On Finite Moments of Full Busy Periods of $GI/G/c$ Queues

Saeed Ghahramani [1]

Abstract

For a $GI/G/c$ queue, a full busy period is a period commencing when an arrival finds $c-1$ customers in the system and ending when, for the first time after that, a departure leaves behind $c-1$ customers in the system. A probabilistic proof for the necessary and sufficient conditions for the finite moment conditions of the full busy periods of $GI/G/c$ queues is presented. The results on which this proof relies are much easier to prove than those previously used.

FULL AND PARTIAL BUSY PERIODS; REMAINING BUSY PERIODS; FAST SINGLE-SERVER

1 Introduction

A $GI/G/c$ queue is a system in which *customers* $C_1, C_2, \dots$ arrive respectively and, letting the inter-arrival time between C_n and C_{n+1} be T_n and the service time of C_n be S_n, $n = 1, 2, 3, \dots$, the random variables in the sequences $\{T_n\}$ and $\{S_n\}$ are mutually independent, the random variables within each sequence are identically distributed and customers are served in the order of their arrival by c *channels* (servers) operating in parallel. The inter-arrival and the service time distributions will be denoted by H and G, respectively, and $\rho = \lambda/c\mu$, where $1/\lambda = E(T_n)$, $1/\mu = E(S_n)$. We assume throughout that $\rho < 1$; under this assumption, the queue is *stable*. For $GI/G/c$ queues, a *full busy period*, first introduced by Kiefer and Wolfowitz in [7], is a period commencing when an arrival finds $c-1$ customers in the system and ending when, for the first time after that, a departure leaves behind $c-1$ customers in the system. By a *partial busy period* we mean a period commencing when an arrival finds no customers in the system and ends when, for the first time after that, a departing customer leaves behind no one in the system. An interval between successive partial busy periods is called an *idle period*, and an interval composed of a partial busy period and the immediately following idle period is called a *busy cycle*. Whitt in [10] has shown that if $P\{T_n - S_n > 0\} > 0$, then, with probability 1, customers find the system empty infinitely often. This means that with probability 1, there are infinitely many partial busy periods and cycles. At epochs where an arrival finds the system empty, the process restarts itself, and the system is regenerative. Let K be the number of customers served during a partial

[1]This short work is dedicated to my teacher, Professor Siavash Shahshahani. It is short only because it is from an administrator who is overwhelmed with administrative work.

busy period, since K is the number of transitions between returns to 0, Whitt also proved that $E(K) < \infty$. Now $B_p \leq \sum_{n=1}^{K} S_n$, where B_p is the corresponding partial busy period. From Wald's equation, we have $E(B_p) < \infty$. This shows that full busy periods also, if they occur, have finite means.

Let $\{X_1, X_2, \ldots\}$ be the sequence of busy cycles, where $\{X_i\}_{i=1}^{\infty}$ is i.i.d. Assuming $\lambda > 0$, we have that $E(X) = \sum_{n=1}^{K} T_n < \infty$. Every cycle contains a random number (possibly zero) of full busy periods. Let B_{ij} be the jth full busy period in cycle i and N_i be the number of full busy periods in the ith cycle. Clearly, $E(N_i) < E(K) < \infty$. We want to ensure that $P(N_i > 0) > 0$, i.e. that full busy periods occur. The following are sufficient conditions: either $P(T_n > \epsilon) > 0$ for any $\epsilon > 0$, or $P(S_n \leq t) < 1$ for any $t > 0$. If both of these conditions fail, then there is an upper bound on the number of arrivals that can occur during a service time and, for sufficiently large c, full busy periods never occur. For any finite c, there are weaker sufficient conditions, depending on c, that ensure the occurrence of a full busy period.

For a $GI/G/c$ queue, for $t \geq 0$, let $V(t)$, called the work in system at epoch t, be the sum of all service times of all customers in queue and the remaining service times of all customers in service at t. Also let

$$J(u,x) = \begin{cases} 1 & \text{if } V(u) > x \\ 0 & \text{otherwise,} \end{cases}$$

then V, the stationary work in system at any random epoch, has the following distribution

$$P(V > x) = \lim_{t \to \infty} \frac{1}{t} \int_0^t J(u,x)du$$

and is called *virtual work*. Kiefer and Wolfowitz in [6] establish conditions for the delay in queue to converge in distribution to a unique stationary distribution and conditions for the stationary distribution to have finite moments. For an elementary proof of their results and related references, see Wolff [11]. Ghahramani in [3] has shown that for a $GI/G/c$ queue with $\lambda > 0$ and $P\{T_n - S_n > 0\} > 0$ for any $r > 0$, $E(S^{r+1}) < \infty$ if and only if $E(V^r) < \infty$.

Thorisson in [9] has established moment and stochastic domination results for delay and recurrence times of a certain regenerative process associated with a certain multichannel queue leading to results of uniform rates of convergence. In [8] he has studied cycle variables of $GI/G/c$ queues and has established conditions for finite geometric moments and *ϕ-moments* for these variables, where $\phi(x) = x^n \psi(x)$ and ψ is a concave function.

Let B_f be a full busy period of a $GI/G/c$ queue. In this paper we prove the following theorem.

Theorem. *For a $GI/G/c$ queue with $\lambda > 0$, and $P(N_i > 0) > 0$ for any $r \geq 1$, $E(S_i^r) < \infty$ implies that $E(B_f^r) < \infty$.*

General moment results for the busy cycle length and for the number of customers served during a busy cycle are given in [8]. The proof we present here is a short probabilistic one, the results on which it relies are much easier to prove than those used by Thorisson.

2 Proof of the Main Result

We will now prove the theorem that we stated above. Let A be the event that *"a full busy period is in progress at a random time"* (see [1] for the precise definition of this notion). Let $1 - F(x)$ be the fraction of full busy periods that have length $> x$. Suppose that a full busy period is in progress at a random time, then given this, B_e, the remaining full busy period, has distribution

$$F_e(x) = \frac{1}{E(B_f)} \int_0^x (1 - F(u))du, \qquad x \geq 0,$$

the equilibrium distribution of F (see [4]). Therefore

$$E(B_e^{r-1}) = \frac{E(B_f^r)}{rE(B_f)},$$

and it suffices to prove that $E(S_i^r) < \infty$ implies that $E(B_e^{r-1}) < \infty$.

To do this, note that similar to the remaining full busy period, the stationary work in system at a random time has also a well defined distribution. Since a full busy period is found to be in progress, we denote by $(V \mid A)$ the corresponding random variable conditioned on this event. We will find an upper bound on B_e, random variable $\tilde{B}$ say, by a combination of two devices:

1. While B_e is in progress, work at each arrival epoch increases by the service time of the arriving customer, and decreases at rate c at all points in between. Suppose we compare work in system during B_e with work in system for a corresponding *"fast single service"* $GI/G/1$ queue with the same initial conditions, $(V \mid A)$ and arrival process. The service times for the $GI/G/1$ system are S_n/c for all n (this is the fast single channel system first introduced by Brumelle in [2]). For the $GI/G/1$ queue, work increases during B_e at the same arrival epochs and decreases at rate c at points in between, i.e. work in system is the *same stochastic process* during B_e. When B_e ends, work in system (usually) is still positive, and the $GI/G/1$ busy period will continue. Hence the remaining $GI/G/1$ busy period is an upper bound on B_e.

2. Now for the $GI/G/1$ queue, let the time between our random epoch and next arrival be T, and the service time of the next arrival be S. Suppose we *shift* the entire arrival process forward by T, so that the next arrival occurs at our

random epoch. Doing this can only make the remaining busy period longer. That is if $\tilde{B}$ denotes the remaining busy period with this shift, then

$$B_e \leq \tilde{B}. \tag{1}$$

With this shift, inter-arrival times are i.i.d., and $\tilde{B}$ has the distribution of an exceptional first service busy period with exceptional first service $S + (V \mid A)$.

Now $E(S^r) < \infty$ implies that $E(V^{r-1}) < \infty$ (see [3] for a short proof of this), so $E(S + (V \mid A))^{r-1} < \infty$ and hence by Lemma 2 of [5], $E(\tilde{B}^{r-1}) < \infty$. Therefore by (1), $E(B_e^{r-1}) < \infty$.

Q. E. D.

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