (zero).

Proof. If M is compact, the theorem follows from the (strong) maximum principle. So we can assume that M is not compact.

By contradiction, suppose that there is $x_0 \in M$ such that $v(x_0) \neq 0$. Then, by Corollary 1.3, we have $\delta \leq \lambda_1(M)$. This is a contradiction. This completes the proof of the theorem. \square

The following theorem is stimulated from the false result of [CIK], Theorem 1, page 196. The counter-example can be obtained by using [CY], Theorem 7, page 351. In fact, for any complete (non-compact) manifold M , there exists a negative function f , defined on M , such that $\Delta f = -\lambda_1(M) f$.

Theorem 1.6. *Let M be an n -dimensional complete Riemannian manifold whose Ricci curvature is bounded from below by some constant k . Suppose that a non-positive function v is defined on an open neighborhood of the closure of $B(x, r) \subset M$, for some $r > 0$, and*

$$\Delta v \geq -\delta v \geq 0,$$

where $\delta > \max\left\{0, -k \frac{(n-1)^2}{4}\right\}$ is a positive constant. Then there is a positive number R , depending on k , n and δ such that either v is identically zero on $B(x, r)$ or $r \leq R$.

Proof. Without loss of generality, we can assume that v is not identically zero on $B(x, r)$. We define $R_{max} \geq r$ as the supremum of all numbers s such that the function v with the properties of the theorem can be defined on $B(x, s)$.

(i) Let $k > 0$. By the Bonnet-Myers theorem [Ch, p. 73], we know that M is compact and the diameter of M is bounded from above by $\pi k^{-1/2}$. So, by the maximum principle, v is identically zero, if $r > \pi k^{-1/2}$. Therefore, we can choose $R := \pi k^{-1/2}$.

(ii) Let $k = 0$. By the Laplacian comparison theorem (see [Chg] or [Ch, p. 74, Thm. 7]), we have ($t > 0$)

$$\lambda_1(B(x, t)) \leq \lambda_1(\mathbf{B}(t)) = \frac{c_n}{t^n},$$

where $\mathbf{B}(t)$ denotes a ball of radius t in the n -dimensional Euclidean space, and c_n is a constant which depends on n . Then, by Corollary 1.3, we have

$$\delta \leq \lambda_1(B(x, s)) \leq \lambda_1(\mathbf{B}(s)) = \frac{c_n}{s^n},$$

for all $0 < s < R_{max}$. So we can choose $R := \left(\frac{c_n}{\delta}\right)^{1/n} + 1$.

(iii) Now, let $k < 0$. By the Laplacian comparison theorem, we have ($t > 0$)

$$\lambda_1(B(x, t)) \leq \lambda_1(\mathbf{B}(t)),$$

where $\mathbf{B}(t)$ is a ball of radius t in the space form of constant curvature k . By a result which is due to McKean (see [Mc] or [Ch, p. 46, Thm. 5]), we know that

$$\lambda_1(\mathbf{B}(t)) \geq -k \frac{(n-1)^2}{4},$$

for $t > 0$, and

$$\lim_{t \rightarrow \infty} \lambda_1(\mathbf{B}(t)) = -k \frac{(n-1)^2}{4}.$$

On the other hand, by Corollary 1.3, we know that $\delta \leq \lambda_1(\mathbf{B}(s))$, for all $0 < s < R_{max}$. Therefore, we have

$$-k \frac{(n-1)^2}{4} < \delta \leq \lambda_1(B(x, s)) \leq \lambda_1(\mathbf{B}(s)),$$

for all $0 < s < R_{max}$, and

$$\lim_{t \rightarrow \infty} \lambda_1(\mathbf{B}(t)) = -k \frac{(n-1)^2}{4}.$$

This completes the proof of the theorem. \square

Corollary 1.7. *Let M be an n -dimensional complete Riemannian manifold whose Ricci curvature is non-negative (or $\lambda_1(M) = 0$). Suppose that a non-positive function v is defined on an open neighborhood of the closure of $B(x, r) \subset M$, for some $r > 0$, and*

$$\Delta v \geq -\delta v \geq 0,$$

where δ is a positive constant. Then there is a positive number R , depending on n and δ such that either v is identically zero on $B(x, r)$ or $r \leq R$.

Proof. It is an immediate consequence of Theorem 1.6. \square

2 A Rigidity Type Theorem

In this section, we extend the rigidity theorem of Markvorsen [M1] to the complete (not necessarily compact) manifolds. See also [R, Question 5.8]. The following lemma is stimulated from [M2, Lemma 2] and [R, Thm. 2.1].

Lemma 2.1. *Let $f : M^n \rightarrow \overline{M}^{n+k}$ be an isometric C^2 -immersion between Riemannian manifolds. Suppose that the image of f is contained in $B(p, R) \setminus (C_p \cup \{p\})$, for some $p \in \overline{M}$ and $R > 0$. Suppose that the Hessian of the distance function on \overline{M} , $r(y) = r_p(y) = d(y, p)$, is bounded from below by $m(r) \geq 0$ on the tangent bundle of $\partial B(p, r)$, i.e. $\text{Hess}(r)(v, v) \geq m(r) \|v\|^2$ for all vectors v in the tangent bundle of $\partial B(p, r)$. Let $\Phi : [0, R] \rightarrow \mathbf{R}$ be a C^2 -function. Then we have*

$$\begin{aligned} \Delta(\Phi(r \circ f))(x) &\geq \left(\sum_{i=1}^n [\Phi''(r \circ f) - \Phi'(r \circ f) m(r \circ f)] |\langle \nabla r, e_i \rangle|^2 \right) \\ &\quad + n \Phi'(r \circ f)(x) [m(r \circ f)(x) - \langle \nabla r, H(f(x)) \rangle], \end{aligned}$$

where $\{e_1, e_2, \dots, e_n\}$ is an orthogonal basis for $T_x M$ and H denotes the mean curvature vector of M in \overline{M} . In particular, if $\Phi''(r) \geq \Phi'(r) m(r)$, we have

$$\Delta(\Phi(r \circ f)) \geq n \Phi'(r \circ f)[m(r \circ f) - \langle \nabla r, H(f(x)) \rangle].$$

Proof. Let γ be a geodesic in M which is parametrized by arc-length and $\gamma(0) = x \in M$ and $\gamma'(0) = e \in T_x M$. By abuse of notation, we denote $f \circ \gamma$ by γ (note that f is an isometric immersion). Define

$$h(s) := r(\gamma(s)) = (r \circ f)(\gamma(s)).$$

Then we have

$$h'(s) = \langle \nabla r, \gamma'(s) \rangle,$$

$$h''(s) = \langle \overline{D}_{\gamma'(s)}^{\nabla r}, \gamma'(s) \rangle + \langle \nabla r, \overline{D}_{\gamma'(s)}^{\gamma'(s)} \rangle,$$

$$h''(s) = \langle \overline{D}_{\gamma'_T(s)}^{\nabla r}, \gamma'_T(s) \rangle + \langle \nabla r, \overline{D}_{\gamma'(s)}^{\gamma'(s)} \rangle,$$

where \overline{D} denotes the covariant derivative of \overline{M} and $\gamma'_T(s)$ denotes the projection of $\gamma'(s)$ on the tangent bundle of $\partial B(p, r(\gamma(s)))$. So, we have

$$\begin{aligned} h''(s) &\geq \left[1 - |\langle \nabla r, \gamma'(s) \rangle|^2 \right] m(r \circ \gamma) - \langle \nabla r, \overline{D}_{\gamma'(s)}^{\gamma'(s)} \rangle \\ &\geq \left[1 - |\langle \nabla r, \gamma'(s) \rangle|^2 \right] m(r \circ \gamma) - \langle \nabla r, \overline{D}_{\gamma'(s)}^{\gamma'(s)} - D_{\gamma'(s)}^{\gamma'(s)} \rangle, \end{aligned} \quad (*)$$

where D denotes the covariant derivative of M and since γ is a geodesic in M , we know that $D_{\gamma'(s)}^{\gamma'(s)} = 0$ (see the proof of Thm. 2.1 in [R]). Now, let $\{e_1, e_2, \dots, e_n\}$ be an orthogonal basis for $T_x M$. Let γ_i be a geodesic in M which is parametrized by arc-length and $\gamma_i(0) = x \in M$ and $\gamma'_i(0) = e_i \in T_x M$, for $i = 1, 2, \dots, n$. Then we have

$$\begin{aligned} \Delta(\Phi(r \circ f))(x) &= \sum_{i=1}^n \langle D_{e_i}^{\nabla(\Phi(r \circ f))}, e_i \rangle \\ &= \sum_{i=1}^n \left[\frac{d^2}{ds^2} (\Phi(r \circ f))(\gamma_i(s)) \right]_{s=0} \\ &= \sum_{i=1}^n \left[\frac{d^2}{ds^2} (\Phi \circ h_i)(s) \right]_{s=0}, \end{aligned}$$

where $h_i(s) := r(\gamma_i(s))$. Then, by the chain rule, we obtain

$$\begin{aligned}
\Delta(\Phi(r \circ f))(x) &= \sum_{i=1}^n \frac{d}{ds} [\Phi'(h_i(s)) h_i'(s)] \\
&= \sum_{i=1}^n [\Phi''(h_i(s)) |h_i(s)|^2 + \Phi'(h_i(s)) h_i''(s)]_{s=0} \\
&= \sum_{i=1}^n [\Phi''(h_i(s)) |\langle \nabla r, \gamma_i'(s) \rangle|^2 + \Phi'(h_i(s)) h_i''(s)]_{s=0} \\
&= \sum_{i=1}^n [\Phi''(r \circ f)(x) |\langle \nabla r, \gamma_i'(0) \rangle|^2 + \Phi'(r \circ f)(x) h_i''(0)].
\end{aligned}$$

Then, by (*), we have

$$\begin{aligned}
\Delta(\Phi(r \circ f))(x) &\geq \sum_{i=1}^n \Phi''(r \circ f)(x) |\langle \nabla r, e_i \rangle|^2 + \Phi'(r \circ f)(x) \\
&\quad \cdot \left[m(r \circ f)(x) (1 - |\langle \nabla r, e_i \rangle|^2) - \langle \nabla r, \overline{D}_{e_i}^{e_i} - D_{e_i}^{e_i} \rangle \right] \\
&\geq \sum_{i=1}^n \left[\Phi''(r \circ f)(x) - \Phi'(r \circ f)(x) m(r \circ f)(x) \right] |\langle \nabla r, e_i \rangle|^2 \\
&\quad + \Phi'(r \circ f)(x) \left[m(r \circ f)(x) - \langle \nabla r, \overline{D}_{e_i}^{e_i} - D_{e_i}^{e_i} \rangle \right] \\
&\geq \left[\sum_{i=1}^n [\Phi''(r) - \Phi'(r) m(r)] |\langle \nabla r, e_i \rangle|^2 \right] \\
&\quad + \left[n \Phi'(r) [m(r) - |\langle \nabla r, H(f(x)) \rangle|] \right].
\end{aligned}$$

□

Proposition 2.2. *Let $f : M^n \rightarrow \overline{M}^{n+k}$ be an isometric C^2 -immersion between Riemannian manifolds. Suppose that the image of f is contained in $B(p, R) \setminus (C_p \cup \{p\})$, for some $p \in \overline{M}$ and $R > 0$. Suppose that the Hessian of the distance function on \overline{M} , $r(y) = r_p(y) = d(y, p)$, is bounded from below by $m(r) \geq 0$ on the tangent bundle of $\partial B(p, r)$, i.e. $\text{Hess}(r)(v, v) \geq m(r) \|v\|^2$ for all vectors v in the tangent bundle of $\partial B(p, r)$. Let $\Phi : [0, R] \rightarrow \mathbf{R}$ be a C^2 -function. Suppose that $\Phi''(r) - \Phi'(r) m(r) \geq 0$ and also we assume that*

$$n \Phi'(r) [m(r) - m(R)] \geq -\delta [\Phi(r) - \Phi(R)],$$

where δ is a positive number. Suppose that $\|H(f(x))\| \leq m(R)$, for all $x \in M$. Then we have

$$\Delta w \geq -\delta w,$$

where $w(x) := \Phi(r \circ f)(x) - \Phi(R)$. In particular, if Φ is non-increasing, we also have $w \leq 0$.

Proof. By Lemma 2.1, we have

$$\Delta(\Phi(r \circ f))(x) \geq n \Phi'(r \circ f)(x) \left[m(r \circ f)(x) - |\langle \nabla r, H(f(x)) \rangle| \right].$$

Then, by the assumptions, we have

$$\Delta(\Phi(r \circ f))(x) \geq -\delta [\Phi(r \circ f)(x) - \Phi(R)].$$

This completes the proof of the proposition. \square

Example 2.3. Let \overline{M} be a complete Riemannian manifold whose sectional curvature is bounded from above by some constant b . Then, by the Hessian comparison theorem [SY, p. 4], we can choose $m(r)$ as follows

$$m(r) := \begin{cases} \sqrt{b} \cot(r \sqrt{b}) & \text{if } b > 0, \\ \frac{1}{r} & \text{if } b = 0, \\ \sqrt{-b} \coth(r \sqrt{-b}) & \text{if } b < 0. \end{cases}$$

Also, it is easy to see that the function

$$\Phi(r) := \begin{cases} \frac{1 - \cos(r \sqrt{b})}{b} & \text{if } b > 0, \\ \frac{r^2}{2} & \text{if } b = 0, \\ \frac{1 - \cosh(r \sqrt{-b})}{b} & \text{if } b < 0, \end{cases}$$

and the number

$$\delta := \begin{cases} \frac{nb}{\sin^2(R \sqrt{b})} & \text{if } b > 0, \\ \frac{n}{R^2} & \text{if } b = 0, \\ \frac{-nb}{\sinh^2(R \sqrt{-b})} & \text{if } b < 0, \end{cases}$$

satisfy the assumptions of Proposition 2.2; the number R is less than or equal to $\frac{\pi}{2\sqrt{b}}$, if $b > 0$. See [M2] and [CIK, p. 206].

Theorem 2.4. *Let $f : M^n \rightarrow \overline{M}^{n+k}$ be an isometric C^2 -immersion between complete and connected Riemannian manifolds. Suppose that the image of f is contained in $B(p, R) \setminus (C_p \cup \{p\})$, for some $p \in \overline{M}$ and $R > 0$. Suppose that the Hessian of the distance function on \overline{M} , $r(y) = r_p(y) = d(y, p)$, is bounded from below by $m(r) \geq 0$ on the tangent bundle of $\partial B(p, r)$, i.e. $\text{Hess}(r)(v, v) \geq m(r) \|v\|^2$ for all vectors v in the tangent bundle of $\partial B(p, r)$. Let $\Phi : [0, R] \rightarrow \mathbf{R}$ be a C^2 -function. Suppose that*

$$\Phi''(r) - \Phi'(r) m(r) \geq 0$$

and also assume that

$$n \Phi'(r) [m(r) - m(R)] \geq -\delta [\Phi(r) - \Phi(R)].$$

Suppose that $\|H(f(x))\| \leq m(R)$, for all $x \in M$, and $m(r)$ is a non-increasing function of r . Moreover, assume that $\delta > \lambda_1(M)$. Then $f(M)$ is contained in the sphere $\partial B(p, R)$.

Proof. It is an immediate consequence of Theorem 1.5 and Proposition 2.2. \square

Theorem 2.5. *Let $f : M^n \rightarrow \overline{M}^{n+k}$ be an isometric C^2 -immersion between complete and connected Riemannian manifolds. Suppose that the image of f is contained in $B(p, R) \setminus (C_p \cup \{p\})$, for some $p \in \overline{M}$ and $R > 0$. Suppose that $\|H(f(x))\| \leq m(R)$, for all $x \in M$, where*

$$m(r) := \begin{cases} \sqrt{b} \cot(r \sqrt{b}) & \text{if } b > 0, \\ \frac{1}{r} & \text{if } b = 0, \\ \sqrt{-b} \coth(r \sqrt{-b}) & \text{if } b < 0. \end{cases}$$

Suppose that the sectional curvature of \overline{M} is bounded from above by some constant b and the Ricci curvature of M is bounded from below by some constant k . Suppose that $R \leq \frac{\pi}{2\sqrt{b}}$, if $b > 0$, and also

$$\max\{0, -k \frac{(n-1)^2}{4}\} < \begin{cases} \frac{nb}{\sin^2(R\sqrt{b})} & \text{if } b > 0, \\ \frac{n}{R^2} & \text{if } b = 0, \\ \frac{-nb}{\sinh^2(R\sqrt{-b})} & \text{if } b < 0. \end{cases}$$

Then $f(M)$ is contained in the sphere $\partial B(p, R)$.

Proof. By the Hessian comparison theorem [SY, p. 4], we know that the Hessian of the distance function on \overline{M} , $r(y) = r_p(y) = d(y, p)$, is bounded from below by $m(r)$ on the tangent bundle of $\partial B(p, r)$, i.e. $Hess(r)(v, v) \geq m(r) \|v\|^2$ for all vectors v in the tangent bundle of $\partial B(p, r)$, for $r \leq R \leq \frac{\pi}{2\sqrt{b}}$, if $b > 0$. Now, the theorem is an immediate consequence of Theorem 1.6, Theorem 2.4 and Example 2.3. \square

Remark 2.6. In Theorem 2.5, when M is not compact, the requirement

$$\max\{0, -k \frac{(n-1)^2}{4}\} < \begin{cases} \frac{nb}{\sin^2(R\sqrt{b})} & \text{if } b > 0, \\ \frac{n}{R^2} & \text{if } b = 0, \\ \frac{-nb}{\sinh^2(R\sqrt{-b})} & \text{if } b < 0, \end{cases}$$

or a lower bound on Ricci (scalar) curvature cannot be omitted. Indeed, there exists an example, which is due to Calabi [BZ, 28.2.7], of a complete minimal ($H \equiv 0$) surface which is contained in a ball of \mathbf{R}^4 .

Corollary 2.7. *Let $f : M^n \rightarrow \overline{M}^{n+k}$ be an isometric C^2 -immersion between complete connected Riemannian manifolds. Suppose that the image of f is contained in $B(p, R) \setminus (C_p \cup \{p\})$, for some $p \in \overline{M}$ and $R > 0$. Suppose that the Ricci curvature*

of M is non-negative and the sectional curvature of \overline{M} is bounded from above by some constant b . Suppose that $\|H(f(x))\| \leq m(R)$, for all $x \in M$, where

$$m(r) := \begin{cases} \sqrt{b} \cot(r \sqrt{b}) & \text{if } b > 0, \\ \frac{1}{r} & \text{if } b = 0, \\ \sqrt{-b} \coth(r \sqrt{-b}) & \text{if } b < 0. \end{cases}$$

Then $f(M)$ is contained in the sphere $\partial B(p, R)$.

Proof. It is an immediate consequence of Theorem 2.5. \square

3 Hypersurfaces of Constant Mean Curvature

Let M be a complete and oriented hypersurface of \mathbf{R}^{n+1} with unit normal vector ν . Suppose that M has constant (parallel) mean curvature, then we have (see for instance [HOS])

$$\Delta \nu = -\|d\nu\|^2 \nu, \quad (3.1)$$

where $\|d\nu\|^2$ is the square norm of the second fundamental form of M (in \mathbf{R}^{n+1}), i.e. $\|d\nu\|^2 := \sum_{i,j=1}^n |\overline{D}_{e_i}^{e_j} \nu|^2$, where $\langle \cdot, \cdot \rangle$ and \overline{D} denote the usual inner product and covariant derivative on \mathbf{R}^{n+1} and $\{e_1, e_2, \dots, e_n\}$ is an orthogonal basis for the tangent space of M .

Theorem 3.1. *Let M be a complete and oriented hypersurface of \mathbf{R}^{n+1} with unit normal vector ν . Suppose that M has constant (parallel) mean curvature. Then*

(i) *Suppose that the image of M under the Gauss map, $\nu(x)$, lies in some closed hemisphere. Then we have*

- *either there is a unit vector $e \in \mathbf{R}^{n+1}$ such that $\langle \nu(x), e \rangle = 0$, for all $x \in M$,*
- *or $\lambda_1(M) \geq \inf_{x \in M} \|d\nu(x)\|^2$.*

(ii) *Suppose that the image of M under the Gauss map, $\nu(x)$, lies in some open hemisphere. Then*

$$\lambda_1(M) \geq \inf_{x \in M} \|d\nu(x)\|^2.$$

Proof. By the assumptions, there is a unit vector $e \in \mathbf{R}^{n+1}$ such that

$$w(x) := \langle \nu(x), e \rangle \leq 0.$$

Then, by (3.1), we obtain

$$\Delta w(x) \geq - \left(\inf_{z \in M} \|d\nu(z)\|^2 \right) w(x).$$

Now, the theorem follows from Corollary 1.3. \square

The following corollary gives a partial affirmative answer to this conjecture that the Gauss image of a complete, oriented and non-compact hypersurface of a Euclidean space of non-zero constant mean curvature cannot lie totally inside an open hemisphere (see [L]).

Corollary 3.2. *Let M be a complete and oriented hypersurface of \mathbf{R}^{n+1} with unit normal vector ν . Suppose that M has constant mean curvature. Suppose that the Ricci curvature of M is non-negative (or $\lambda_1(M) = 0$). Then*

(i) *Suppose that the image of M under the Gauss map, $\nu(x)$, lies in some closed hemisphere. Then, either there is a unit vector $e \in \mathbf{R}^{n+1}$ such that $\langle \nu(x), e \rangle = 0$ for all $x \in M$, or M has zero mean curvature.*

(ii) *Suppose that the image of M under the Gauss map, $\nu(x)$, lies in some open hemisphere. Then M has zero mean curvature.*

The following corollary sharpens and generalizes Theorem 3 of [L].²

Corollary 3.3. *Let M be a complete and oriented hypersurface of \mathbf{R}^{n+1} with unit normal vector ν . Suppose that M has constant mean curvature H_0 . Suppose that $\lambda_1(M) < n|H_0|^2$. Suppose that the image of M under the Gauss map, $\nu(x)$, lies in some closed hemisphere. Then there is a unit vector $e \in \mathbf{R}^{n+1}$ such that $\langle \nu(x), e \rangle = 0$, for all $x \in M$.*

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²In fact, the proof of Theorem 3 of [L] is not correct as it stands. The author uses the wrong formula $\Delta u = Su$, while the correct formula is $\Delta u = -Su$. But if we replace u by $-u$ (or e to $-e$), we can obtain the desired result.

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Postscript. After this paper was written, the author learned that Zhou [Z] realized the mistakes of the papers [CIK] and [AC]. The counter-example of Zhou [Z, p. 51] does not cover dimension 2 and the domain of function is not the entire manifold. Moreover, the statement of Lemma 2.3, [Z, p. 54], is not correct. For correct statement (and proof) of Lemma 2.3, [Z, p. 54], see [B, Thm. 1]. See also [CZ].

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