A Survey of Cyclic Cohomology for Hopf Algebras

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For Siavash Shahshahani on the occasion of his 60th birthday.

Abstract

We review recent progress in the study of cyclic cohomology of Hopf algebras and Hopf algebroids, starting with the pioneering work of Connes-Moscovici.

1 Introduction

It is well known that the theory of characteristic classes of vector bundles, more precisely the Chern character, can be extended to the noncommutative geometry, thanks to the noncommutative Chern-Weil theory of Connes [5, 3, 10]. In order to have a similar extension for quantum principal bundles, for example Hopf-Galois extensions, one needs first appropriate analogues of group and Lie algebra cohomology Hopf algebras. The recent works of Connes-Moscovici [9, 7, 6] on the index theory of transversely elliptic operators, more precisely their definition of cyclic cohomology of Hopf algebras, provides one with such a theory.

It is the goal of the present article to review the developments in the study of cyclic cohomology of Hopf algebras, starting with the pioneering work of Connes-Moscovici [9, 7, 6]. We will present a dual cyclic theory for Hopf algebras, first defined in [16], and independently in [29]. One motivation is that, as it was observed by M. Crainic [11], cyclic cohomology of cosemisimple Hopf algebras, e.g. the algebra of polynomial functions on a compact quantum group, due to existence of Haar integral, is always trivial. In other words it behaves in much the same way as continuous group cohomology. Let $HP^\bullet$ and $\widetilde{HP}_\bullet$ denote the resulting periodic cyclic (co)homology groups in the sense of [9] and [16], respectively. We present two very general results: for any commutative Hopf algebra $\mathcal{H}$, $HP^\bullet(\mathcal{H})$ decomposes into direct sums of Hochschild cohomology groups of the coalgebra $\mathcal{H}$ with trivial coefficients, and for any cocommutative $\mathcal{H}$, $\widetilde{HP}_\bullet(\mathcal{H})$ decomposes as Hochschild homology groups of algebra $\mathcal{H}$ with trivial coefficients. So far very few examples of computations of $HP^\bullet$ and $\widetilde{HP}_\bullet$ for quantum groups are known. We present what is known in Sections 3 and 4.

In Section 5 we review the main results on cyclic cohomology of extended Hopf algebras known so far, following [6, 18]. Extended Hopf algebras are closely related to Hopf algebroids. It seems that now the question of finding an appropriate algebraic framework to define cyclic cohomology of Hopf algebroids is settled by [18].

Finally, in Section 6 we present some of the results obtained in [1] on cyclic cohomology of smash products.

It was not our intention to cover all aspects of this new branch of noncommutative geometry. For applications to transverse index theory and for the whole theory one should consult the original Connes-Moscovici articles [9, 8, 6] as well as their review article [7]. We also recommend [30] for a general introduction to applications of Hopf algebras in noncommutative geometry. Much remains to be done in this area. For example, the relation between cyclic homology of Hopf algebras and developments in Hopf-Galois theory (see e.g. Montgomery's book [24]) remain to be explored. Also, what is missing is a general conjecture about the nature of Hopf cyclic homology of the algebra of polynomial functions (or smooth functions, provided they are defined) of quantum groups.

2 Preliminaries on Hopf algebras

In this paper algebra means an associative, not necessarily commutative, unital algebra over a fixed commutative ground ring k. Similar convention applies to *coalgebras*, *bialgebras* and *Hopf algebras*. The undecorated tensor product $\otimes$ means the tensor product over k. If $\mathcal{H}$ is a Hopf algebra, we denote its coproduct by $\Delta : \mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}$, its counit by $\epsilon : \mathcal{H} \longrightarrow k$, its unit by $\eta : k \longrightarrow \mathcal{H}$ and its antipode by $S : \mathcal{H} \longrightarrow \mathcal{H}$. We will use Sweedler's notation $\Delta(h) = h^{(1)} \otimes h^{(2)}$, $(\Delta \otimes id)\Delta(h) = h^{(1)} \otimes h^{(2)} \otimes h^{(3)}$, etc., where summation is understood.

If $\mathcal{H}$ is a Hopf algebra, the word $\mathcal{H}$-module means a module over the underlying algebra of $\mathcal{H}$. Similarly, an $\mathcal{H}$-comodule is a comodule over the underlying coalgebra of $\mathcal{H}$. The same convention applies to $\mathcal{H}$-bimodules and $\mathcal{H}$-bicomodules. The category of (left) $\mathcal{H}$-modules has a tensor product defined via the coproduct of $\mathcal{H}$: if M and N are left $\mathcal{H}$-modules, their tensor product $M \otimes N$ is again an $\mathcal{H}$-module via

$$h(m \otimes n) = h^{(1)}m \otimes h^{(2)}n.$$

Similarly, if M and N are left $\mathcal{H}$-comodules, the tensor product $M \otimes N$ is again an $\mathcal{H}$-comodule via

$$\Delta(m \otimes n) = m^{(-1)}n^{(-1)} \otimes m^{(0)} \otimes n^{(0)}.$$

We take the point of view, standard in noncommutative geometry, that a noncommutative space is encoded by an algebra or by a coalgebra. The idea of *symmetry*, i.e. action of a group on a space, can be expressed by the action/coaction of a Hopf algebra on an algebra/coalgebra. Thus four possibilities arise. Let $\mathcal{H}$ be a Hopf algebra. An algebra A is called a left *$\mathcal{H}$-module algebra* if it is a left $\mathcal{H}$-module and the multiplication map $A \otimes A \longrightarrow A$ and the unit map are morphisms of $\mathcal{H}$-modules. That is

$$h(ab) = h^{(1)}(a)h^{(2)}(b), \qquad h(1) = \epsilon(h)1,$$

for $h \in \mathcal{H}, a, b \in A$. Similarly an algebra A is called a $\mathcal{H}$-comodule algebra, if A is a left $\mathcal{H}$-comodule and the multiplication and the unit maps are morphisms of

$\mathcal{H}$-comodules. In a similar fashion an $\mathcal{H}$-module coalgebra is a coalgebra C which is a left $\mathcal{H}$-module, and the comultiplication $\Delta : C \longrightarrow C \otimes C$ and the counit map are $\mathcal{H}$-module maps. Finally an $\mathcal{H}$-comodule coalgebra is a coalgebra C which is an $\mathcal{H}$-comodule and the coproduct and counit map are comodule maps.

The *smash product* $A\#\mathcal{H}$ of an $\mathcal{H}$-module algebra A with $\mathcal{H}$ is, as a k-module, $A \otimes \mathcal{H}$ with the product

$$(a \otimes g)(b \otimes h) = a(g^{(1)}b) \otimes g^{(2)}h.$$

It is an associative algebra under the above product.

Examples

- 1. For $\mathcal{H} = U(\mathfrak{g})$, the enveloping algebra of a Lie algebra, A is an $\mathcal{H}$-module algebra iff $\mathfrak{g}$ acts on A by derivations, i.e. we have a Lie algebra map $\mathfrak{g} \longrightarrow Der(A)$.

- 2. For $\mathcal{H} = kG$, the group algebra of a (discrete) group G, A is a $\mathcal{H}$-module algebra iff G acts on A via automorphisms $G \longrightarrow Aut(A)$. The smash product $A\#\mathcal{H}$ is then isomorphic to the crossed product algebra $A \rtimes G$.

- 3. For any Hopf algebra $\mathcal{H}$, the algebra $A = \mathcal{H}$ is an $\mathcal{H}$-comodule algebra where the coaction is afforded by comultiplication $\mathcal{H} \longrightarrow \mathcal{H} \otimes \mathcal{H}$. Similarly, the coalgebra $\mathcal{H}$ is an $\mathcal{H}$-module coalgebra where the action is given by the multiplication $\mathcal{H} \otimes \mathcal{H} \to \mathcal{H}$. These are analogues of the action of a group on itself by translations.

- 4. By a theorem of Kostant [28], any cocommutative Hopf algebra $\mathcal{H}$ over an algebraically closed field of characteristic zero is isomorphic (as a Hopf algebra) to a smash product $\mathcal{H} = U(P(\mathcal{H}))\#kG(\mathcal{H})$, where $P(\mathcal{H})$ is the Lie algebra of primitive elements of $\mathcal{H}$ and $G(\mathcal{H})$ is the group of all grouplike elements of $\mathcal{H}$ and $G(\mathcal{H})$ acts on $P(\mathcal{H})$ by inner automorphisms $(g, h) \mapsto ghg^{-1}$, for $g \in G(\mathcal{H})$ and $h \in P(\mathcal{H})$.

3 Cyclic modules

Cyclic co/homology was first defined for (associative) algebras through explicit complexes or bicomplexes. Soon after, Connes introduced the notion of cyclic module and defined cyclic homology of cyclic modules [10]. The motivation was to define cyclic homology of algebras as a derived functor. Since the category of algebras and algebra homomorphisms is not an additive category, the standard (abelian) homological algebra is not enough. In Connes's approach, the category of cyclic modules appears as "abelianization" of the category of algebras with the embedding defined by the functor $A \mapsto A^{\natural}$, explained below. For an alternative approach one can consult ([13]), where cyclic cohomology is shown to be the nonabelian derived functor of the functor of traces on A. It was soon realized that cyclic modules and the flexibility they afford are indispensable tools in the theory. A recent example is the cyclic homology of Hopf algebras which cannot be defined as the cyclic homology of an algebra or coalgebra.

In this section we recall the theory of cyclic and paracyclic modules and their cyclic homologies. We also consider the doubly graded version, i.e. biparacyclic modules and the generalized Eilenberg-Zilber theorem [10, 13, 14].

For $r \geq 1$ an integer or $r = \infty$, let Λ^r denote the r-cyclic category. An *r-cyclic object* in a category $\mathcal{C}$ is a contravariant functor $\Lambda^r \to \mathcal{C}$. Equivalently, we have a sequence $X_n, n \geq 0$, of objects of $\mathcal{C}$ and morphisms called face, degeneracy and cyclic operators

$$\delta_i : X_n \to X_{n-1}, \quad \sigma_i : X_n \to X_{n+1}, \quad \tau : X_n \to X_n \qquad 0 \leq i \leq n$$

such that (X, δ_i, σ_i) is a simplicial object and the following extra relations are satisfied:

$$\begin{aligned} \delta_i \tau &= \tau \delta_{i-1} && 1 \leq i \leq n \\ \delta_0 \tau &= \delta_n \\ \sigma_i \tau &= \tau \sigma_{i-1} && 1 \leq i \leq n \\ \sigma_0 \tau &= \tau^2 \sigma_n \\ \tau^{r(n+1)} &= \mathrm{id}_n. \end{aligned}$$

For $r = \infty$, the last relation is replaced by the empty relation and we have a *paracyclic* object. For $r = 1$, a Λ^1 object is a *cyclic object.*

A cocyclic object is defined in a dual manner. Thus a cocyclic object in $\mathcal{C}$ is a covariant functor $\Lambda^1 \to \mathcal{C}$. Let k be a commutative ground ring. A cyclic module over k is a cyclic object in the category of k-modules. We denote the category of cyclic k-modules by Λ_k.

Next, let us recall that a *biparacyclic* object in a category $\mathcal{C}$ is a contravariant functor $\Lambda^\infty \times \Lambda^\infty \to \mathcal{C}$. Equivalently, we have a doubly graded set of objects $X_{n,m}$, $n, m \geq 0$, in $\mathcal{C}$ with horizontal and vertical face, degeneracy and cyclic operators $\delta_i, \sigma_i, \tau, d_i, s_i, t$ such that each row and each column is a paracyclic object in $\mathcal{C}$ and vertical and horizontal operators commute. A biparacyclic object X is called *cylindrical* if the operators $\tau^{m+1}, t^{n+1} : X_{m,n} \to X_{m,n}$ are inverse of each other. If X is cylindrical then it is easy to see that its *diagonal*, $d(X)$, defined by $d(X)_n = X_{n,n}$ with face, degeneracy and cyclic maps $\delta_i d_i$, $\sigma_i s_i$ and τt is a cyclic object.

We give a few examples of cyclic modules that will be used in this paper. The first example is the most fundamental example which motivated the whole theory.

1. Let A be an algebra. The cyclic module $A^\natural$ is defined by $A^\natural_n = A^{\otimes(n+1)}, n \geq 0$,

with the face, degeneracy and cyclic operators defined by

$$\begin{aligned}
\delta_i(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &= a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\
\delta_n(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &= a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1} \\
\sigma_i(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &= a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes \cdots \otimes a_n \\
\tau(a_0 \otimes a_1 \otimes \cdots \otimes a_n) &= a_n \otimes a_0 \cdots \otimes a_{n-1}.
\end{aligned}$$

The underlying simplicial module of $A^\natural$ is a special case of the following simplicial module. Let M be an A-bimodule. Let $C_n(A, M) = M \otimes A^{\otimes n}, \quad n \geq 0$. For $n = 0$, we put $C_0(A, M) = M$. Then the following faces and degeneracies δ_i, σ_i define a simplicial module structure on $C_\bullet(A, M)$:

$$\begin{aligned}
\delta_0(m \otimes a_1 \otimes \cdots \otimes a_n) &= m a_1 \otimes a_2 \otimes \cdots \otimes a_n \\
\delta_i(m \otimes a_1 \otimes \cdots \otimes a_n) &= m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\
\delta_n(m \otimes a_1 \otimes \cdots \otimes a_n) &= a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1} \\
\sigma_0(m \otimes a_1 \otimes \cdots \otimes a_n) &= m \otimes 1 \otimes a_1 \otimes \cdots \otimes a_n \\
\sigma_i(m \otimes a_1 \otimes \cdots \otimes a_n) &= m \otimes a_1 \otimes \cdots \otimes a_i \otimes 1 \otimes \cdots \otimes a_n \quad 1 \leq i \leq n.
\end{aligned}$$

Obviously, for $M = A$ we obtain $A^\natural$. In general, there is no cyclic structure on $C_\bullet(A, M)$.

2. let C be a coalgebra. The cocyclic module $C_\natural$ is defined by $C_\natural^n = C^{\otimes n+1}, \quad n \geq 0$, with coface, codegeneracy and cyclic operators:

$$\begin{aligned}
\delta_i(c_0 \otimes c_1 \otimes \cdots \otimes c_n) &= c_0 \otimes \cdots \otimes c_i^{(1)} \otimes c_i^{(2)} \otimes c_n \quad 0 \leq i \leq n \\
\delta_{n+1}(c_0 \otimes c_1 \otimes \cdots \otimes c_n) &= c_0^{(2)} \otimes c_1 \otimes \cdots \otimes c_n \otimes c_0^{(1)} \\
\sigma_i(c_0 \otimes c_1 \otimes \cdots \otimes c_n) &= c_0 \otimes \ldots c_i \otimes \varepsilon(c_{i+1}) \otimes \cdots \otimes c_n \quad 0 \leq i \leq n-1 \\
\tau(c_0 \otimes c_1 \otimes \cdots \otimes c_n) &= c_1 \otimes c_2 \otimes \cdots \otimes c_n \otimes c_0,
\end{aligned}$$

where as usual $\Delta(c) = c^{(1)} \otimes c^{(2)}$ (Sweedler's notation). The underlying cosimplicial module for $C_\natural$ is a special case of the following cosimplicial module. Let M be a C-bicomodule and $C^n(C, M) = M \otimes C^{\otimes n}$. The following coface and codegeneracy operators define a cosimplicial module.

$$\begin{aligned}
\delta_0(m \otimes c_1 \otimes \cdots \otimes c_n) &= m^{(0)} \otimes m^{(1)} \otimes c_1 \cdots \otimes c_n \\
\delta_i(m \otimes c_1 \otimes \cdots \otimes c_n) &= m \otimes c_1 \cdots \otimes c_i^{(0)} \otimes c_i^{(1)} \otimes c_n \quad \text{for} \quad 1 \leq i \leq n \\
\delta_{n+1}(m \otimes c_1 \otimes \cdots \otimes c_n) &= m_{(0)} \otimes c_1 \otimes \cdots \otimes c_n \otimes m_{(-1)} \\
\sigma_i(m \otimes c_1 \otimes \cdots \otimes c_n) &= m \otimes c_1 \ldots \varepsilon(c_{i+1}) c_i \otimes \cdots \otimes c_n \quad 0 \leq i \leq n-1,
\end{aligned}$$

where we have denoted the left and right comodule maps by $\Delta_l(m) = m_{(-1)} \otimes m_{(0)}$ and $\Delta_r(m) = m^{(0)} \otimes m^{(1)}$. Let

$$d = \sum_{i=0}^{n+1} (-1)^i \delta_i : C^n(C, M) \to C^{n+1}(C, M).$$

Then $d^2 = 0$. The cohomology of the complex $(C^\bullet(C, M), d)$ is the Hochschild cohomology of the coalgebra C with coefficients in the bicomodule M. For $M = C$, we obtain the Hochschild complex of $C_\natural$. Another special case occurs with $M = k$ and $\Delta_r : k \to k \otimes C \cong C$ and $\Delta_l : k \to C \otimes k \cong C$, are given by $\Delta_r(1) = 1 \otimes g$ and $\Delta_l(1) = h \otimes 1$, where $g, h \in C$ are grouplike elements. The differential $d : C^n \to C^{n+1}$ in the latter case is given by

$$d(c_1 \otimes c_2 \cdots \otimes c_n) = g \otimes c_1 \otimes \cdots \otimes c_n + \sum_{i=1}^{n} (-1)^i c_1 \otimes \cdots \otimes \Delta(c_i) \otimes \cdots \otimes c_n + (-1)^{n+1} c_1 \otimes \cdots \otimes c_n \otimes h.$$

3. Let $g : A \to A$ be an automorphism of an algebra A. The paracyclic module $A_g^\natural$ is defined by $A_{g,n}^\natural = A^{\otimes(n+1)}$ with the same cyclic structure as $A^\natural$, except the following changes

$$\begin{aligned} \delta_n(a_0 \otimes a_1 \cdots \otimes a_n) &= g(a_n)a_0 \otimes \cdots \otimes a_{n-1} \\ \tau(a_0 \otimes a_1 \cdots \otimes a_n) &= g(a_n) \otimes a_0 \otimes \cdots \otimes a_{n-1}. \end{aligned}$$

One can check that $A_g^\natural$ is a Λ^∞-module and if $g^r = id$, then it is a Λ^r-module. For $g = id$, we obtain example 1.

Next, let us indicate how one defines the Hochschild, cyclic and periodic cyclic homology of a cyclic module. This is particularly important since the cyclic homology of Hopf algebras is naturally defined as the cyclic homology of some cyclic modules associated with them. Given a cyclic module $M \in \Lambda_k$, its cyclic homology group $HC_n(M)$, $n \geq 0$, is defined (in [10]) by

$$HC_n(M) := Tor_n^{\Lambda_k}(M, k^\natural),$$

and similarly the cyclic cohomology groups of M are defined by

$$HC^n(M) := Ext_{\Lambda_k}^n(M, k^\natural).$$

Using a specific projective resolution for $k^\natural$, one obtains the following bicomplex to compute cyclic homology. Given a cyclic module M, consider the following first quadrant bicomplex, called the *cyclic bicomplex* of M.

$$\begin{array}{ccccccc}
\vdots & & \vdots & & \vdots & & \\
M_2 & \xleftarrow{1-\tau} & M_2 & \xleftarrow{N} & M_2 & \xleftarrow{1-\tau} & \dots \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \\
M_1 & \xleftarrow{1+\tau} & M_1 & \xleftarrow{N} & M_1 & \xleftarrow{1+\tau} & \dots \\
\downarrow b & & \downarrow -b' & & \downarrow b & & \\
M_0 & \xleftarrow{1-\tau} & M_0 & \xleftarrow{N} & M_0 & \xleftarrow{1-\tau} & \dots
\end{array}$$

We denote this bicomplex by $CC^+(M)$. The operators b, b' and N are defined by

$$\begin{aligned} b &= \sum_{i=0}^{n}(-1)^i\delta_i \\ b' &= \sum_{i=0}^{n-1}(-1)^i\delta_i \\ N &= \sum_{i=0}^{n}(-1)^{ni}\tau^i. \end{aligned}$$

Using the simplicial and cyclic relations, one can check that $b^2 = b'^2 = 0$, $b(1-(-1)^n\tau) = (1-(-1)^{n-1}\tau)b'$ and $b'N = Nb'$. The *Hochschild homology* of M, denoted by $HH_\bullet(M)$, is the homology of the first column $(M_\bullet, b)$. The *cyclic homology* of M, denoted by $HC_\bullet(M)$, is the homology of the total complex $TotCC^+(M)$.

To define the *periodic cyclic* homology of M, we extend the first quadrant bicomplex $CC^+(M)$ to the left and denote it by $CC(M)$. Let $TotCC(M)$ denote the "total complex" where instead of direct sums we use direct product,

$$TotCC(M)_n = \prod_{i=0}^{\infty} M_i.$$

It is obviously a 2-periodic complex and its homology is called the periodic cyclic homology of M and denoted by $HP_\bullet(M)$.

The complex $(M_\bullet, b')$ is acyclic with contracting homotopy $\sigma_{-1} = \tau\sigma_n$. One can then show that $CC^+(M)$ is homotopy equivalent to Connes's (b, B) bicomplex

$$\begin{array}{ccccc} \vdots & & \vdots & & \vdots \\ M_2 & \xleftarrow{B} & M_1 & \xleftarrow{B} & M_0 \\ \downarrow b & & \downarrow b & & \\ M_1 & \xleftarrow{B} & M_0 & & \\ \downarrow b & & & & \\ M_0 & & & & \end{array}$$

Where $B : M_n \to M_{n+1}$ is Connes's boundary operator defined by $B = (1-(-1)^n\tau)\sigma_{-1}N$.

Finally we arrive at the 3rd definition of cyclic homology by noticing that if k is a field of characteristic zero, then the rows of $CC^+(M)$ are acyclic in positive degree and its homology in dimension zero is

$$C_n^\lambda(M) = \frac{M_n}{(1-(-1)^n\tau)M_n}.$$

It follows that the total homology, i.e. cyclic homology of M can be computed, if k is a field of characteristic zero, as the homology of Connes's cyclic complex $(C^{\lambda}_{\bullet}(M), b)$

Now, if A is an associative algebra, its Hochschild, cyclic and periodic cyclic homology, are defined as the corresponding homology of the cyclic module $A^{\natural}$. We denote these groups by $HH_{\bullet}(A)$, $HC_{\bullet}(A)$ and $HP_{\bullet}(A)$, respectively. Similarly, if C is a coalgebra, its Hochschild, cyclic and periodic cyclic cohomology are defined as the corresponding homology of the cocyclic module $C_{\natural}$.

Our next goal is to recall the generalized Eilenberg-Zilber theorem for cylindrical modules from [14, 17]. This is needed in Section 6 to derive a spectral sequence for cyclic homology of smash products.

A *parachain complex* $(M_{\bullet}, b, B)$ is a chain complex $(M_{\bullet}, b)$ endowed with a map $B : M_{\bullet} \to M_{\bullet+1}$ such that $B^2 = 0$ and $T = 1 - (bB + Bb)$ is an invertible operator. For example, a *mixed complex* is a parachain complex such that $bB + Bb = 0$. Given a mixed complex M one can define its (b, B)-bicomplex as the Connes's (b, B) bicomplex. One can thus define the Hochschild, cyclic and periodic cyclic homology of mixed complexes. The definition of a *bi-parachain complex* should be clear. Given a bi-parachain complex $X_{p,q}$, one defines its total complex $TotX$ by

$$(TotX)_n = \oplus X_{p,q}, \quad b = b_v + b_h, \quad B = B_v + TB_h,$$

where v and h refers to horizontal and vertical differentials. One can check that $TotX$ is a parachain complex [14].

Now if X is a cylindrical module and $C(X)$ is the bi-parachain complex obtained by forming the associated mixed complexes horizontally and vertically, then one can check that $Tot(C(X))$ is indeed a mixed complex. On the other hand we know that the diagonal $d(X)$ is a cyclic module and hence its associated chain complex $C(d(X))$ is a mixed complex.

The following theorem was first proved in [14] using topological arguments. A purely algebraic proof can be found in [16].

Theorem 3.1. *([14, 16]) Let X be a cylindrical module. There is a quasi-isomorphism of mixed complexes $f_0 + uf_1 : Tot(C(X)) \to C(d(X))$ such that f_0 is the shuffle map.*

4 Cyclic cohomology of Hopf algebras

Thanks to the recent works of Connes-Moscovici [9, 8, 6], the following principle has emerged. A reasonable co/homology theory for Hopf algebras and Hopf algebra like objects in noncommutative geometry should address the following two issues:

- It should reduce to group co/homology or Lie algebra co/homology for $\mathcal{H} = kG$, $k[G]$ or $U(\mathfrak{g})$; Hopf algebras naturally associated to (Lie) groups or Lie algebras.

- There should exist a characteristic map, connecting the cyclic cohomology of a Hopf algebra $\mathcal{H}$ to the cyclic cohomology of an algebra A on which it acts.

For example, for any $\mathcal{H}$-module algebra A and an invariant trace $\tau : A \longrightarrow \mathbb{C}$, there should exist a map

$$\gamma : HC^\bullet(\mathcal{H}) \longrightarrow HC^\bullet(A).$$

Let us explain both points, starting with the first. It might seem that given a Hopf algebra $\mathcal{H}$, the Hochschild homology of the algebra $\mathcal{H}$ might be a good candidate for a homology theory for $\mathcal{H}$ in noncommutative geometry. After all, one knows that for a Lie algebra $\mathfrak{g}$ and a $U(\mathfrak{g})$-bimodule M,

$$H_\bullet(\mathfrak{g}, M^{ad}) \cong H_\bullet(U(\mathfrak{g}), M)$$

where the action of $\mathfrak{g}$ on M is given by $g \cdot m = gm - mg$ [20]. Thus Hochschild homology of $U(\mathfrak{g})$ can be recovered from the Lie algebra homology of $\mathfrak{g}$. Conversely, if M is a $\mathfrak{g}$-module we can turn it into a $U(\mathfrak{g})$-bimodule where the left action is induced by $\mathfrak{g}$-action and the right action is by augmentation: $mX = \epsilon(X)m$. It follows that $H_\bullet(\mathfrak{g}, M) \cong H_\bullet(U(\mathfrak{g}), M)$, which shows that the Lie algebra homology can also be recovered from Hochschild homology. In particular $H_\bullet(\mathfrak{g}, k) \cong H_\bullet(U(\mathfrak{g}), k)$. Similarly, if G is a (discrete) group and M is a kG-bimodule then $H_\bullet(G; M^{ad}) \cong HH_\bullet(kG, M)$ where the action of G on $M^{ad} = M$ is given by $gm = gmg^{-1}$.

In [16] these type of results were extended to all Hopf algebras in the following way. Let $\mathcal{H}$ be a Hopf algebra and M a left $\mathcal{H}$-module. One defines groups $H_\bullet(\mathcal{H}, M)$ as the left derived functor of the functor of coinvariants from $\mathcal{H}$-mod$\to k$-mod,

$$M \mapsto M_{\mathcal{H}} := M/ \text{ submodule generated by } \{hm - \epsilon(h)m \mid h \in \mathcal{H}, \quad m \in M\}.$$

Obviously, $M_{\mathcal{H}} = k \otimes_{\mathcal{H}} M$ which shows that $H_\bullet(\mathcal{H}, M) \cong Tor_\bullet^{\mathcal{H}}(k, M)$. For $\mathcal{H} = kG$ or $U(\mathfrak{g})$, one obtains group and Lie algebra homologies.

Now let $\mathcal{H}$ be a Hopf algebra and M be an $\mathcal{H}$-bimodule. We can convert M to a new left $\mathcal{H}$ -module $M^{ad} = M$, where the action of $\mathcal{H}$ is given by

$$h \cdot m = h^{(2)} m S(h^{(1)}).$$

Proposition 4.1. *([16])(Mac Lane isomorphism for Hopf algebras) Under the above hypotheses there is a canonical isomorphism*

$$H_n(\mathcal{H}, M) \cong H_n(\mathcal{H}; M^{ad}) = Tor_n^{\mathcal{H}}(k, M^{ad}),$$

where the left hand side is Hochschild homology.

Note that the result is true for all Hopf algebras irrespective of being (co)commutative or not.

This suggests to define $H_\bullet(\mathcal{H}, k)$, where k is an $\mathcal{H}$-bimodule via augmentation map, in analogy with the group homology. This is not, however, a reasonable candidate as can be seen by considering $\mathcal{H} = k[G]$, the coordinate ring of an affine algebraic group. Then by the Hochschild-Kostant-Rosenberg theorem $HH_\bullet(k[G]; k) \cong \wedge^\bullet(Lie(G))$ and hence is independent of the group structure.

Next we discuss the second point above. Some interesting cyclic cocycles were defined by Connes in the context of Lie algebra homology and group cohomology. For example let A be an algebra and $\delta_1, \delta_2 : A \to A$ two commuting derivations. Let $\tau : A \to \mathbb{C}$ be an *invariant trace* in the sense that τ is a trace and $\tau(\delta_1(a)) = \tau(\delta_2(a)) = 0$ for all $a \in A$. Then one can directly check that the following is a cyclic 2-cocycle on A [5]:

$$\varphi(a_0, a_1, a_2) = \tau(a_0(\delta_1(a_1)\delta_2(a_2) - \delta_2(a_1)\delta_1(a_2))).$$

This cocycle is non-trivial. For example, if $A = A_\theta$ is the algebra of smooth noncommutative torus and $e \in A_\theta$ is the smooth Rieffel projection, then $\varphi(e, e, e) = \pm q$, where $\tau(e) = \mid p - q\theta \mid$ [5].

For a second example let G be a (discrete) group and c be a normalized group cocycle on G with trivial coefficients. Then one can easily check that the following is a cyclic cocycle on the group algebra $\mathbb{C}G$ [7]:

$$\varphi(g_0, g_1 \dots, g_n) = \begin{cases} c(g_1, g_2 \dots, g_n) & \text{if } g_0 g_1 \dots g_n = 1 \\ 0 & \text{otherwise} \end{cases}$$

It is highly desirable to understand the origin of these formulas, put them in a conceptual context and generalize them. For example we need to know in the case where a Lie algebra $\mathfrak{g}$ acts by derivations on an algebra A, $\mathfrak{g} \to Der(A)$, if there is a map

$$\gamma : H_\bullet(\mathfrak{g}, \mathbb{C}) \to HC^\bullet(A).$$

Now let us indicate how the cohomology theory defined by Connes-Moscovici [9, 8] and its dual version in [16] resolve both issues. Let $\mathcal{H}$ be a Hopf algebra. Let δ be *character* and σ a *group like* element on $\mathcal{H}$, i.e. $\delta : \mathcal{H} \to k$ is an algebra map and $\sigma : k \to \mathcal{H}$ a coalgebra map. Following [9, 8], we say (δ, σ) is a *modular pair* if $\delta\sigma = id_k$ and a *modular pair in involution* if, in addition, $(\sigma^{-1}\widetilde{S})^2 = id_{\mathcal{H}}$ where the *twisted antipode* $\tilde{S}$ is defined by

$$\widetilde{S}(h) = \sum_{(h)} \delta(h^{(1)})S(h^{(2)}).$$

Given $\mathcal{H}$, and (δ, σ), Connes-Moscovici define a cocyclic module $\mathcal{H}^\natural_{(\delta,\sigma)}$ as follows. Let $\mathcal{H}^{\natural,0}_{(\delta,\sigma)} = k$ and $\mathcal{H}^{\natural,n}_{(\delta,\sigma)} = \mathcal{H}^{\otimes n}$, $n \geq 1$. The coface, codegeneracy and cyclic operators δ_i, σ_i, τ are defined by

$$\begin{aligned}
\delta_0(h_1 \otimes \cdots \otimes h_n) &= 1_{\mathcal{H}} \otimes h_1 \otimes \cdots \otimes h_n \\
\delta_i(h_1 \otimes \cdots \otimes h_n) &= h_1 \otimes \cdots \otimes \Delta(h_i) \otimes \cdots \otimes h_n \quad \text{for } 1 \leq i \leq n \\
\delta_{n+1}(h_1 \otimes \cdots \otimes h_n) &= h_1 \otimes \cdots \otimes h_n \otimes \sigma \\
\sigma_i(h_1 \otimes \cdots \otimes h_n) &= h_1 \otimes \cdots \otimes \epsilon(h_{i+1}) \otimes \cdots \otimes h_n \quad \text{for } 0 \leq i \leq n \\
\tau(h_1 \otimes \cdots \otimes h_n) &= \Delta^{n-1}\widetilde{S}(h_1) \cdot (h_2 \otimes \cdots \otimes h_n \otimes \sigma).
\end{aligned}$$

These formulas were discovered in [9] and then proved in full generality in [8]. In [11], M. Crainic gave an alternative approach based on Cuntz-Quillen formalism of cyclic homology [12]. Note that the cosimplicial module $\mathcal{H}^{\natural}_{(\delta,\sigma)}$ is the cosimplicial module associated to the coalgebra $\mathcal{H}$ with coefficients in k via the unit map and σ. The passage from the cyclic homology of (co)algebras to the cyclic homology of Hopf algebras is remarkably similar to passage from de Rham cohomology to Lie algebra cohomology. The key idea in both cases is *invariant cohomology.*

It is not difficult to see that the above complex is an exact analogue of *invariant cohomology* in noncommutative geometry. In fact, under the multiplication map $\mathcal{H}\otimes\mathcal{H}\to\mathcal{H}$ the coalgebra $\mathcal{H}$ is an $\mathcal{H}$-module coalgebra. Let $\hat{\mathcal{H}}_{\natural}$ be the cocyclic module of the coalgebra $\mathcal{H}$. The cocyclic module $\hat{\mathcal{H}}_{\natural}$ becomes a cocyclic $\mathcal{H}$-module via the diagonal action $\mathcal{H}\otimes\hat{\mathcal{H}}_{\natural}\to\hat{\mathcal{H}}_{\natural}$. We have $\hat{\mathcal{H}}_{\natural}{}^{\delta}=\mathcal{H}^{\natural}_{(\delta,1)}$ where $\hat{\mathcal{H}}_{\natural}{}^{\delta}$ is the space of δ-coinvariants.

The cohomology groups $HP^{\bullet}_{(\delta,\sigma)}(\mathcal{H})$ are so far computed for the following Hopf algebras. For quantum universal enveloping algebras no examples are known except for $U_q(sl_2)$ that we recall below.

1. If $\mathcal{H}=\mathcal{H}_n$ is the Connes-Moscovici Hopf algebra, we have [9]

$$HP^n_{(\delta,1)}(\mathcal{H})\cong\bigoplus_{i=n\ (\mathrm{mod}\ 2)}H^i(\mathfrak{a}_n,\mathbb{C})$$

   where $\mathfrak{a}_n$ is the Lie algebra of formal vector fields on $\mathbb{R}^n$.

2. If $\mathcal{H}=U(\mathfrak{g})$ is the enveloping algebra of a Lie algebra $\mathfrak{g}$, we have [9]

$$HP^n_{(\delta,1)}(\mathcal{H})\cong\bigoplus_{i=n\ (\mathrm{mod}\ 2)}H_i(\mathfrak{g},\mathbb{C}_\delta)$$

3. If $\mathcal{H}=\mathbb{C}[G]$ is the coordinate ring of a nilpotent affine algebraic group G, we have [9]

$$HP^n_{(\epsilon,1)}(\mathcal{H})\cong\bigoplus_{i=n\ (\mathrm{mod}\ 2)}H^i(\mathfrak{g},\mathbb{C}),$$

   where $\mathfrak{g}=Lie(G)$.

4. If $\mathcal{H}$ admits a normalized left Haar integral, then [11]

$$HP^1_{(\delta,\sigma)}(\mathcal{H})=0,\qquad HP^0_{(\delta,\sigma)}(\mathcal{H})=k.$$

   Recall that a linear map $\int:\mathcal{H}\to k$ is called a normalized left Haar integral if for all $h\in\mathcal{H}$, $\int(h)=\int(h^{(1)})h^{(2)}$ and $\int(1)=1$. Compact quantum groups, finite dimensional Hopf algebras over a filed of characteristic zero, and group algebras are known to admit normalized Haar integral in the above sense. In the latter case $\int:kG\to k$ sending $g\mapsto 0$ for all $g\neq e$ and $e\mapsto 1$ is a Haar integral. Note that G need not to be finite.

5. If $\mathcal{H} = U_q(sl_2(k))$ is the quantum universal algebra of $sl_2(k)$, we have [11],

$$HP^0_{(\epsilon,\sigma)}(\mathcal{H}) = 0, \quad HP^1_{(\epsilon,\sigma)}(\mathcal{H}) = k \oplus k.$$

6. Let $\mathcal{H}$ be a commutative Hopf algebra. The periodic cyclic cohomology of the cocyclic module $\mathcal{H}^\natural_{(\epsilon,1)}$ can be computed in terms of the Hochschild homology of coalgebra $\mathcal{H}$ with trivial coefficients.

Proposition 4.2. *([16]) Let $\mathcal{H}$ be a commutative Hopf algebra. Its periodic cyclic cohomology in the sense of Connes-Moscovici is given by*

$$HP^n_{(\epsilon,1)}(\mathcal{H}) = \bigoplus_{i=n \ (mod\ 2)} H^i(\mathcal{H}, k).$$

For example, if $\mathcal{H} = k[G]$ is the algebra of regular functions on an affine algebraic group G, the coalgebra complex of $\mathcal{H} = k[G]$ is isomorphic to the group cohomology complex of G where instead of regular cochains one uses regular functions $G \times G \times \cdots \times G \to k$. Denote this cohomology by $H^i(G,k)$. It follows that

$$HP^n_{(\epsilon,1)}(k[G]) = \bigoplus_{i=n \ (\mathrm{mod}\ 2)} H^i(G, k).$$

As is remarked in [7], if the Lie algebra $\mathrm{Lie}(G) = \mathfrak{g}$ is nilpotent, it follows from Van Est's theorem that $H^i(G,k) \cong H^i(\mathfrak{g}, k)$. This gives an alternative proof of Prop., 4 and Remark 5 in [7].

Let A be an $\mathcal{H}$-module algebra and $Tr : A \to \mathbb{C}$ a *δ-invariant* linear map, i.e., $Tr(h(a)) = \delta(h)Tr(a)$ for $h \in \mathcal{H},\ a \in A$. Equivalently, Tr satisfies the *integration by part* property:

$$Tr(h(a)b) = Tr(a\tilde{S}(h)(b)).$$

In addition we assume $Tr(ab) = Tr(b\sigma a)$. Given $(A, \mathcal{H}, Tr)$, Connes-Moscovici show that the following map, called the *the characteristic map*, defines a morphism of cyclic modules $\gamma : \mathcal{H}^\natural_{\delta,\sigma} \to A^\natural$, where $A^\natural = \hom(A_\natural, k)$ is the cocyclic module associated to A,

$$\gamma(h_1 \otimes \cdots \otimes h_n)(a_0, a_1, \ldots, a_n) = Tr(a_0 h_1(a_1) \ldots h_n(h_n)).$$

We therefore have well-defined maps

$$\gamma : HC^\bullet_{(\delta,\sigma)}(\mathcal{H}) \to HC^\bullet(A)$$
$$\gamma : HP^\bullet_{(\delta,\sigma)}(\mathcal{H}) \to HP^\bullet(A).$$

Examples show that, in general, this map is non-trivial. For example let $\mathfrak{g}$ be an abelian n-dimensional Lie algebra acting by derivations on an algebra A. Let $\delta_i \in Der(A)$ be the family of derivations corresponding to a basis $X_1, \ldots, X_n$ of $\mathfrak{g}$, and $Tr : A \to k$ an invariant trace on A, i.e. $Tr\delta_i(a) = 0, \quad 1 \le i \le n$. We have $H_i(\mathfrak{g}, k) \cong \wedge^i \mathfrak{g}$. In particular $H_n(\mathfrak{g}, k)$ is 1-dimensional. The inclusion

$$H_n(\mathfrak{g}, k) \hookrightarrow \bigoplus_{i=n \bmod 2} H_i(\mathfrak{g}, k) \cong HP^n_{(\epsilon,1)}(U(\mathfrak{g}))$$

combined with the characteristic map γ defines a map

$$\gamma : H_n(\mathfrak{g}, k) \cong k \to HC^n(A).$$

The image of $X_1 \wedge X_2 \wedge \cdots \wedge X_n$ under γ is the cyclic n-cocycle φ given by

$$\varphi(a_0, a_1, \ldots, a_n) = \sum_{\sigma \in S_n} (-1)^n Tr(a_0 \delta_1(a_{\sigma(1)}) \delta_2(a_{\sigma(2)}) \ldots \delta_n(a_{\sigma(n)})).$$

The rest of this section is devoted to a dual cyclic theory for Hopf algebras which was defined, independently, in [16, 29]. There is a need for a dual theory to be developed. This is needed, for example, when one studies coactions of Hopf algebras (or quantum groups) on noncommutative spaces, since the original Connes-Moscovici theory works for actions only. A more serious problem is the fact that if $\mathcal{H}$ has normalized left Haar integral then its cyclic cohomology in the sense of Connes-Moscovici is trivial in positive dimensions [11], but the dual theory is non-trivial.

In [16] we associated a cyclic module to any Hopf algebra $\mathcal{H}$ over k if $\mathcal{H}$ has a modular pair (δ, σ) such that $\widehat{S}^2 = id_{\mathcal{H}}$, where $\widehat{S}(h) = \delta(h^{(2)})\sigma S(h^{(1)})$. This cyclic module can be seen as the dual of the cocyclic module introduced in [8] by A. Connes and H. Moscovici. Using ϵ and δ one can endow k with an $\mathcal{H}$-bimodule structure, i.e.,

$$\delta \otimes id : \mathcal{H} \otimes k \to k \quad and \quad id \otimes \epsilon : k \otimes \mathcal{H} \to k.$$

Our cyclic module as a simplicial module is exactly the Hochschild complex of $\mathcal{H}$ with coefficients in k where k is an $\mathcal{H}$-bimodule as above. So if we denote our cyclic module by $\widetilde{\mathcal{H}}^{(\delta,\sigma)}_{\natural}$, we have $\widetilde{\mathcal{H}}^{(\delta,\sigma)}_{\natural n} = \mathcal{H}^{\otimes n}$, for $n > 0$ and $\widetilde{\mathcal{H}}^{(\delta,\sigma)}_{\natural 0} = k$. Its faces and degeneracies are as follows:

$$\begin{aligned}
\delta_0(h_1 \otimes h_2 \otimes ... \otimes h_n) &= \epsilon(h_1) h_2 \otimes h_3 \otimes ... \otimes h_n \\
\delta_i(h_1 \otimes h_2 \otimes ... \otimes h_n) &= h_1 \otimes h_2 \otimes ... \otimes h_i h_{i+1} \otimes ... \otimes h_n \\
\delta_n(h_1 \otimes h_2 \otimes ... \otimes h_n) &= \delta(h_n) h_1 \otimes h_2 \otimes ... \otimes h_{n-1} \\
\sigma_0(h_1 \otimes h_2 \otimes ... \otimes h_n) &= 1 \otimes h_1 \otimes ... \otimes h_n \\
\sigma_i(h_1 \otimes h_2 \otimes ... \otimes h_n) &= h_1 \otimes h_2 ... \otimes h_i \otimes 1 \otimes h_{i+1} ... \otimes h_n \\
\sigma_n(h_1 \otimes h_2 \otimes ... \otimes h_n) &= h_1 \otimes h_2 \otimes ... \otimes 1.
\end{aligned}$$

To define a cyclic module it remains to introduce an action of cyclic group on our module. Our candidate is

$$\tau_n(h_1 \otimes h_2 \otimes ... \otimes h_n) = \sum \delta(h_n^{(2)}) \sigma S(h_1^{(1)} h_2^{(1)} ... h_{n-1}^{(1)} h_n^{(1)}) \otimes h_1^{(2)} \otimes ... \otimes h_{n-1}^{(2)}.$$

Theorem 4.1. *([16]) Let $\mathcal{H}$ be a Hopf algebra over k with a modular pair (δ,σ) such that $\widehat{S}^2 = id_{\mathcal{H}}$. Then $\widetilde{\mathcal{H}}_\natural^{(\delta,\sigma)}$ with operators given above defines a cyclic module. Conversely, if (δ,σ) is a modular pair such that $\widetilde{\mathcal{H}}_\natural^{(\delta,\sigma)}$ is a cyclic module, then $\widehat{S}^2 = id_{\mathcal{H}}$.*

Now let A be an $\mathcal{H}$-comodule algebra. To define the characteristic map we need an analogue of an invariant trace.

Definition 4.1. *A linear map $Tr : A \to k$ is called δ-trace if*

$$Tr(ab) = \sum_{(b)} Tr(b^{(0)}a)\delta(b^{(1)}) \qquad\qquad \forall a,b \in A.$$

It is called σ-invariant if for all $a,b \in A$,

$$\sum_{(b)} Tr(ab^{(0)})(b^{(1)}) = \sum_{(a)} Tr(a^{(0)}b)S_\sigma(a^{(1)})$$

or equivalently

$$Tr(a^{(0)})a^{(1)} = Tr(a)\sigma.$$

Consider the map $\gamma : A_\natural \to \widetilde{\mathcal{H}}_\natural^{(\delta,\sigma)}$ defined by

$$\gamma(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = Tr(a_o a_1^{(0)} \dots a_n^{(0)})a_1^{(1)} \otimes a_2^{(1)} \otimes \dots a_n^{(1)}.$$

It is proved in [16] that γ is a morphism of cyclic modules.

Corollary 4.1. *Under the above conditions, γ induces the following canonical maps:*

$$\gamma :\ HC_\bullet(A) \to \widetilde{HC}_\bullet^{(\delta,\sigma)}(\mathcal{H})$$
$$\gamma :\ HP_\bullet(A) \to \widetilde{HP}_\bullet^{(\delta,\sigma)}(\mathcal{H}).$$

Next, we state a theorem which computes the cyclic homology of cocommutative Hopf algebras.

Theorem 4.2. *([16]) If $\mathcal{H}$ is a cocommutative Hopf algebra, then*

$$\widetilde{HC}_n^{(\delta,1)}(\mathcal{H}) = \bigoplus_{i\geq 0} H_{n-2i}(\mathcal{H},k_\delta),$$

where k_δ is the one dimensional module defined by δ.

Example 4.1. *Let $\mathfrak{g}$ be a Lie algebra over k and $U(\mathfrak{g})$ be its enveloping algebra. One knows that $H_n(U(\mathfrak{g});k) = H_n(\mathfrak{g};k)$ [20]. So by Theorem 4.2 we have*

$$\widetilde{HC}_n^{(\delta,1)}(\mathfrak{g}) = \bigoplus_{i\geq 0} H_i(\mathfrak{g};k_\delta).$$

Example 4.2. *Let G be a discrete group and $\mathcal{H} = kG$ its group algebra. Then from theorem 4.2 we have*

$$\widetilde{HC}_n^{(\epsilon,1)}(kG) \cong \bigoplus_{i\geq 0} H_{n-2i}(G,k)$$
$$\widetilde{HP}_n^{(\epsilon,1)}(kG) \cong \bigoplus_{i=n\ (mod\ 2)} H_i(G,k).$$

Example 4.3. *Let G be a discrete group and $\mathcal{H} = \mathbb{C}G$. Then the algebra $\mathcal{H}$ is a comodule algebra for the Hopf algebra $\mathcal{H}$ via coproduct map $\mathcal{H} \longrightarrow \mathcal{H}\otimes\mathcal{H}$. The map $Tr : \mathbb{C}G \to \mathbb{C}$ defined by*

$$Tr(g) = \begin{cases} 1 & g = e \\ 0 & g \neq e \end{cases}$$

is a δ-invariant σ-trace for $\delta = \epsilon$, $\sigma = 1$. The dual characteristic map $\gamma^* : \widetilde{HC}^n_{(\epsilon,1)}(\mathbb{C}G) \to HC^n(\mathbb{C}G)$ combined with the inclusion $H^n(G,\mathbb{C}) \hookrightarrow \widetilde{HC}^n_{(\epsilon,1)}(\mathbb{C}G)$ is exactly the map $H^n(G,\mathbb{C}) \to HC^n(\mathbb{C}G)$ described earlier in this section.

It would be very interesting to compute the Hopf cyclic homology $\widetilde{HC}_\bullet$ for compact quantum groups. Of course, one should look at algebra of polynomial or smooth functions on compact quantum groups, the C^*-completion being uninteresting from cyclic theory point of view. In the following we recall two results that are known so far about quantum groups.

Let k be a field of characteristic zero and $q \in k$, $q \neq 0$ and q not a root of unity. The Hopf algebra $\mathcal{H} = A(SL_q(2,k))$ is defined as follows. As an algebra it is generated by symbols a, b, c, d, with the following relations:

$$ba = qab, \quad ca = qac, \quad db = qbd, \quad dc = qcd,$$

$$bc = cb, \quad ad - q^{-1}bc = da - qbc = 1.$$

The coproduct, counit and antipode of $\mathcal{H}$ are defined by

$$\Delta(a) = a\otimes a + b\otimes c, \quad \Delta(b) = a\otimes b + b\otimes c$$

$$\Delta(c) = c\otimes a + d\otimes c, \quad \Delta(d) = c\otimes b + d\otimes d$$

$$\epsilon(a) = \epsilon(d) = 1, \quad \epsilon(b) = \epsilon(c) = 0,$$

$$S(a) = d, \quad S(d) = a, \quad S(b) = -qb, \quad S(c) = -q^{-1}c.$$

For more details about $\mathcal{H}$ we refer to [19]. Because $S^2 \neq id$, to define our cyclic structure we need a modular pair (σ, δ) in involution. Let δ be as follows:

$$\delta(a) = q, \quad \delta(b) = 0, \quad \delta(c) = 0, \quad \delta(d) = q^{-1}.$$

And $\sigma = 1$. Then we have $\widetilde{S}^2_{(1,\delta)} = id$.

For computing cyclic homology we should at first compute the Hochschild homology $H_*(\mathcal{H},k)$ where k is an $\mathcal{H}$-bimodule via δ, ϵ for left and right action of $\mathcal{H}$, respectively.

One knows $H_*(\mathcal{H},k) = Tor_*^{\mathcal{H}^e}(\mathcal{H},k)$, where $\mathcal{H}^e = \mathcal{H} \otimes \mathcal{H}^{op}$. So we need a resolution for k, or $\mathcal{H}$ as $\mathcal{H}^e$-module. We take advantage of the free resolution for $\mathcal{H}$ given by Masuda et al. [16]. By a lengthy computation one can check that $H_0(\mathcal{H},k) = 0$, $H_1(\mathcal{H},k) = H_2(\mathcal{H},k) = k \oplus k$, and $H_n(\mathcal{H},k) = 0$ for all $n \geq 3$. Moreover we find that the operator $B = (1-\tau)\sigma N : H_1(\mathcal{H},k) \longrightarrow H_2(\mathcal{H},k)$ is bijective and we obtain:

Theorem 4.3. *([16]) For any $q \in k$ which is not a root of unity one has $\widetilde{HC}_1(A(SL_q(2,k))) = k \oplus k$ and $\widetilde{HC}_n(A(SL_q(2,k))) = 0$ for all $n \neq 1$. In particular, $\widetilde{HP}_0(A(SL_q(2,k))) = \widetilde{HP}_1(A(SL_q(2,k))) = 0$.*

The above theorem shows that Theorem 4.2 is not true for non-cocommutative Hopf algebras.

The quantum universal enveloping algebra $U_q(sl(2,k))$ is an k-Hopf algebra which is generated as an k-algebra by symbols σ, σ^{-1}, x, y subject to the following relations

$$\sigma\sigma^{-1} = \sigma^{-1}\sigma = 1,\ \ \sigma x = q^2 x\sigma,\ \ \sigma y = q^{-2} y\sigma,\ \ xy - yx = \frac{\sigma - \sigma^{-1}}{q - q^{-1}}.$$

The coproduct, counit and antipode of $U_q(sl(2,k))$ are defined by:

$$\Delta(x) = x \otimes \sigma + 1 \otimes x,\ \ \Delta(y) = y \otimes 1 + \sigma^{-1} \otimes y,\ \ \Delta(\sigma) = \sigma \otimes \sigma,$$

$$S(\sigma) = \sigma^{-1},\ \ S(x) = -x\sigma^{-1},\ \ S(y) = -\sigma y,$$

$$\varepsilon(\sigma) = 1,\ \varepsilon(x) = \varepsilon(y) = 0.$$

It is easy to check that $S^2(a) = \sigma a \sigma^{-1}$, so that $(\sigma^{-1},\varepsilon)$ is a modular pair in involution. As the first step to compute its cyclic homology we should find its Hochschild homology group with trivial coefficients. (k is a $U_q(sl(2,k))$ bimodule via ε.) We define a free resolution for $\mathcal{H} = U_q(sl(2,k))$ as a $\mathcal{H}^e$-module as follows

$$(*) \qquad \mathcal{H} \xleftarrow{\mu} M_0 \xleftarrow{d_0} M_1 \xleftarrow{d_1} M_2 \xleftarrow{d_2} M_3 \ldots$$

where M_0 is $\mathcal{H}^e$, M_1 is the free $\mathcal{H}^e$-module generated by symbols $1\otimes e_\sigma, 1\otimes e_x, 1\otimes e_y$, M_2 is the free $\mathcal{H}^e$-module generated by symbols $1\otimes e_x \wedge e_\sigma, 1\otimes e_y \wedge e_\sigma, 1 \otimes e_x \wedge e_y$, and finally M_3 is generated by $1 \otimes e_x \wedge e_y \wedge e_\sigma$ as a free $\mathcal{H}^e$-module. We let $M_n = 0$ for all $n \geq 4$. We claim that with the following boundary operators, $(*)$ is a free resolution for $\mathcal{H}$:

$$
\begin{aligned}
&d_0(1\otimes e_x)=x\otimes 1-1\otimes x\\
&d_0(1\otimes e_y)=y\otimes 1-1\otimes y\\
&d_0(1\otimes e_\sigma)=\sigma\otimes 1-1\otimes \sigma\\
&d_1(1\otimes e_x\wedge e_\sigma)=(\sigma\otimes 1-1\otimes q^2\sigma)\otimes e_\sigma-(q^2x\otimes 1-1\otimes x)\otimes e_x\\
&d_1(1\otimes e_y\wedge e_\sigma)=(\sigma\otimes 1-1\otimes q^{-2}\sigma)\otimes e_\sigma-(q^{-2}y\otimes 1-1\otimes y)\otimes e_y\\
&d_1(1\otimes e_x\wedge e_y)=(y\otimes 1-1\otimes y)\otimes e_x-(x\otimes 1-1\otimes x)\otimes e_y\\
&\qquad\qquad+\frac{1}{q-q^{-1}}(\sigma^{-1}\otimes\sigma^{-1}+1\otimes 1)\otimes e_\sigma\\
&d_2(1\otimes e_x\wedge e_y\wedge e_\sigma)=(y\otimes 1-1\otimes q^2y)\otimes e_x\wedge e_\sigma\\
&\qquad -q^2(q^2x\otimes 1-1\otimes x)\otimes e_y\wedge e_\sigma+q^2(\sigma\otimes 1-1\otimes\sigma)\otimes e_y\wedge e_x
\end{aligned}
$$

To show that this complex is a resolution, we need a homotopy map. First we recall that the set $\{\sigma^l x^m y^n \mid l\in\mathbb{Z}, m,n\in\mathbb{N}_0\}$ is a P.B.W. type basis for $\mathcal{H}$ [19]. Let

$$\phi(a,b,n)=(a^{n-1}\otimes 1+a^{n-1}\otimes b\ldots a\otimes b^{n-1}+1\otimes b^{n-1})$$

where $n\in\mathbb{N}, a\in\mathcal{H}, b\in\mathcal{H}^o$, and $\phi(a,b,0)=0$, and $\omega(p)=1$ if $p\geq 0$ and 0 otherwise.

The following maps define a homotopy map for $(*)$, i.e. $sd+ds=1$:

$$
\begin{aligned}
&S_{-1}:\mathcal{H}\to M_0,\\
&S(a)=1\otimes a,\\
&S_0:M_0\to M_1,\\
&S_0(\sigma^l x^m y^n\otimes b)=(1\otimes b)((\sigma^l x^m\otimes 1)\phi(y,y,n)\otimes e_y+\\
&\qquad +(\sigma^l\otimes y^n)\phi(x,x,m)\otimes e_x)+\omega(l)(1\otimes x^m y^n)\phi(\sigma,\sigma,l)\otimes e_\sigma\\
&\qquad\qquad +(\omega(l)-1)(1\otimes x^m y^n)\phi(\sigma^{-1},\sigma^{-1},-l)(\sigma^{-1}\otimes\sigma^{-1}\otimes e_\sigma),\\
&S_1:M_1\to M_2,\\
&S_1(\sigma^l x^m y^n\otimes b\otimes e_y)=0,\\
&S_1(\sigma^l x^m y^n\otimes b\otimes e_x)=(1\otimes b)((\sigma^l x^m\otimes 1)\phi(y,y,n)\otimes e_x\wedge e_y\\
&+\frac{1-q^{2n}}{(q-q^{-1})(1-q^2)}(\sigma^l\otimes y^{n-1})\phi(x,x,m)(\sigma^{-1}\otimes\sigma^{-1}+q^{-2}\otimes 1)\otimes e_x\wedge e_\sigma\\
&\qquad +\frac{1}{q-q^{-1}}(\sigma^l x^m\otimes 1)\phi(y,y,n-1)(\sigma^{-1}\otimes\sigma{-1}+q^2\otimes 1)\otimes e_y\wedge e_\sigma),\\
&S_1(\sigma^l x^m y^n\otimes b\otimes e_\sigma)=(1\otimes b)(q^2(\sigma^l x^m\otimes 1)\phi(y,q^2y,n)\otimes e_y\wedge e_\sigma\\
&\qquad\qquad\qquad +q^{2(n-1)}(\sigma^l\otimes y^n)\phi(x,q^{-2}x,m)\otimes e_x\wedge e_\sigma),
\end{aligned}
$$

$$S_2 : M_2 \to M_3,$$
$$S_2(a \otimes b \otimes e_x \wedge e_y) = 0,$$
$$S_2(a \otimes b \otimes e_y \wedge e_\sigma) = 0,$$
$$S_2(\sigma^l x^m y^n \otimes b \otimes e_x \wedge e_\sigma) = (1 \otimes b)(\sigma^l x^m \otimes 1)\phi(y, q^2 y, n) \otimes e_x \wedge e_y \wedge e_\sigma,$$
$$S_n = 0 : M_n \to M_{n+1} \quad \text{for} \quad n \geq 3.$$

Again, by a rather long, but straightforward computation, we can check that $ds + sd = 1$. By using the definition of Hochschild homology as $Tor^{\mathcal{H}^e}(\mathcal{H}, k)$ we have the following theorem:

Theorem 4.4. *([16]) $H_0(U_q(sl(2,k)), k) = k$ and $H_n(U_q(sl(2,k)), k) = 0$ for all $n \neq 0$ where k is $U_q(sl(2,k))$-bimodule via ε for both sides.*

Corollary 4.2. *$\widetilde{HC}_n(U_q(sl(2,k))) = k$ when n is even, and 0 otherwise.*

5 Cohomology of Hopf algebroids

In their study of index theory for transversely elliptic operators and in order to treat the general non-flat case, Connes and Moscovici [6] had to replace their Hopf algebra $\mathcal{H}_n$ by a so-called "extended Hopf algebra" $\mathcal{H}_{FM}$. In fact $\mathcal{H}_{FM}$ is neither a Hopf algebra nor a Hopf algebroid in the sense of [21], but it has enough structure to define a cocyclic module similar to Hopf algebras [9, 8, 7].

In attempting to define a cyclic cohomology theory for Hopf algebroids in general, we were led instead to define a closely related concept that we call an extended Hopf algebra. This terminology is already used in [6]. All examples of interest, including the Connes-Moscovici algebra $\mathcal{H}_{FM}$, are extended Hopf algebras.

Our first goal in this section is to recall the definition of *extended Hopf algebra* from [18]. This is closely related to, but different from, *Hopf algebroids* in [21, 31]. The reason we prefer this concept to Hopf algebroids is that it is not clear how to define cyclic homology of Hopf algebroids, but it can be defined for extended Hopf algebras as we will recall from [18]. The whole theory is motivated by [6].

Broadly speaking, extended Hopf algebras and Hopf algebroids are quantizations (i.e. not necessarily commutative or cocommutative analogues) of groupoids and Lie algebroids. This should be compared with the point of view that Hopf algebras are quantization of groups and Lie algebras. Commutative Hopf algebroids were defined as cogroupoid objects in the category of commutative algebras in [26], the main example being algebra of functions on a groupoid. The concept was later generalized to allow noncommutative total algebras. A decisive step was taken in [21] where both total and base algebra are allowed to be noncommutative. From the point of view of noncommutative geometry, however , this definition is too restrictive because it demands the existing of a section $\gamma : H \otimes_R H \to H \otimes H$ and at the same time its coproduct $\Delta : H \to H \otimes_R H$ is not an "anticoalgebra" map.

To define a cocyclic module one does not need γ, but one needs an *antipode pair* $(S, \widetilde{S})$ as we define below. Motivated by this observation and also the fundamental work of [6], we were led to define extended Hopf algebras and their cocyclic modules.

Recall from [21, 31] that a *bialgebroid* $(H, R, \Delta, \varepsilon)$ consist of

1: An algebra H, an algebra R, an algebra homomorphism $\alpha : R \to H$, and an algebra antihomomorphism $\beta : R \to H$ such that the images of α and β commute in H. It follows that H can be regarded as R-bimodule via $axb = \alpha(a)\beta(b)x$, $a, b \in R$ $x \in H$.
   H is called the *total algebra*, R the *base algebra*, α the *source map* and β the *target map*.

2: A coproduct, i.e. an (R, R)-bimodule map $\Delta : H \to H \otimes_R H$ with $\Delta(1) = 1 \otimes_R 1$ satisfying the following conditions

   i) Coassociativity :

$$(\Delta \otimes_R id_H)\Delta = (id_H \otimes_R \Delta)\Delta : H \to H \otimes_R H \otimes_R H.$$

   ii) Compatibility with product:

$$\Delta(a)(\beta(r) \otimes 1 - 1 \otimes \alpha(r)) = 0 \text{ in} H \otimes_R H \text{ for any } r \in R \quad a \in H$$

$$\Delta(ab) = \Delta(a)\Delta(b) \text{ for any } a, b \in H.$$

3: A counit, i.e. an (R, R)-bimodule map $\epsilon : H \to R$ satisfying
   $\epsilon(1_H) = 1_R$ and $(\epsilon \otimes_R id_H)\Delta = (id_H \otimes_R \epsilon)\Delta = id_H : H \to H$.

Definition 5.1. *Let $(H, R, \alpha, \beta, \Delta, \varepsilon)$ be a k-bialgebroid. We call it a Hopf algebroid if there is a bijective map $S : H \to H$ which is a antialgebra map satisfying the following conditions,*

i) $S\beta = \alpha$.

ii) $m_H(S \otimes id)\Delta = \beta\epsilon S : H \to H$.

iii) There exists a linear map $\gamma : H \otimes_R H \to H \otimes H$ satisfying $\pi \circ \gamma = id_{H \otimes_R H} : H \otimes_R H \to H \otimes_R H$ and $m_H(id \otimes S)\gamma\Delta = \alpha\epsilon : H \to H$ where $\pi : H \otimes H \to H \otimes_R H$ is the natural projection.

Definition 5.2. *Let (H, R) be a bialgebroid. An antipode pair $(S, \widetilde{S})$ consists of maps $S, \widetilde{S} : H \to H$ such that*

(i) S and $\widetilde{S}$ are antialgebra maps.

(ii) $\widetilde{S}\beta = S\beta = \alpha$.

(iii) $m_H(S \otimes id)\Delta = \beta\epsilon S : H \to H$ *and* $m_H(\widetilde{S} \otimes id)\Delta = \beta\epsilon\widetilde{S} : H \to H$.

(iv) S is an anticoalgebra map, i.e.

$$\Delta S(h) = \sum S(h^{(2)}) \otimes_R S(h^{(1)}) \quad \textit{for all } h \in H.$$

(v) $\widetilde{S}$ is a twisted anticoalgebra map, i.e.

$$\Delta \widetilde{S}(h) = \sum S(h^{(2)}) \otimes_R \tilde{S}(h^{(1)}) \quad \textit{for all } h \in H.$$

Remark. The exchange operator $H \otimes_R H \to H \otimes_R H$, $x \otimes_R y \mapsto y \otimes_R x$, is not well-defined in general. It is, however, part of our assumption that for all $h \in H$, $\sum S(h^{(2)}) \otimes_R S(h^{(1)})$ and $\sum S(h^{(2)}) \otimes_R \tilde{S}(h^{(1)})$ are well-defined and equal to $\Delta(S(h))$ and $\Delta\tilde{S}(h)$ respectively. This happens in all examples of interest in this paper. We have relaxed the condition *(iii)* of Definition 5.1, the existence of a section, and added the extra conditions on S and $\widetilde{S}$. This is motivated by the fact that many examples do not admit a section and also the section is not needed to define a cocyclic module. The extra conditions are needed to define a cocyclic module.

Let (H, R) be a bialgebroid with an operator $S : H \to H$ such that S satisfies axioms *(i),(ii)* of Definition 5.1 and S is an anticoalgebra map. Assume R is commutative and $\alpha, \beta : R \to Z(H)$, where Z denotes the center. Let δ be a *character* in the sense that $\delta : H \to R$ is an algebra map, and $\delta\beta = id_R$. Let $\widetilde{S} = \delta \star S$ be the convolution of δ and S, so that

$$\widetilde{S}(h) = \sum \beta\delta(h^{(1)}) S(h^{(2)}).$$

Then $\widetilde{S}$ is well defined and we have:

Lemma 5.1. *By above construction $(S, \widetilde{S})$ is an antipode pair on H.*

Now let $\mathcal{H}_{FM}$ be the Connes-Moscovici algebra [6]. The operators S and δ are defined by

$$\begin{array}{lll} S(Y_i^j) = -Y_i^j, & S(X_k) = -X_k + \delta_{kj}^i Y_i^j & S(\delta_{kj}^i) = -\delta_{kj}^i \\ \delta(Y_i^j) = \delta_i^j, & \delta(X_k) = 0 & \delta(\delta_{kj}^i) = 0. \end{array}$$

One can check that $\delta \star S$ is equal to the twisted antipode $\widetilde{S}$ of Connes-Moscovici. Since $\widetilde{S}^2 = id_H$, we have

Corollary 5.1. *The Connes-Moscovici algebra $\mathcal{H}_{FM}$ is an extended Hopf algebra.*

We give a few more examples of extended Hopf algebras.

1. Let $\mathcal{H}$ be a Hopf algebra over k with a modular pair $(\delta, 1)$ in involution. Then $(\mathcal{H}, \alpha, \beta, \Delta, \epsilon, S, \widetilde{S})$ is an extended Hopf algebra, where $\alpha = \beta : k \longrightarrow \mathcal{H}$ is the unit map. More generally, given any algebra R, let $H = R \otimes \mathcal{H} \otimes R^o$.

With the following structure H is a Hopf algebroid and extended Hopf algebra over R:

$$\begin{aligned}
&\alpha(a) = a \otimes 1 \otimes 1, \quad \beta(a) = 1 \otimes 1 \otimes a,\\
&\Delta(a \otimes h \otimes b) = a \otimes h^{(1)} \otimes 1 \otimes_R 1 \otimes h^{(2)} \otimes b\\
&\epsilon(a \otimes h \otimes b) = \epsilon(h)ab, \quad S(a \otimes h \otimes b) = (b \otimes S(h) \otimes a)\\
&\widetilde{S}(a \otimes h \otimes b) = (b \otimes \widetilde{S}(h) \otimes a)
\end{aligned}$$

2. The universal enveloping algebra, $U(L, R)$, of a Lie-Rinehart algebra (L, R) is an example of an extended Hopf algebra over the algebra R where the other structures are:

$$\left.\begin{array}{ll}
\Delta(X) = X \otimes_R 1 + 1 \otimes_R X & \Delta(r) = r \otimes_R 1\\
\epsilon(X) = 0 & \epsilon(r) = r\\
S(X) = -X & S(r) = r
\end{array}\right\} \forall r \in R \text{and} X \in L.$$

The source and target maps are the natural embeddings, and $\widetilde{S} = S$. Lie-Rinehart algebras are defined further in this section.

3. Let $\mathcal{G}$ be a groupoid over a finite base (i.e., a category with a finite set of objects, such that each morphism is invertible). Then the groupoid algebra $H = k\mathcal{G}$ is generated by morphism $g \in \mathcal{G}$ with unit $1 = \sum_{X \in \mathcal{O}bj(\mathcal{G})} id_X$, and the product of two morphisms is equal to their composition if the latter is defined and 0 otherwise. It becomes a Hopf algebroid over $R = k\mathcal{S}$, where $\mathcal{S}$ is the subgroupoid of $\mathcal{G}$ whose objects are those of $\mathcal{G}$ and $\mathcal{M}or(X, Y) = id_X$ whenever $X = Y$ and $\emptyset$ otherwise, for all $X, Y \in \mathcal{O}bj(\mathcal{G})$.
The Hopf algebroid structure of H is given by:
$\alpha = \beta : R \longrightarrow H$ are natural embeddings.
The coproduct map $\Delta : H \longrightarrow H \otimes_R H$ is $\Delta(g) = g \otimes_R g$.
The counit map $\epsilon : H \longrightarrow R$ by $\epsilon(g) = id_{target(g)}$.
The antipode pair $\widetilde{S} = S : H \longrightarrow H$ by $S(g) = g^{-1}$.
The section map $\gamma : H \otimes_R H \longrightarrow H \otimes H$ by $\gamma(h \otimes_R g) = h \otimes g$.

It can be easily checked that H is both a Hopf algebroid and an extended Hopf algebra.

Given an extended Hopf algebra (H, R) we define a cocyclic module $H_\natural^\bullet$ as follows:

$$H_\natural^0 = R, \text{ and } H_\natural^n = H \otimes_R \otimes_R \cdots \otimes_R H \qquad (n \text{ factors}), \quad n \geq 1.$$

The coface, codegeneracy and cyclic actions δ_i, σ_i and τ are defined by

$$\begin{aligned}
\delta_0(h_1 \otimes_R \cdots \otimes_R h_n) &= 1_H \otimes_R h_1 \otimes_R \cdots \otimes_R h_n\\
\delta_i(h_1 \otimes_R \cdots \otimes_R h_n) &= h_1 \otimes_R \cdots \otimes_R \Delta(h_i) \otimes_R \cdots \otimes_R h_n \text{ for } 1 \leq i \leq n\\
\delta_{n+1}(h_1 \otimes_R \cdots \otimes_R h_n) &= h_1 \otimes_R \cdots \otimes_R h_n \otimes_R 1_H\\
\sigma_i(h_1 \otimes_R \cdots \otimes_R h_n) &= h_1 \otimes_R \cdots \otimes_R \epsilon(h_{i+1}) \otimes_R \cdots \otimes_R h_n \text{ for } 0 \leq i \leq n\\
\tau(h_1 \otimes_R \cdots \otimes_R h_n) &= \Delta^{n-1}\widetilde{S}(h_1) \cdot (h_2 \otimes \cdots \otimes h_n \otimes 1_H).
\end{aligned}$$

These formulas were obtained in [6] by transporting a cocyclic submodule of $A_\natural^\bullet$ via a faithful trace to $\mathcal{H}_{FM}{}_\natural^\bullet$, where A is an algebra on which $\mathcal{H}_{FM}$ acts. In [18] we proved directly that these formulas define a cocyclic module for any extended Hopf algebra.

Theorem 5.1. *[18] For any extended Hopf algebra (H, R), the above formulas define a cocyclic module on $H_\natural^\bullet$.*

The periodic cyclic cohomology of the universal enveloping algebra of Lie-Rinehart algebras is computed in [18]. Lie-Rinehart algebras interpolate between Lie algebras and commutative algebras, exactly in the same way that groupoids interpolate between groups and spaces. In fact Lie-Rinehart algebras can be considered as the infinitesimal analogue of groupoids. In the following all algebras are unital, and all modules are unitary. For more information on Lie-Rinehart algebras one can see [2, 15, 27].

Let k be a commutative ring. A Lie-Rinehart algebra over k is a pair (L, R) where R is a commutative k-algebra, L is a k-Lie algebra and a left R- module, L acts on R by derivations $\rho : L \longrightarrow \mathcal{D}er_k(R)$ such that $\rho[X, Y] = [\rho(X), \rho(Y)]$ for all X, Y in L and the action is R-linear, and the Leibniz property holds:

$$[X, aY] = a[X, Y] + \rho(X)(a)Y \quad \text{for all} \quad X, Y \in L \text{ and } a \in R.$$

Instead of $\rho(X)(a)$ we simply write $X(a)$.

Example 5.1. *Let $R = C^\infty(M)$ be the algebra of smooth functions on a manifold M and $L = C^\infty(TM) = \mathcal{D}er_{\mathbb{R}}(C^\infty(M))$, the Lie algebra of vector fields on M. Then (L, R) is a Lie-Rinehart algebra, where the action $\rho :\ L = \mathcal{D}er_{\mathbb{R}}(R) \longrightarrow \mathcal{D}er_{\mathbb{R}}(R)$ is the identity map.*

Example 5.2. *Let $R = C^\infty(M)$ and (L, R) a Lie-Rinehart algebra such that L is a finitely generated projective R-module. Then it follows from Swan's theorem that $L = C^\infty(E)$ is the space of smooth sections of a vector bundle over M. Since $\rho : C^\infty(E) \longrightarrow C^\infty(TM)$ is R-linear, it is induced by a bundle map $\rho : E \to TM$. In this way we recover Lie algebroids as a particular example of Lie-Rinehart algebras.*

Next we recall the definition of the homology of a Lie-Rinehart algebra [27]. This homology theory is a simultaneous generalization of Lie algebra homology and de Rham homology. Let (L, R) be a Lie-Rinehart algebra. A module over (L, R) is a left R-module M and a left Lie L-module $\varphi : L \to End_k(M)$, denoted by $\varphi(X)(m) = X(m)$ such that for all $X \in L$, $a \in R$ and $m \in M$,

$$\begin{aligned} X(am) &= aX(m) + X(a)m \\ (aX)(m) &= a(X(m)). \end{aligned}$$

Alternatively, we can say an (L, R)-module is an R-module endowed with a *flat connection* defined by $\nabla_X(m) = X(m),\ \ X \in L,\ m \in M.$

Let $C_n = C_n(L,R;M) = M \otimes_R \mathcal{A}lt_R^n(L)$, where $\mathcal{A}lt_R^n(L)$ denotes the n-th exterior power of the R-module L over R. Let $d : C_n \longrightarrow C_{n-1}$ be the differential defined by

$$\begin{aligned} d(m \otimes X_1 \wedge \cdots \wedge X_n) &= \textstyle\sum_{i=1}^{n} (-1)^{i-1} X_i(m) \otimes X_1 \wedge \cdots \wedge \hat{X}_i \cdots \wedge X_n \\ &+ \textstyle\sum_{1 \le i < j \le n} (-1)^{i+j} m \otimes [X_i, X_j] \wedge X_1 \cdots \wedge \hat{X}_i \cdots \wedge \hat{X}_j \cdots \wedge X_n. \end{aligned}$$

It is easy to check that $d^2 = 0$ and thus we have a complex (C_n, d). The homology of this complex is, by definition, the homology of the Lie-Rinehart algebra (L,R) with coefficients in M and we denote this homology by $H_*(R,L;M)$. To interpret this homology theory as a derived functor, Rinehart in [27] introduced the *universal enveloping algebra* of a Lie-Rinehart algebra (L,R). It is an associative k-algebra, denoted by $U(L,R)$, such that the category of (L,R)-modules as defined above is equivalent to the category of $U(L,R)$-modules. It is defined as follows.

One can see easily that the following bracket defines a k-Lie algebra structure on $R \oplus L$:

$$[r + X, s + Y] = [X, Y] + X(s) - Y(r) \ \text{ for } r, s \in R \text{ and } X, Y \in L.$$

Let $\tilde{U} = U(R \oplus L)$ be the enveloping algebra of the Lie algebra $R \oplus L$, and let $\tilde{U}^+$ be the subalgebra generated by the canonical image of $R \oplus L$ in U. Then $U(L,R) = \tilde{U}^+/I$, where I is the two sided ideal generated by the set $\{(r.Z)' - r'Z' \mid r \in R \text{ and } Z \in R \oplus L\}$. In [27] Rinehart showed that if L is a projective R-module, then

$$H_*(L,R;M) \cong Tor_*^{U(L,R)}(R,M).$$

Next we compute the cyclic cohomology groups of the extended Hopf algebra $U(L,R)$ of a Lie-Rinehart algebra (L,R). Let $S(L)$ be the symmetric algebra of the R-module L. It is an extended Hopf algebra over R. In fact it is the enveloping algebra of the pair (L,R) where L is an abelian Lie algebra acting by zero derivations on R. Let $\wedge(L)$ be the exterior algebra of the R-module L. The following lemma computes the Hochschild cohomology of the cocyclic module $S(L)_\natural$.

Lemma 5.2. *Let R be a commutative k-algebra and let L be a flat R-module. Then*

$$HH^*(S(L)_\natural) \cong \wedge^*(L).$$

The following proposition computes the periodic cyclic cohomology of the extended Hopf algebra $U(L,R)$ associated to a Lie-Rinehart algebra (L,R) in terms of its Rinehart homology. It extends a similar result for enveloping algebra of Lie algebras from [9].

Proposition 5.1. *([18]) If L is a projective R-module, then we have*

$$HP^n(U(L,R)) = \bigoplus_{i=n \ mod\ 2} H_i(L,R;R),$$

where HP^* means periodic cyclic cohomology.

Corollary 5.2. *Let M be a smooth closed manifold and $\mathcal{D}$ be the algebra of differential operators on M. It is an extended Hopf algebra and its periodic cyclic homology is given by*

$$HP_n(\mathcal{D}) = \bigoplus_{i=n \pmod 2} H^i_{dR}(M).$$

Proof. We have $\mathcal{D} = U(L, R)$, where $L = C^\infty(TM)$ and $R = C^\infty(M)$. Dualizing the above proposition, we obtain

$$HP_n(\mathcal{D}) = \bigoplus_{i=n \ (\mathrm{mod}\ 2)} H^i(L, R) = \bigoplus_{i=n \ (\mathrm{mod}\ 2)} H^i_{dR}(M).$$

□

Definition 5.3. *(Haar system for bialgebroids) Let (H, R) be a bialgebroid. Let $\tau : H \longrightarrow R$ be a right R-module map. We call τ a left Haar system for H if*

$$\sum_{(h)} \alpha(\tau(h^{(1)}))h^{(2)} = \alpha(\tau(h))1_H$$

and $\alpha\tau = \beta\tau$. We call τ a normal left Haar system if $\tau(1_H) = 1_R$.

We give a few examples of Haar systems. Let H be the Hopf algebroid of a groupoid with finite base. Then it is easy to see that $\tau : H \to R$ defined by $\tau(id_x) = id_x$ for all $x \in Obj(\mathcal{G})$ and 0 otherwise is a normal Haar system for H. This example can be generalized. Let $H = C_c^\infty(G)$ be the convolution algebra of a smooth e′tale (Hausdorff) groupoid. The map $\tau : C_c^\infty(G) \to C_c^\infty(G_0)$, defined as the transpose of the map $G_0 \to G$, $x \mapsto id_x$, is a normal Haar system for $C_c^\infty(G)$. In a related example, one can directly check that the map $\tau : A_\theta \to \mathbb{C}[U, U^{-1}]$ defined by

$$\tau(U^n V^m) = \delta_{m,0} U^n$$

is a normal Haar system for the noncommutative torus A_θ.

Proposition 5.2. *Let H be an extended Hopf algebra that admits a normal left Haar system. Then $HC^{2i+1}(H) = 0$ and $HC^{2i}(H) = \ker\{\alpha - \beta\}$ for all $i \geq 0$.*

Finally in this section we compute the Hopf periodic cyclic cohomology of commutative Hopf algebroids in terms of Hochschild cohomology. Given an extended Hopf algebra (H, R), we denote the Hochschild cohomology of the cocyclic module $H_\natural$ by $H^i(H, R)$. It is the cohomology of the complex

$$R \xrightarrow{d_0} H \xrightarrow{d_1} H \otimes_R H \xrightarrow{d_2} H \otimes_R H \otimes_R H \xrightarrow{d_3} \dots$$

where the first differential is $d_0 = \alpha - \beta$ and d_n is given by

$$d_n(h_1 \otimes_R \cdots \otimes_R h_n) = 1_H \otimes_R h_1 \otimes_R \cdots \otimes_R h_n + \sum_{i=1}^{n} (-1)^i h_1 \otimes_R \cdots \otimes_R \Delta(h_i) \otimes_R \cdots \otimes_R h_n + (-1)^{n+1} h_1 \otimes_R \cdots \otimes_R h_n \otimes_R 1_H.$$

Theorem 5.2. *([18]) Let (H, R) be a commutative Hopf algebroid. Then its periodic Hopf cyclic cohomology is given by*

$$HP^n(H) \cong \bigoplus_{i=n \ (mod\ 2)} H^i(H, R).$$

□

6 Cohomology of smash products

A celebrated problem in cyclic homology theory is to compute the cyclic homology of the crossed product algebra $A \ltimes G$, where the group G acts on the algebra A by automorphisms. If G is a discrete group, there is a spectral sequence, due to Feigin and Tsygan [13], which converges to the cyclic homology of the crossed product algebra. This result generalizes Burghelea's calculation of the cyclic homology of a group algebra [20]. In [14] Getzler and Jones gave a new proof of this spectral sequence using their Eilenberg-Zilber theorem for cylindrical modules. In [1], this spectral sequence has been extended to all Hopf algebras with invertible antipode. In this section we recall this result.

Let $\mathcal{H}$ be a Hopf algebra and A an $\mathcal{H}$-module algebra. We define a bicomplex, in fact a cylindrical module, $A\natural\mathcal{H}$ as follows: Let

$$(A\natural H)p,q = \mathcal{H}^{\otimes(p+1)} \otimes A^{\otimes(q+1)} \qquad p,q \geq 0.$$

The vertical and horizontal operators, $\tau^{p,q}$, $\delta^{p,q}$, $\sigma^{p,q}$ and $t^{p,q}$,$d^{p,q}$, $s^{p,q}$ are defined by

$$\tau^{p,q}(g_0,\dots,g_p \mid a_0,\dots,a_q) = (g_0^{(1)},\dots,g_p^{(1)} \mid S^{-1}(g_0^{(0)}g_1^{(0)}\dots g_p^{(0)})\cdot a_q, a_0,\dots,a_{q-1})$$
$$\delta_i^{p,q}(g_0,\dots,g_p \mid a_0,\dots,a_q) = (g_0,\dots,g_p \mid a_0,\dots,a_ia_{i+1},\dots,a_q) \quad 0 \leq i < q$$
$$\delta_q^{p,q}(g_0,\dots,g_p \mid a_0,\dots,a_q) = (g_0^{(1)},\dots,g_p^{(1)} \mid (S^{-1}(g_0^{(0)}g_1^{(0)}\dots g_p^{(0)})\cdot a_q)a_0,\dots,a_{q-1})$$
$$\sigma_i^{p,q}(g_0,\dots,g_p \mid a_0,\dots,a_q) = (g_0,\dots,g_p \mid a_0,\dots,a_i,1,a_{i+1},\dots,a_q) \quad 0 \leq i \leq q$$
$$t^{p,q}(g_0,\dots,g_p \mid a_0,\dots,a_q) = (g_p^{(q+1)},g_0,\dots,g_{p-1} \mid g_p^{(0)}\cdot a_0,\dots,g_p^{(q)}\cdot a_q)$$
$$d_i^{p,q}(g_0,\dots,g_p \mid a_0,\dots,a_q) = (g_0,\dots,g_ig_{i+1},\dots,g_p \mid a_0,\dots,a_q) \quad 0 \leq i < q$$
$$d_q^{p,q}(g_0,\dots,g_p \mid a_0,\dots,a_q) = (g_p^{(q+1)}g_0,g_1,\dots,g_{p-1} \mid g_p^{(0)}\cdot a_0,\dots,g_p^{(q)}\cdot a_q)$$
$$s_i^{p,q}(g_0,\dots,g_p \mid a_0,\dots,a_q) = (g_0,\dots,g_i,1,g_{i+1},\dots,g_p \mid a_0,\dots,a_q) \quad 0 \leq i \leq q.$$

Remark. The cylindrical module $A\natural\mathcal{H}$ in [1] is defined for all Hopf algebras. For applications, however, one has to assume that S is invertible. The above formulas are essentially isomorphic to those in [1], when S is invertible.

Theorem 6.1. *([1]) Endowed with the above operations, $A\natural\mathcal{H}$ is a cylindrical module.*

□

Corollary 6.1. *The diagonal $d(A\natural\mathcal{H})$ is a cyclic module.*

□

Our next task is to identify the diagonal $d(A\natural\mathcal{H})$ with the cyclic module of the smash product $(A\#\mathcal{H})_\natural$. Define a map $\phi : (A\#\mathcal{H})_\natural \to d(A\natural\mathcal{H})$ by

$$\phi(a_0 \otimes g_0, \dots, a_n \otimes g_n) =$$
$$(g_0^{(1)}, g_1^{(2)}, \dots, g_n^{(n+1)} \mid S^{-1}(g_0^{(0)} g_1^{(1)} \dots g_n^{(n)}) \cdot a_0, S^{-1}(g_1^{(0)} g_2^{(1)} \dots g_n^{(n-1)}) \cdot a_1, \dots$$
$$, S^{-1}(g_{n-1}^{(0)} g_n^{(1)}) \cdot a_{n-1}, S^{-1}(g_n^{(0)}) \cdot a_n)$$

By a long computation one shows that ϕ is a morphism of cyclic modules [1].

Theorem 6.2. *([1]) We have an isomorphism of cyclic modules $d(A\natural\mathcal{H}) \cong (A\#\mathcal{H})_\natural$.*

Proof. Define a map $\psi : d(A\natural\mathcal{H}) \to (A\#\mathcal{H})_\natural$ by $\psi(g_0, \dots, g_n \mid a_0, \dots, a_n) =$

$$((g_0^{(0)} g_1^{(0)} \dots g_n^{(0)}) \cdot a_0 \otimes g_0^{(1)}, (g_1^{(1)} \dots g_n^{(1)}) \cdot a_1 \otimes g_1^{(2)}, \dots, g_n^{(n)} \cdot a_n \otimes g_n^{(n+1)}).$$

Then one can check that $\phi \circ \psi = \psi \circ \phi = id$. □

Now we are ready to give an spectral sequence to compute the cyclic homology of the smash product $A\#\mathcal{H}$. By using the Eilenberg-Zilber theorem for cylindrical modules, we have:

Theorem 6.3. *There is a quasi-isomorphism of mixed complexes*

$$Tot((A\natural\mathcal{H})) \cong d(A\natural\mathcal{H}) \cong (A\#\mathcal{H})^\natural,$$

and therefore an isomorphism of cyclic homology groups,

$$HC_\bullet(Tot(A\natural\mathcal{H})) \cong HC_\bullet(A\#H).$$

□

Next, we show that one can identify the E^2-term of the spectral sequence obtained from the column filteration. To this end, we define an action of $\mathcal{H}$ on the first row of $A\natural\mathcal{H}$, denoted by $A^\natural_{\mathcal{H}} = \{\mathcal{H} \otimes A^{\otimes(n+1)}\}_{n\geq 0}$, by

$$h \cdot (g \mid a_0, \dots, a_n) = (h^{(n+1)} \cdot g \mid h^{(0)} \cdot a_0, \dots, h^{(n)} \cdot a_n)$$

where $h^{(n+1)} \cdot g = h^{(n+1)} g\, S^{-1}(h^{(n+2)})$ is an action of $\mathcal{H}$ on itself. We let $C_\bullet^{\mathcal{H}}(A)$ be the space of coinvariants of $\mathcal{H} \otimes A^{\otimes(n+1)}$ under the above action. So in $C_\bullet^{\mathcal{H}}(A)$, we have

$$h \cdot (g \mid a_0, \dots, a_n) = \epsilon(h)(g \mid a_0, \dots, a_n).$$

We define the following operators on $C_\bullet^{\mathcal{H}}(A)$,

$$\tau_n(g \mid a_0, \dots, a_n) = (g^{(1)} \mid (S^{-1}(g^{(0)}) \cdot a_n), a_0, \dots, a_{n-1})$$
$$\delta_i(g \mid a_0, \dots, a_n) = (g \mid a_0, \dots, a_i a_{i+1}, \dots, a_n)$$
$$\delta_n(g \mid a_0, \dots, a_n) = (g^{(1)} \mid (S^{-1}(g^{(0)}) \cdot a_n) a_0, a_1, \dots, a_{n-1})$$
$$\sigma_i(g \mid a_0, \dots, a_n) = (g \mid a_0, \dots, a_i, 1, a_{i+1}, \dots, a_n).$$

Proposition 6.1. *([1]) $C^{\mathcal{H}}_{\bullet}(A)$ with the operators defined above is a cyclic module.* □

Let M be a left $\mathcal{H}$-module. Then M is an H-bimodule if we let $\mathcal{H}$ act on the right on M via the counit map: $m.h = \varepsilon(h)m$. We denote the resulting Hochschild homology groups by $H_{\bullet}(\mathcal{H}, M)$. Explicitly it is computed from the complex $C_p(\mathcal{H}, M) = \mathcal{H}^{\otimes p} \otimes M, \quad p \geq 0$, with the differential $\delta : C_p(\mathcal{H}, M) \to C_{p-1}(\mathcal{H}, M)$ defined by

$$\begin{aligned}\delta(g_1, g_2, \dots, g_p, m) &= \epsilon(g_1)(g_2, \dots, g_p, m) \\ &+ \textstyle\sum_{i=1}^{p-1}(-1)^i(g_1, \dots, g_i g_{i+1}, \dots, g_p, m) + (-1)^p(g_1, \dots, g_{p-1}, g_p \cdot m).\end{aligned}$$

Let $C_q(A^{\natural}_{\mathcal{H}}) = \mathcal{H}^{\otimes q} \otimes A^{\natural}_{\mathcal{H}}$ and let $\mathcal{H}$ act on it by $h \cdot (g_1, \dots, g_p \mid m) = (g_1, \dots, g_p \mid h \cdot m)$, where the action of $\mathcal{H}$ on $A^{\natural}_{\mathcal{H}}$ is given by conjugation. So we can construct $H_p(\mathcal{H}, C_q(A^{\natural}_{\mathcal{H}}))$.

Now we can show that our original cylindrical complex $(A\natural\mathcal{H}, (\delta, \sigma, \tau), (d, s, t))$ can be identified with the cylindrical complex $(\mathsf{C}_p(\mathcal{H}, \mathsf{C}_q(A^{\natural}_{\mathcal{H}}), (\mathfrak{d}, \mathfrak{s}, \mathfrak{t}), (\bar{\mathfrak{d}}, \bar{\mathfrak{s}}, \bar{\mathfrak{t}}))$ under the transformations $\beta : (A\natural\mathcal{H})_{p,q} \to \mathsf{C}_p(\mathcal{H}, \mathsf{C}_q(A^{\natural}_{\mathcal{H}}))$ and $\gamma : \mathsf{C}_p(\mathcal{H}, \mathsf{C}_q(A^{\natural}_{\mathcal{H}})) \to (A\natural\mathcal{H})_{p,q}$ defined by

$$\beta(g_0, \dots, g_p \mid a_0, \dots, a_q) = (g_1^{(0)}, \dots, g_p^{(0)} \mid g_0 g_1^{(1)} \dots g_p^{(1)} \mid a_0, \dots, a_q)$$
$$\gamma(g_1, \dots, g_p \mid g \mid a_0, \dots, a_q) = (gS^{-1}(g_1^{(1)} \dots g_p^{(1)}), g_1^{(0)}, \dots, g_p^{(0)} \mid a_0, \dots, a_q).$$

One checks that $\beta\gamma = \gamma\beta = id$. To compute the homologies of the mixed complex $(Tot(C(A\natural\mathcal{H}), b + \bar{b} + \mathbf{u}(B + \bar{B}))$ we filter it by the subcomplexes (column filteration)

$$\mathsf{F}^i_{pq} = \sum_{q \leq i}(\mathcal{H}^{\otimes(p+1)} \otimes A^{\otimes(q+1)}).$$

Theorem 6.4. *([1]) The E^0-term of the spectral sequence is isomorphic to the complex*

$$\mathsf{E}^0_{pq} = (\mathsf{C}_p(\mathcal{H}, \mathsf{C}_q(A^{\natural}_{\mathcal{H}})), \delta)$$

and the E^1-term is

$$\mathsf{E}^1_{pq} = (H_p(\mathcal{H}, \mathsf{C}_q(A^{\natural}_{\mathcal{H}})), \mathfrak{b} + \mathbf{u}\mathfrak{B})).$$

The E^2-term of the spectral sequence is

$$\mathsf{E}^2_{pq} = HC_q(H_p(\mathcal{H}, \mathsf{C}_q(A^{\natural}_{\mathcal{H}}))),$$

the cyclic homologies of the cyclic module $H_p(\mathcal{H}, \mathsf{C}_q(A^{\natural}_{\mathcal{H}}))$.

References

[1] R. Akbarpour, and M. Khalkhali, Hopf Algebra Equivariant Cyclic Homology and Cyclic Homology of Crossed Product Algebras. arXiv: math.KT/0011248.

[2] A. Cannas da Silva, and A. Weinstein, Geometric models for noncommutative algebras. Berkeley Mathematics Lecture Notes, 10. American Mathematical Society, Providence, RI; Berkeley Center for Pure and Applied Mathematics, Berkeley, CA, 1999.

[3] A. Connes, Noncommutative geometry. Academic Press, Inc., San Diego, CA, 1994.

[4] A. Connes, Noncommutative differential geometry. Inst. Hautes tudes Sci. Publ. Math. No. 62 (1985), 257–360.

[5] A. Connes, C^* alg ebres et géométrie différentielle. (French) C. R. Acad. Sci. Paris Sér. A-B 290 (1980), no. 13, A599–A604.

[6] A. Connes and H. Moscovici, Differential cyclic cohomology and Hopf algebraic structures in transverse geometry arXiv:math.DG/012167 20 Feb 2001.

[7] A. Connes and H. Moscovici, Cyclic cohomology and Hopf algebra symmetry. Conference Mosh Flato 1999 (Dijon). Lett. Math. Phys. 52 (2000), no. 1, 1–28.

[8] A. Connes and H. Moscovici, Cyclic Cohomology, Hopf Algebras, and the Modular Theory, math.QA/9905013.

[9] A. Connes and H. Moscovici, Hopf algebras, Cyclic Cohomology and the transverse index theorem, Comm. Math. Phys. 198 (1998), no. 1, 199–246.

[10] A. Connes, Cohomologie cyclique et foncteurs Ext^n . (French) (Cyclic cohomology and functors Ext^n) C. R. Acad. Sci. Paris Sr. I Math. 296 (1983), no. 23, 953–958.

[11] M. Crainic, Cyclic cohomology of Hopf algebras. J. Pure Appl. Algebra 166 (2002), no. 1-2, 29–66.

[12] J. Cuntz, and D. Quillen, Algebra extensions and nonsingularity. J. Amer. Math. Soc. 8 (1995), no. 2, 251–289.

[13] B. Feĭgin, and B. L. Tsygan, Additive K-theory. K-theory, arithmetic and geometry (Moscow, 1984–1986), 67–209, Lecture Notes in Math., 1289, Springer, Berlin, 1987.

[14] E. Getzler, and J. D. S Jones, The cyclic homology of crossed product algebras. J. Reine Angew. Math. 445 (1993), 161–174.

[15] J. Huebschmann, Poisson cohomology and quantization. J. Reine Angew. Math. 408 (1990), 57–113.

[16] M. Khalkhali, and B. Rangipour, A new cyclic module for Hopf algebras. arXiv:math.KT/0010153.

[17] M. Khalkhali, and B. Rangipour, On the generalized cyclic Eilenberg-Zilber theorem. arXiv:math.QA/0106167.

[18] M. Khalkhali, and B. Rangipour, On cohomology of Hopf algebroids. arXiv: math.KT/0105105.

[19] A. Klimyk, and K. Schmüdgen, Quantum groups and their representations. Texts and Monographs in Physics. Springer-Verlag, Berlin, 1997.

[20] J. L. Loday, Cyclic Homology. Springer-Verlag, (1992).

[21] J. H. Lu, Hopf algebroids and quantum groupoids. Internat. J. Math. 7 (1996), no. 1, 47–70.

[22] S. Majid, Foundations of quantum group theory. Cambridge University Press, Cambridge, 1995.

[23] G. Maltsiniotis, Groupoids quantiques. C.R. Acad. Sci. Paris, 314 (1992), 249-252.

[24] S. Montgomery, Hopf algebras and their actions on rings. CBMS Regional Conference Series in Mathematics, 82.

[25] J. Mrčun, The Hopf algebroids of functions on étale groupoids and their principal Morita equivalence. J. Pure Appl. Algebra 160 (2001), no. 2-3, 249–262.

[26] D. C. Ravenel, Complex cobordism and stable homotopy groups of spheres. Pure and Applied Mathematics, 121. Academic Press Inc., Orlando, FL, 1986.

[27] G. Rinehart, Differential forms on general commutative algebras. Trans. Amer. Math. Soc. 108 1963 195–222.

[28] M. Sweedler, Hopf algebras. Mathematics Lecture Note Series W. A. Benjamin, Inc., New York 1969.

[29] R. Taillefer, Cyclic Homology of Hopf Algebras. K-Theory 24 (2001), no. 1, 69–85.

[30] J. C. Varilly, Hopf algebras in noncommutative geometry. hep-th/0109077.

[31] P. Xu, Quantum groupoids. Comm. Math. Phys. 216 (2001), no. 3, 539–581.