

# Algebro-Geometric Approach to Abelian Integrals in Differential Equations

*Hossein Movasati*<sup>1</sup>

*For Siavash Shahshahani on the occasion of his 60th birthday.*

## Abstract

The aim of this paper is to introduce the theory of Abelian integrals for holomorphic foliations in a complex manifold of dimension two. We will show the importance of Picard-Lefschetz theory and the classification of relatively exact 1-forms in this theory. Finally we will calculate Melnikov functions under some generic conditions.

## 0 Introduction

Let us be given a differential equation

$$(1) \quad \frac{dy}{dx} = \frac{P(x, y)}{Q(x, y)}$$

in the real plane  $\mathbb{R}^2$ , where  $P$  and  $Q$  are two polynomials in  $x$  and  $y$ . Let  $H(P, Q)$  denote the number of limit cycles of the above differential equation and

$$H_n = \max\{H(P, Q) \mid \deg(P), \deg(Q) \leq n\}$$

The Hilbert sixteenth problem claims that  $H_n$  is a finite number. An equation (1) has a first integral or is called integrable if there are two polynomials  $F$  and  $G$  in  $\mathbb{R}^2$  such that  $\frac{F}{G}$  is constant on its solutions. In this case the equation (after reducing) has the form

$$\frac{dy}{dx} = -\frac{GF_x - FG_x}{GF_y - FG_y}$$

When  $G \equiv 1$  the equation is called Hamiltonian. Let  $\mathcal{S}'_n$  (resp.  $\mathcal{S}_n$ ) denote the space of equations (1) defined by real (resp. complex) polynomials  $P$  and  $Q$ ,  $\deg(P), \deg(Q) \leq n$ . Every element of  $\mathcal{S}'_n$  is parameterized by the coefficients of  $P$  and  $Q$ .

One of the first attempts to solve the mentioned problem was made by two Russian mathematicians I. G. Petrovskiĭ and E. M. Landis ([LP1],[LP2]). They

---

<sup>1</sup>Supported by IMPA-Brazil, IPM-Iran

Keywords: Holomorphic foliation - Picard-Lefschetz theory

Math. classification: 57R30 - 14D99 - 32G34

complexified the equation (1) and considered the equation in  $\mathbb{C}^2$  and tried to find another set of cycles  $\mathcal{C}(P, Q)$  in the solutions of (1) whose cardinality is not less than the number of limit cycles in the related real equation. Then they tried to calculate a uniform upper bound for the cardinality of  $\mathcal{C}(P, Q)$  (and hence  $H_n$ ) by perturbation of integrable equations. Let  $\{\delta_t\}_{t \in (\mathbb{R}, 0)}$  be a family of closed solutions of an integrable equation. They observed that the bifurcation of limit cycles from this family of closed cycles is related with the zeros of a certain Abelian integrals.

Although their work had errors, the idea of using Abelian integrals to count the number of limit cycles of a perturbed Hamiltonian equation has been one of the useful tools in approaching the Hilbert Sixteenth Problem. For this see the book [Ro] and its references. The aim of this text is to introduce an algebro-geometric approach to these Abelian integrals. Instead of  $\mathbb{C}^2$  we consider an arbitrary two dimensional compact complex manifolds and instead of Hamiltonian fibrations we consider the fibrations of meromorphic functions on  $M$ . In sections 1 and 2 holomorphic foliations and the Picard-Lefschetz theory of meromorphic functions on  $M$  are introduced. In section 3 we see how Abelian integrals appear in the formula of the first Melnikov function. In section 4 we will encounter with relatively exact 1-forms and finally in section 5 we will calculate higher order Melnikov functions under some generic conditions.

Initial pages of this text was written in IMPA-Brazil. I would like to inform my sincere thanks to C. Camacho, A. Lins Neto and P. Sad, my teachers in IMPA, for their support. The final version of this text was obtained in IPM-Iran. Here I express my special acknowledgment to S. Shahshahani.

## 1 Holomorphic Foliations

Let  $M$  be a compact complex manifold of dimension two. A holomorphic foliation in  $M$  with isolated singularities is given by a collection of holomorphic 1-forms  $\omega_\alpha$  defined on  $U_\alpha$ ,  $\alpha \in I$ , where  $\{U_\alpha\}_{\alpha \in I}$  is an open covering of  $M$ , and such that

$$(2) \quad \omega_\alpha = g_{\alpha\beta} \omega_\beta, \quad \alpha, \beta \in I$$

where  $g_{\alpha\beta}$  is a holomorphic without zero function in  $U_\alpha \cap U_\beta$ . Furthermore we assume that the set of points in which  $\omega_\alpha$  is zero has codimension greater than one (discrete set). In other words  $\omega_\alpha$  has not zero divisor.

Therefore for any foliation  $\mathcal{F}$  there is associated a line bundle  $L$  given by the transition functions  $\{g_{\alpha\beta}\}_{\alpha, \beta \in I} \in H^1(M, \mathcal{O}^*) = Pic(M)$ . We also say that the foliation  $\mathcal{F}$  is of degree  $L$ . The data (2) can be considered as a holomorphic section  $\omega \in H^0(M, \Omega^1 \otimes L)$  without zero divisor, where  $\Omega^1$  is the cotangent bundle of  $M$ . Since  $M$  is compact,  $H^0(M, \Omega^1 \otimes L)$  is a finite dimensional space (this is a simple corollary of Grauert direct image theorem). By  $\mathcal{F}(\omega)$  we mean that the foliation  $\mathcal{F}$  is given by the 1-form  $\omega \in H^0(M, \Omega^1 \otimes L)$ . It is easy to check that for any foliation  $\mathcal{F}$  there exists a unique line bundle  $L$  in  $M$  such that  $\mathcal{F}$  is given by a holomorphic

without zero divisor element of  $H^0(M, \Omega^1 \otimes L)$ . From now on we fix a line bundle  $L$  and assume that  $H^0(M, \Omega^1 \otimes L)$  has a holomorphic without zero divisor section. In this case the set of 1-forms  $\omega \in H^0(M, \Omega^1 \otimes L)$  without zero divisor is an open subset of  $H^0(M, \Omega^1 \otimes L)$ .

If  $\omega \in H^0(M, \Omega^1 \otimes L)$  has a zero divisor we use the following trick: Let  $\omega \in H^0(M, \Omega^1 \otimes L)$  be a holomorphic section with the zero divisor  $Z$ . Let  $L_Z$  be the line bundle associated to  $Z$  and  $s \in H^0(M, L_Z)$  be the holomorphic section with the zero divisor  $Z$ . Now  $\frac{\omega}{s}$  is a holomorphic without zero divisor section of  $H^0(M, \Omega \wedge L \wedge L_Z^{-1})$ , and so, we can substitute  $L$  for  $L \wedge L_Z^{-1}$ .

Two holomorphic without zero divisor sections  $\omega, \omega' \in H^0(M, \Omega^1 \otimes L)$  induce the same foliation if and only if  $\omega = c\omega'$ , where  $c$  is a constant. Therefore the space of foliations of degree  $L$ , namely  $\mathcal{F}(M, L)$ , form an open subset of the projective space

$$\text{Proj}(H^0(M, \Omega^1 \otimes L))$$

Many times we need the foliation  $\mathcal{F}$  to be given by a meromorphic 1-form in  $M$ . In these cases we assume that  $H^0(M, L) \neq 0$  and choose a non-zero section  $s$  of  $H^0(M, L)$ . Now the foliation  $\mathcal{F}(\omega), \omega \in H^0(M, \Omega^1 \otimes L)$  is given by the meromorphic 1-form  $\frac{\omega}{s}$  in  $M$ .

**Definition 1.1** The meromorphic section  $s \in H^0(M, L)$  is called the integrating factor of  $\mathcal{F}(\omega), \omega \in H^0(M, \Omega^1 \otimes L)$  if  $\frac{\omega}{s}$  is a closed meromorphic function in  $M$ .  $\mathcal{F}(\omega)$  is called integrable if there exists an integrating factor  $s \in H^0(M, L)$  such that  $\frac{\omega}{s} = df$ , where  $f$  is a meromorphic function on  $M$ . In this case  $f$  is constant on the leaves of  $\mathcal{F}$ . We also say that  $f$  is a first integral of  $\mathcal{F}$ .

Let  $\mathcal{F}$  be an integrable holomorphic foliation given by a without zero holomorphic 1-form  $\omega \in H^0(M, \Omega^1 \otimes L)$ . Let also  $f$  be a first integral of  $\mathcal{F}$ . The fibers of  $f$  have the same linear bundle which we denote it by  $L_A$ , where  $A$  is a generic fiber of  $f$ . Let  $A_1, A_2, \dots, A_s$  be irreducible components of the fibers of  $f$  in which  $f$  has multiplicities greater than one, namely  $n_1, n_2, \dots, n_s$ , respectively. It is easy to see that

$$L = L_{2A - \sum_1^s (n_i - 1)A_i}$$

## 2 Picard-Lefschetz Theory

This section is mainly based on the papers [La] and [Ho2]. For more information the reader is referred to that papers. Throughout the text when we do not write the coefficients used in the homology we mean homology with coefficients in  $\mathbb{Z}$ .

Let  $f$  be a meromorphic function on a complex manifold  $M$  of dimension two. The indeterminacy set  $\mathcal{R}$  of  $f$  contains the points of  $M$  in which  $f$  has the form  $\frac{0}{0}$ .  $\mathcal{R}$  is a discrete set and the following holomorphic function is well-defined:

$$f : M - \mathcal{R} \rightarrow \mathbb{P}^1$$

We use the following notations

$$L_K = f^{-1}(K), \quad M_K = \overline{L_K}, \quad K \subset \mathbb{P}^1$$

For any point  $c \in \mathbb{P}^1$ , by  $L_c$  and  $M_c$  we mean the sets  $L_{\{c\}}$  and  $M_{\{c\}}$ , respectively. Throughout the text by a compact  $f$ -fiber we mean  $M_t$  and by a  $f$ -fiber only we mean  $L_t$ . Performing a finite number of blow-ups (see [CaSa]) at the points of  $\mathcal{R}$  and using Ehresmann's Fibration Theorem (see [La]), we can see that there exists a finite subset  $C = \{c_1, c_2, \dots, c_r\}$  of  $\mathbb{P}^1$  such that  $f$  fibers  $M - \mathcal{R}$  locally trivially over  $B = \mathbb{P}^1 - C$ , i.e. for every point  $b \in B$  there is a neighborhood  $U$  of  $b$  and a  $C^\infty$ -diffeomorphism  $\phi : U \times f^{-1}(c) \rightarrow f^{-1}(U)$  such that  $f \circ \phi = \pi_1 =$  the first projection. We say that  $C$  is the set of critical values of  $f$ . The regularity of  $f$  along a fiber  $L_c$  does not imply that it is fiber bundle over a neighborhood of  $c$ . This situation happens when  $M_c$  has a tangency point with the divisor of the blow-up in which the leaves are separated from each other. In this case we say that the critical point related to the value  $c$  is in  $\mathcal{R}$ .

Now we are able to use the Picard-Lefschetz theory for understanding the topology of the fibers of  $f$ . Let  $\lambda$  be a path in  $B$  with the initial and end points  $b_0$  and  $b_1$ . There is an isotopy

$$H : L_{b_0} \times [0, 1] \rightarrow L_\lambda$$

such that for all  $x \in L_{b_0}$ ,  $t \in [0, 1]$ ,

$$H(x, 0) = x, \quad H(x, t) \in L_{\lambda(t)}$$

For every  $t \in [0, 1]$  the map  $h_t = H(\cdot, t)$  is a homeomorphism between  $L_{b_0}$  and  $L_{\lambda(t)}$ . The different choices of  $H$  and paths homotopic to  $\lambda$  in  $B$  would give the class of isotopic maps

$$\{h_\lambda : L_{b_0} \rightarrow L_{b_1}\}$$

where  $h_\lambda(\cdot) = H(\cdot, 1)$ . The class  $\{h_\lambda : L_{b_0} \rightarrow L_{b_1}\}$  defines the map

$$h_\lambda : H_1(L_{b_0}) \rightarrow H_1(L_{b_1})$$

For any regular value  $b$  of  $f$ , we can define

$$h : \pi_1(B, b) \times H_1(L_b) \rightarrow H_1(L_b)$$

$$h(\lambda, \cdot) = h_\lambda(\cdot)$$

$\pi = \pi_1(B, b)$  is called the monodromy group and its action  $h$  on  $H_1(L_b)$  is called the action of monodromy on the first homology group of  $L_b$ . We also say that  $H_1(L_c)$  is a  $\pi$ -module.

**Definition 2.1** Let  $K$  be a subset of  $B$  and  $b$  be a point in  $K \setminus C$ . Any relative 2-cycle of  $L_K$  modulo  $L_b$  is called a 2-thimble above  $(K, b)$  and its boundary in  $L_b$  is called a vanishing 1-cycle above  $K$ .

Suppose that  $f$  has a nondegenerate critical point at  $p_i \in M - \mathcal{R}$  and  $f(p_i) = c_i$ . Let  $\lambda_i$  be a path in  $B \cup \{c_i\}$  connecting a regular value  $b$  to  $c_i$ . This path passes through  $c_i$  only in its end point. The leaf  $L_b$  contains a cycle which vanishes above  $\lambda_i$  and is called the Lefschetz vanishing cycle. Roughly speaking, when the value of  $x$  varies from  $b$  to  $c_i$ , the Lefschetz vanishing cycle in  $L_x$  moves in the leaves and arrives to the leaf  $L_{c_i}$  at the point  $p_i$ . The locus of this movement is exactly the related 2-thimble.

Fix a point  $\infty \in \mathbb{P}^1$ , which may be a critical value. Let  $C' = \{c_1, c_2, \dots, c_r\}$  be the subset of  $\mathbb{C} = \mathbb{P}^1 - \{\infty\}$  containing critical values of  $f$ . Consider a system of  $r$  paths  $\lambda_1, \dots, \lambda_r$  starting from  $b$  and ending at  $c_1, c_2, \dots, c_r$ , respectively, and such that:

1. each path  $\lambda_i$  has no self intersection points;
2. two distinct paths  $\lambda_i$  and  $\lambda_j$  meet only at their common origin  $\lambda_i(0) = \lambda_j(0) = b$ .

This system of paths is called a distinguished system of paths. Let  $K$  be the union of these paths and small disks around  $c_i$ 's. The set of vanishing cycles above  $K$  in  $L_b$  is called a distinguished set of vanishing cycles related to the critical points  $c_1, c_2, \dots, c_r$ .

**Theorem 2.1** (Theorem 2.2.1 [Ho2]) *Suppose that  $H_1(M - M_\infty, \mathbb{Q}) = 0$ . Then a distinguished set of vanishing 1-cycles related to the critical points in the set  $C \setminus \{\infty\} = \{c_1, c_2, \dots, c_r\}$  generates  $H_1(L_b, \mathbb{Q})$ .*

Again note that in the above theorem  $\infty$  can be a critical value of  $f$ . If  $M_\infty$  is smooth and its homology class in  $H_1(M, \mathbb{Q})$  is not zero (for example if  $M_\infty$  is a smooth hyperplane section) then the condition  $H_1(M, \mathbb{Q}) = 0$  implies that  $H_1(M - M_\infty, \mathbb{Q}) = 0$ . The reason is as follows: By Leray (or Thom-Gysin) isomorphism (see [Ch] p. 537) we have  $H_2(M, M - M_\infty) \simeq H_0(M_\infty) \simeq \mathbb{Z}$ . Now we write the long exact sequence of the pair  $(M, M - M_\infty)$ :

$$\dots \rightarrow H_2(M - M_\infty) \xrightarrow{i} H_2(M) \rightarrow \mathbb{Z} \xrightarrow{j} H_1(M - M_\infty) \rightarrow 0$$

Therefore  $H_1(M - M_\infty) = \mathbb{Z}/n\mathbb{Z}$  or  $\mathbb{Z}$  for some natural number  $n$ . In the first case we have  $H_1(M - M_\infty, \mathbb{Q}) = 0$ . In the second case  $j$  is one to one and so  $i$  is surjective. But this means that the intersection of  $M_\infty$  with any 2-cycle in  $H_2(M)$  is zero. By Poincaré duality the class of  $M_\infty$  in  $H_2(M, \mathbb{Q})$  must be zero which is a contradiction.

**Definition 2.2** *The cycle  $\delta$  in a regular fiber  $L_b$  is called simple if the action of  $\pi$  on  $\delta$  generates  $H_1(L_b, \mathbb{Q})$ .*

Note that in the above definition we have considered the homology group with rational coefficients. Of course, not all cycles are simple. For instance if the meromorphic function in a local coordinate  $(x, y)$  around  $q \in \mathcal{R}$  has the form  $\frac{x}{y}$ , then the cycle around  $q$  in each leaf has this property that it is fixed under the action of monodromy, therefore it cannot be simple. In the next paragraph we are going to introduce some fibrations with simple cycles.

**Lefschetz pencil:** The hyperplanes of  $\mathbb{P}^n$  are points of the dual projective space  $\check{\mathbb{P}}^n$ . We use the following notation:

$$H_y \subset \mathbb{P}^n, \quad y \in \check{\mathbb{P}}^n$$

Let  $M$  be a closed irreducible smooth subvariety of  $\mathbb{P}^n$ . Its dual variety  $\check{M}$  consists of all points  $y$  in  $\check{\mathbb{P}}^n$  such that  $H_y$  is tangent to  $M$  at some point.  $\check{M}$  may have singularities. A Lefschetz pencil in  $\mathbb{P}^n$  consists of all hyperplanes which contain a fixed  $(n-2)$ -dimensional projective space  $A$ , which is called the axis of the pencil. We denote a pencil by

$$\{H_t\}_{t \in G}$$

or  $G$  itself, where  $G$  is a projective line in  $\check{\mathbb{P}}^n$ . The pencil  $\{H_t\}_{t \in G}$  is in general position with respect to  $M$  if  $G$  is in general position with respect to  $\check{M}$ . Sometimes we parameterize the line  $G \cong \mathbb{P}^1$  and consider the meromorphic function  $f$  on  $M$  induced by the pencil. The meromorphic function associated to a pencil in general position is called the generic Lefschetz meromorphic function and has the following properties:

1. In a local coordinate system  $(x, y)$  around an indeterminacy point  $q$  of  $f$ ,  $f$  has the form  $\frac{x}{y}$ ;
2.  $f$  has  $r = \text{deg}(\check{M})$  nondegenerate critical points  $p_1, \dots, p_r$  in  $M - A$  such that  $f(p_i) = c_i$ 's are distinct values in  $\mathbb{P}^1$ ;
3. For every critical value  $c_i$  of  $f$ , the hyperplane  $H_{c_i}$  has a unique tangency of order two with  $M$  which lies out of  $A$ . The other hyperplanes are transverse to  $M$  (see [La]).

**Theorem 2.2** *Suppose that  $H_1(M - M_\infty, \mathbb{Q}) = 0$  and  $H_1(M_\infty) \neq 0$ , where  $M_\infty$  is a regular fiber of a generic Lefschetz meromorphic function  $f$ . Then every Lefschetz vanishing cycle in a regular fiber of  $f$  is simple.*

Note that the regular compact fibers of  $f$  have the same topology and so  $H_1(M_\infty) \neq 0$  means that the genus of any regular compact fiber is greater than 0 and so the fibration is not rational.

Almost all the arguments to prove the above theorem exist in [La]. That article have worked with the topology of  $M_t$ 's, and not  $L_t$ 's. Therefore we have proved above theorem in [Ho2] Theorem 2.3.2.

Let  $\mathcal{F}$  be a foliation in  $\mathbb{P}^2$  with a first integral of the type  $\frac{F^p}{G^q}$ , where  $F$  and  $G$  are two relatively prime irreducible polynomials in an affine chart  $\mathbb{C}^2$  of  $\mathbb{P}^2$ ,  $\frac{\deg(F)}{\deg(G)} = \frac{q}{p}$  and  $\text{g.c.d.}(p, q) = 1$ . Assume that  $\mathcal{F}$  has the following generic properties:

1.  $\{F = 0\}$  and  $\{G = 0\}$  are smooth and intersect each other transversally;
2. The critical points of  $\frac{F^p}{G^q}$  in  $\mathbb{P}^2 \setminus (\{F = 0\} \cup \{G = 0\})$  are nondegenerate with distinct images.

Let  $c_1, c_2, \dots, c_r$  denote these critical values. In [Ho2] it is proved that these conditions are generic, i.e. there exists a dense open subset in the space of coefficients of  $F$  and  $G$  such that all polynomials  $F$  and  $G$  whose coefficients are chosen from this set satisfy conditions 1 and 2. The meromorphic function  $\frac{F^p}{G^q}$  is a fiber bundle over  $\mathbb{P}^1 - \{c_1, c_2, \dots, c_r, 0, \infty\}$ . It has multiplicity  $p$  along  $F = 0$  and  $q$  along  $G = 0$ .

**Theorem 2.3** ([Ho2]) *If  $\deg(F) + \deg(G) > 4$  then every Lefschetz vanishing cycle in a regular fiber of  $\frac{F^p}{G^q}$  is simple.*

The condition  $\deg(F) + \deg(G) > 4$  is equivalent to this fact that the genus of a generic compact fiber of  $\frac{F^p}{G^q}$  is not zero. If  $p = q = 1$  then  $\frac{F^p}{G^q}$  is a Lefschetz meromorphic function and the above theorem is a particular case of Theorem 2.2. In the case where  $G$  is a linear polynomial we have  $p = 1$  and  $q = \deg(F)$ . Considering  $G = 0$  as the line at infinity in  $\mathbb{P}^2$ , we have the Hamiltonian fibrations in  $\mathbb{C}^2$ . In this case the above theorem is proved by Ilyashenko [II] using a theorem of Zizcenko.

### 3 Deformation of Holomorphic Foliations and Abelian Integrals

Let  $\mathcal{F}$  be an integrable holomorphic foliation in  $\mathcal{F}(M, L)$  and  $\mathcal{F}_\epsilon(\omega_\epsilon) \in \mathcal{F}(M, L), \epsilon \in (\mathbb{C}, 0)$  a holomorphic deformation of  $\mathcal{F} = \mathcal{F}_0$ . The set  $H^0(M, \Omega^1 \otimes L)$  is a vector space and so we can write

$$\omega_\epsilon = \omega_0 + \epsilon\omega_1 + \epsilon^2\omega_2 + \dots, \quad \omega_i \in H^0(M, \Omega^1 \otimes L)$$

The 1-form  $\omega_1$  is called the tangent vector of the deformation.

Let  $\delta$  be a cycle in a leaf of  $\mathcal{F}$  and  $\Sigma \simeq (\mathbb{C}, 0)$  a holomorphic section to  $\mathcal{F}$  in a point  $p \in \delta$ . Let also  $s$  be an integrating factor of  $\omega_0$  whose zero divisor does not intersect  $\delta$  and

$$\frac{\omega_0}{s} = df$$

where  $f$  is a meromorphic function on  $M$ .

Throughout the text we assume that the transverse section  $\Sigma$  is parameterized by  $t = f|_\Sigma$ . Assume that the holonomy of  $\mathcal{F}$  along  $\delta$  is identity. Note that if  $f$  has multiplicity one along the leaf containing  $\delta$  then the holonomy is always identity. We can consider  $\mathcal{F}_\epsilon$ 's as a codimension two foliation  $\tilde{\mathcal{F}} = \{\mathcal{F}_\epsilon\}_{\epsilon \in (\mathbb{C}, 0)}$  in  $M \times (\mathbb{C}, 0)$

and  $\Sigma \times (\mathbb{C}, 0)$  as a transverse section to  $\bar{\mathcal{F}}$ . So we have the holonomy map defined by

$$H : \Sigma \times (\mathbb{C}, 0) \rightarrow \Sigma \times (\mathbb{C}, 0)$$

$$H(t, \epsilon) = (h_\epsilon(t), \epsilon)$$

$h_\epsilon(t)$  is a holomorphic function in  $\epsilon$  and  $t$  and is called the holonomy of  $\mathcal{F}_\epsilon$  along the path  $\delta$  (note that by hypothesis  $h_0(t) \equiv t$ ). We write

$$h_\epsilon(t) - t = M_1(t)\epsilon + M_2(t)\epsilon^2 + \cdots + M_i(t)\epsilon^i + \cdots, \quad M_i(t) = \left. \frac{\partial^i h_\epsilon}{\partial \epsilon^i} \right|_{\epsilon=0}$$

$M_i$  is called the  $i$ -th Melnikov function of the deformation along the path  $\delta$ . Let  $M_1 \equiv M_2 \equiv \cdots \equiv M_{k-1} \equiv 0$  and  $M_k \neq 0$ . It is a well known fact that the multiplicity of  $M_k$  at  $t = 0$  is the number of limit cycles which appear around  $\delta$  after the deformation. This fact shows the importance of these functions in the local study of Hilbert's 16-th problem. The following proposition gives us a nice formula for the first Melnikov function.

**Proposition 3.1** *The first Melnikov function is given by*

$$M_1(t) = - \int_{\delta_t} \frac{\omega_1}{s}$$

where  $\omega_1$  is the tangent vector of the deformation and  $\delta_t$  is the lifting up of  $\delta$  in the leaf through  $t \in \Sigma$ .

**Proof:** The proof is completely formal in the literature of differential equations (see [Ro] and [Fr]). The deformed foliation is given by the meromorphic 1-form

$$(3) \quad df + \epsilon \frac{\omega_1}{s} + O(\epsilon^2)$$

Let  $\delta_{t, h_\epsilon(t)}$  be a path in the leaf of  $\mathcal{F}_\epsilon$  through  $t$  which connects  $t$  to  $h_\epsilon(t)$  along the path  $\delta$ . Since  $\Sigma$  is parametrized by  $t = f|_\Sigma$ , by integrating the 1-form (3) over the path  $\delta_{t, h_\epsilon(t)}$  we have

$$h_\epsilon(t) - t + \epsilon \left( \int_{\delta_t} \frac{\omega_1}{s} + O(\epsilon) \right) + O(\epsilon^2) = 0$$

The coefficient of  $\epsilon$  in the above equality gives us the desired equality.  $\square$

We want to have an explicit formula for higher Melnikov functions. For this purpose we must classify a certain class of relatively exact 1-forms modulo an integrable foliation.

## 4 Relatively Exact 1-forms

First let us give the definition of a relatively exact 1-forms modulo a foliation.



**Definition 4.1** Let  $\mathcal{F}$  be a foliation in  $M$ . A meromorphic 1-form  $\omega_1$  on  $M$  is called relatively exact modulo  $\mathcal{F}$  if the restriction of  $\omega_1$  to each leaf  $\mathcal{L}$  of  $\mathcal{F}$  is exact, i.e. there is a meromorphic function  $f$  on  $\mathcal{L}$  such that  $\omega_1|_{\mathcal{L}} = df$ .

In fact we are interested in the above definition when  $\mathcal{F}$  is integrable. It is easy to check that a meromorphic 1-form  $\omega_1$  is relatively exact modulo  $\mathcal{F}$  if and only if

$$(4) \quad \int_{\delta} \omega_1 = 0$$

for all closed cycles in the leaves of  $\mathcal{F}$ , where this integral is well-defined.

Relatively exact 1-forms have been studied by many authors. Ilyashenko in [Il] proves that if the integral of a polynomial 1-form of degree  $n$  along a continuous family of level lines of a Morse type polynomial of two variables and of degree  $n$  is identically equal to zero, then the form is relatively exact modulo the polynomial and then he proves that it must be exact. This is generalized to higher dimensions in [Pu]. The classification of relatively exact polynomial 1-forms modulo an arbitrary polynomial is done by P. Bonnet in [Bo] and L. Gavrilov in [Ga]. J. Muciño in [Mu] has classified a certain class of relatively exact 1-forms modulo a Lefschetz pencil. In a generalization of Ilyashenko’s result to integrable foliations in  $M$ , I had to classify another types of relatively exact 1-forms in [Ho1].

Let  $S$  be a complex curve. The meromorphic function  $f : M \rightarrow S$  is called noncomposite if a general compact  $f$ -fiber is irreducible. It is easy to see that  $f : M \rightarrow S$  is noncomposite if and only if  $f$  cannot be factored as a composite

$$(5) \quad M \xrightarrow{f'} S' \xrightarrow{i} S$$

where  $S'$  is a complex curve and  $i$  is a holomorphic map of degree greater than one.

Let  $\mathcal{F}$  be an integrable foliation. There exists a complex curve  $S$  and a noncomposite meromorphic function  $f : M \rightarrow S$  such that  $f$  is constant in the leaves of  $\mathcal{F}$  (see [Go]). We say that  $f$  is a noncomposite first integral of  $\mathcal{F}$ .

Let  $f : M \rightarrow S$  be noncomposite as above. Any other first integral  $f' : M \rightarrow S'$  of  $\mathcal{F}$  is factored as

$$M \xrightarrow{f} S \xrightarrow{i} S'$$

where  $i$  is a holomorphic function.

Poincaré in his article [Po] has studied integrable foliations in  $\mathbb{P}^2$  and has proved (pp. 52-53) that every integrable foliation  $\mathcal{F}$  in  $\mathbb{P}^2$  has a noncomposite first integral  $M \rightarrow \mathbb{P}^1$  (this is also a consequence of Stein factorization theorem).

Let  $\mathcal{F}$  be an integrable foliation with the noncomposite first integral  $f : M \rightarrow S$  and  $\omega_1$  a meromorphic 1-form in  $M$  with the pole divisor

$$D = \sum_{i=1}^n n_i D_i$$

In what follows when we say that a meromorphic object (function, 1-form, ...)  $Z$  has the pole divisor  $D$ , we mean that

$$\text{pol}(Z) + D \geq 0$$

Let

$$D_0 = \sum_{i=1}^n D_i$$

be the reduced part of  $D$ .

**Theorem 4.1** *Keeping the notations used above, assume that*

1. *All  $f$ -fibers are connected (for us a  $f$ -fiber does not contain the indeterminacy points of  $f$ );*
2.  *$D$  is  $\mathcal{F}$ -invariant;*
3. *Every component of  $D$  is an irreducible  $f$ -fiber ( $f$  may have multiplicity along some  $D_i$ );*
4. *There exists a non  $\mathcal{F}$ -invariant Riemann surface embedded in  $M - \mathcal{R}$ .*

*Then every relatively exact meromorphic 1-form  $\omega_1$  modulo  $\mathcal{F}$  with the pole divisor  $D$  has the form*

$$(6) \quad \omega_1 = dg + \omega$$

*where  $g$  is a meromorphic function on  $M$  with the pole divisor  $D$  and  $\omega$  is a meromorphic 1-form in  $M$  inducing the foliation  $\mathcal{F}$  and with the pole divisor  $D + D_0$ .*

The above theorem is no more true if we assume that some  $f$ -fibers are disconnected. P. Bonnet in [Bo] gives the example  $f = x(1 + xy)$  in  $\mathbb{C}^2$  having the disconnected fiber  $f = 0$ . The 1-forms  $y^{k+1}dx + xy^k dy$ ,  $k > 0$  are relatively exact modulo  $f$  but they are not of the form (7). Without the hypothesis of connectedness of  $f$ -fibers the classification of relatively exact polynomial 1-forms modulo a polynomial is done in [Bo]. In [Ho1] we have classified relatively exact 1-forms in  $\mathbb{P}^2$  with non-invariant divisors. The fourth condition is trivial for an algebraic manifold with a Lefschetz pencil in it. I do not know yet that the third and fourth conditions in the above theorem are really necessary or not. The above classification of relatively exact 1-forms will suffice for our purpose.

**Proof:** Let  $\mathcal{R}$  be the indeterminacy set of  $f$  and  $C$  a non  $\mathcal{F}$ -invariant Riemann surface in  $U = M - \mathcal{R}$ . For any point  $x \in U$  let

$$L_x \cap C = \{p_1, p_2, \dots, p_e\}$$

where  $L_x = f^{-1}(g(x))$  is the fiber through  $x$  and  $e$  is the intersection number of  $C$  with a generic  $f$ -fiber ( $p_i$ 's are counted with multiplicity). Define

$$g : M \setminus (\cup_{i=1}^n D_i) \rightarrow \mathbb{C}$$

$$g(x) = \frac{1}{e} \left( \sum \int_x^{p_i} \omega_1 \right)$$

where  $\int_x^{p_i}$  is an integral over a path in  $L_x$  which connects  $x$  to  $p_i$ . Since the  $f$ -fibers are connected,  $\int_x^{p_i}$  is well-defined. The idea of the definition of  $g$  comes from the paper [Mu]. According to the hypothesis this integral does not depend on the choice of the path connecting  $x$  to  $p_i$  on  $L_x$ . The function  $g$  is a well-defined holomorphic function in  $M \setminus (\cup_{i=1}^n D_i)$ . We claim that  $g$  is a meromorphic function on  $M$  with pole divisor  $D$ . According to Levi extension theorem it is enough to prove that  $g$  is meromorphic in  $U$ .

For instance let us prove that  $g$  has a pole of order at most  $n_i$  at  $U \cap D_i$ . Let  $m_i$  be the multiplicity of  $f$  along  $D_i$ ,  $c_i = f(D_i)$  be the value associated to  $D_i$  and  $x \in U \cap D_i$ . In a small neighborhood of the path connecting  $x$  to  $p_i$  the function  $(f - c_i)^{\frac{n_i}{m_i}}$  is a univalued holomorphic function and

$$\int_x^{p_i} \omega_1 = (f - c_i)^{-\frac{n_i}{m_i}} \int_x^{p_i} (f - c_i)^{\frac{n_i}{m_i}} \omega_1$$

$(f - c_i)^{\frac{n_i}{m_i}} \omega_1$  is a holomorphic 1-form along  $U \cap D_i$  and therefore integrals of this type has a pole of order at most  $n_i$  at  $\{D_i = 0\}$ . This implies that  $g$  has pole of order at most  $n_i$  at  $\{D_i = 0\}$ . Note that in the above formulas we have chosen a local chart  $z(c), c \in (S, c_i)$  around  $c_i$  and instead of  $z \circ f - z(c_i)$  we have simply written  $f - c_i$ .

Every integral  $\int_x^{p_i} \omega_1$  satisfies the equation

$$d\left(\int_x^{p_i} \omega_1\right) \wedge \omega_0 = \omega_1 \wedge \omega_0$$

where  $\omega_0$  is a meromorphic 1-form inducing  $\mathcal{F}$ . The function  $g$  is the mean value of these integrals and so

$$\begin{aligned} dg \wedge \omega_0 &= \omega_1 \wedge \omega_0 \Rightarrow (\omega_1 - dg) \wedge \omega_0 = 0 \\ &\Rightarrow \omega_1 = dg + \omega \end{aligned}$$

where  $\omega$  is a meromorphic 1-form inducing  $\mathcal{F}$  and with the pole divisor  $D + D_0$ .  $\square$

Let  $\mathcal{F}$  be a holomorphic foliation in  $M$  considered in Theorem 4.1. Assume that  $\mathcal{F}$  has a non-composite meromorphic first integral  $f$  ( $S = \mathbb{P}^1$ ). Denote by  $D$  a generic fiber of  $f$ .

**Corollary 4.1** *Every relatively exact 1-form  $\omega_1$  in  $M$  with the pole divisor  $nD$  has the form*

$$\omega_1 = dg + pdf$$

where  $g$  and  $p$  are meromorphic functions with the pole divisors  $nD$  and  $(n - 1)D$ , respectively.

## 5 Melnikov Functions

In this section we follow the notations introduced in the first lines of Section 3. Let us suppose that  $f$  is non-composite and  $D = M_\infty$  is a generic compact  $f$ -fiber. The integrating factor  $s$  of  $\omega_0$  with

$$\frac{\omega_0}{s} = df$$

has  $2D$  as the zero divisor. For simplicity let us write  $\omega_\epsilon$  instead of  $\frac{\omega_\epsilon}{s}$  and  $\omega_i$  instead of  $\frac{\omega_i}{s}$ ,  $i = 1, 2, \dots$ .

**Theorem 5.1** *Suppose that  $\delta$  is a simple cycle defined in Section 2. If  $M_1 \equiv M_2 \equiv \dots \equiv M_k \equiv 0$  then*

$$M_{k+1}(t) = - \int_{\delta_t} \left( \sum_{i=1}^k p_i \omega_{k+1-i} + \omega_{k+1} \right)$$

where  $p_i$  and  $g_i$  are meromorphic functions in  $M$  with the pole divisors  $iD$  and  $(i+1)D$ , respectively, and are defined recursively by

$$\omega_i + p_i df + dg_i = - \sum_{j=1}^{i-1} p_j \omega_{i-j}, \quad i = 1, 2, \dots, k$$

**Proof:** The proof essentially follows from [Ro], Proposition 6, p. 73. We prove by induction on  $k$ . The case  $k = 1$  is proved in Proposition 3.1. Let us suppose that the theorem is true for  $k - 1$ , i.e. if  $M_1 \equiv M_2 \equiv \dots \equiv M_{k-1} \equiv 0$  then

$$M_k(t) = - \int_{\delta_t} \left( \sum_{i=1}^{k-1} p_i \omega_{k-i} + \omega_k \right)$$

Now suppose that  $M_k \equiv 0$ . Since  $\delta$  is a simple cycle, the 1-form

$$(7) \quad - \left( \sum_{i=1}^{k-1} p_i \omega_{k-i} + \omega_k \right)$$

is a relatively exact 1-form with the pole divisor  $(k+1)D$  and so by corollary 4.1 there exist  $p_k$  and  $g_k$  with the pole divisors  $kD$  and  $(k+1)D$ , respectively, such that

$$- \left( \sum_{i=1}^{k-1} p_i \omega_{k-i} + \omega_k \right) = dg_k + p_k df$$

A direct expansion gives

$$\left( 1 + \sum_{i=1}^k p_i \epsilon^i \right) \omega_\epsilon = d \left( f - \sum_{i=1}^k g_i \epsilon^i \right) + \left( \sum_{i=1}^k p_i \omega_{k+1-i} + \omega_{k+1} \right) \epsilon^{k+1} + O(\epsilon^{k+2})$$

Let  $\delta_{t, h_\epsilon(t)}$  be a path in the leaf of  $\mathcal{F}_\epsilon$  through  $t$  which connects  $t$  to  $h_\epsilon(t)$  along the path  $\delta$ . Since  $\Sigma$  is parameterized by  $t = f|_\Sigma$ , integrating the above equality over the path  $\delta_{t, h_\epsilon(t)}$  we have

$$(h_\epsilon(t) - t) - \left( \sum_{i=1}^k g_i \epsilon^i \right) \Big|_t^{h_\epsilon(t)} + \epsilon^{k+1} \int_{\delta_{t, h_\epsilon(t)}} \left( \sum_{i=1}^k p_i \omega_{k+1-i} + \omega_{k+1} \right) + O(\epsilon^{k+2}) = 0$$

$\int_{\delta_{t, h_\epsilon(t)}} = \int_{\delta_t} + O(\epsilon)$  and so by putting zero the coefficient of  $\epsilon^{k+1}$  in the above formula we get the desired equality.  $\square$

## References

- [AGV] Arnold, V.I.; Gusein-Zade, S.M.; Varchenko, A.N. Singularities of differentiable maps. Vol. II. Monodromy and asymptotics of integrals. Monographs in Mathematics, 83. Birkhäuser Boston, Inc., Boston, MA, 1988.
- [Bo] Bonnet, P.; Description of the Module of Relatively Exact 1-Forms Modulo a Polynomial  $f$  on  $\mathbb{C}^2$ , Prépublication ou Rapport de Recherche, n° 184 (1999).
- [CaSa] Camacho, C.; Sad, P. Pontos singulares de equações diferenciais analíticas. 16° Colóquio Brasileiro de Matemática. Instituto de Matemática Pura e Aplicada, Rio de Janeiro, 1987.
- [Ch] Chéniot, D.; Vanishing cycles in a pencil of hyperplane sections of a non-singular quasi-projective variety. Proc. London Math. Soc. (3) 72 (1996), no. 3, 515–544.
- [Fr] Françoise J.P.; Successive derivatives of a first return map, application to the study of quadratic vector fields, Ergodic Theory Dynam. Systems 16 (1996), no. 1, 87–9.
- [Ga] Gavrilov, L.; Petrov modules and zeros of Abelian integrals. Bull. Sci. Math. 122 (1998), no. 8, 571–584.
- [Go] Gomez-Mont X.; Integrals for Holomorphic Foliations with Singularities Having all Leaves Compact, Ann. Inst. Fourier, Grenoble 30, 2 (1989), 451-459.
- [GrRe] Grauert H.; Remmert R.; Theorie der Steinschen Räume.[Theory of Stein spaces] Grundlehren der Mathematischen Wissenschaften Springer-Verlag, Berlin-New York, 1977.
- [Il] Ilyashenko Yu.S.; The Origin of Limit Cycles Under Perturbation of Equation  $\frac{dw}{dz} = -\frac{R_z}{R_w}$ , Where  $R(z, w)$  Is a Polynomial, Math. USSR , Sbornik, Vol. 7, (1969), No. 3.

- [LP1] Landis, E.M.; Petrovskii, I.G. On the number of limit cycles of the equation  $dy/dx = P(x, y)/Q(x, y)$ , where  $P$  and  $Q$  are polynomials. Amer. Math. Soc. Transl. (2) 14 1960 181–199
- [LP2] Petrovskii, I.G.; Landis, E.M. On the number of limit cycles of the equation  $dy/dx = P(x, y)/Q(x, y)$ , where  $P$  and  $Q$  are polynomials of the second degree. 1958 Amer. Math. Soc. Transl., Ser. 2, Vol. 10 pp. 177–221.
- [Al] Lins Neto A.; Scárdua B.A.; Foleações Algébricas Complexas, 21° Colóquio Brasileiro de Matemática.
- [Ho1] Movasati H.; On Deformation of Foliations with a Center in the Projective Space, Doctorate thesis in IMPA.
- [Ho2] Movasati H.; On the topology of foliations with a first integral. Bol. Soc. Brasil. Mat. (N.S.) 31 (2000), no. 3, 305–336.
- [La] Lamotke K.; The Topology of Complex Projective Varieties After S. Lefschetz, Topology, Vol 20, 1981.
- [Mu] Muciño R.J.; Deformations of holomorphic foliations having a meromorphic first integral. J. Reine Angew. Math. 461 (1995), 189–219.
- [Po] Poincaré H.; Sur L'Intégration Algébrique des Équations Différentielles du Premier Ordre et du Premier Degré, Rendiconti del circolo Matematico di Palermo, t. 5, p. 161-191 (1891).
- [Pu] Pushkar I.A.; A multidimensional generalization of Ilyashenko's theorem on abelian integrals. Funktsional. Anal. i Prilozhen. 31 (1997), no. 2, 34–44, 95; translation in Funct. Anal. Appl. 31 (1997), no. 2, 100–108.
- [Ro] Roussarie R.; Bifurcation of planar vector fields and Hilbert's sixteenth problem. Progress in Mathematics, 164. Birkhäuser Verlag, Basel, 1998.