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Logarithmic Operators in Conformal Field Theory and The \mathcal{W}_∞ -algebra

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Abstract

It is shown explicitly that the correlation functions of Conformal Field Theories (CFT) with the logarithmic operators are invariant under the differential realization of Borel subalgebra of \mathcal{W}_∞ -algebra. This algebra is constructed by tensor-operator algebra of differential representation of ordinary $sl(2, \mathbb{C})$. This method allows us to write differential equations which can be used to find general expression for three and four-point correlation functions possessing logarithmic operators. The operator product expansion (OPE) coefficients of general logarithmic CFT are given up to third level.

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1 Introduction

It has been shown by Gurarie [1], that in the OPE of two given local fields which has at least two fields with the same conformal dimension, one can find some operators with a special property, known as logarithmic operators. As discussed in [1], these operators together with the ordinary operators form the basis of the Jordan cell for the operator L_0 . In some interesting physical theories, for example dynamics of polymers [2], WZW model on the $GL(1,1)$ super-group [3], percolation [4], edge excitation in fractional quantum Hall effect [5], one can find naturally such operators. Recently the role of such operators has been considered in the study of some physical problems, for instance: 2D-magnetohydrodynamic turbulence [6, 7, 8], 2D-turbulence [9, 10], $c_{p,1}$ models [19, 11], gravitationally dressed CFT's [12, 13] and in some critical disordered models [15, 16]. They also play an important role in so called unifying \mathcal{W} algebra [17] and in the description of normalizable zero modes for string backgrounds [14, 18].

The basic properties of logarithmic operators are, that they form part of the basis of the Jordan cell for L_0 whose in the correlators posses logarithmic singularity. It has been shown that in rational minimal models such a situation, i.e. two fields with the same conformal dimensions, does not occur [7]. The modular invariant partition functions for such theories with $c_{eff} = 1$ and the fusion rules of logarithmic conformal field theories (LCFT) are considered in [19, 20].

In this paper, we consider the symmetry algebra of the correlation functions with logarithmic factor. We take the modified operator L_0 according to ref. [1] and show that if certain conditions on the two point function of ordinary fields with logarithmic partner is satisfied, such correlation functions remain invariant under $sl(2, \mathbb{C})$ algebra. Here, we introduce another method for calculating logarithmic operators. This method is obtained by finding differential equations from some combinations of ordinary l_n 's differential realization. We show that this combination is related to the Borel subalgebra of the \mathcal{W}_λ -algebra, which is the wedge subalgebra of \mathcal{W}_∞ .

The structure of this paper is as follows: In section 2, we give a summary of LCFT and its relation to \mathcal{W}_∞ -algebra. In section 3, we calculate the general behavior of three and

four-point correlation functions of such theories. In section 4, we give the OPE coefficients up to third level of such CFT's.

2 The Logarithmic Operators and \mathcal{W}_∞ -algebra

According to [1], the OPE of two fields A and B , which their fusion rule [21] contains two fields Φ and Ψ of equal dimension, has a logarithmic term,

$$A(z, \bar{z})B(0, 0) = |z|^{2(\Delta_\Phi - \Delta_A - \Delta_B)} \{ \Psi(0, 0) + \dots + \log |z| [\Phi(0, 0) + \dots] \}, \quad (1)$$

where dots denote the descendants of fields Ψ and Φ . To see this, it is sufficient to look at the four-point function [21]:

$$\langle A(z_1)B(z_2)A(z_3)B(z_4) \rangle \sim \frac{1}{(z_1 - z_3)^{2\Delta_A}} \frac{1}{(z_2 - z_4)^{2\Delta_B}} \frac{1}{[x(1-x)]^{\Delta_A + \Delta_B - \Delta_\Phi}} F(x) \quad (2)$$

where the cross ratio x is given by:

$$x = \frac{(z_1 - z_2)(z_3 - z_4)}{(z_1 - z_3)(z_2 - z_4)}. \quad (3)$$

In degenerate models $F(x)$ satisfies a second order linear differential equation [21].

It should be noted that since logarithmic fields are not chiral, we can not separate holomorphic and anti-holomorphic parts. In this paper $(z_i - z_j)$ is set to $|z_i - z_j|^2$, for simplicity.

According to [7], the hypergeometric equation governing the correlator of the two fields in whose OPE, two other fields Ψ and Φ with conformal dimension Δ_Ψ and $\Delta_\Psi + \epsilon$ appear, admits two solutions

$${}_2F_1(a, b, c, x) \quad (4)$$

$$x^\epsilon {}_2F_1(a + \epsilon, b + \epsilon, c + 2\epsilon, x) \quad (5)$$

where a , b and c are sums of conformal dimension. Clearly in the limit of $\epsilon \rightarrow 0$ these two solutions coincide. However, it can be shown that [22] another independent solution exists which involves logarithms and can be generated by standard methods. Therefore, the following two independent solutions can be constructed according to [1,10].

$$\sum b_n x^n + \log x \sum a_n x^n. \quad (6)$$

Next, consistency of equation (1) and (2) requires that

$$\langle A(z_1)B(z_2)\Psi(z_3) \rangle = \langle A(z_1)B(z_2)\Phi(z_3) \rangle \left\{ \log \frac{(z_1 - z_2)}{(z_1 - z_3)(z_2 - z_3)} + \lambda \right\}, \quad (7)$$

$$\langle \Psi(z)\Psi(0) \rangle = -\frac{2}{z^{2\Delta_\Psi}} [\log z + \lambda'], \quad (8)$$

$$\langle \Psi(z)\Phi(0) \rangle = \frac{1}{z^{2\Delta_\Phi}}, \quad (9)$$

where λ and λ' are constants. In this notation Ψ is the logarithmic operator and Φ is the ordinary one. Let us now consider the action of $sl(2, \mathbb{C})$ on the above correlators. In the absence of logarithmic operators (e.g., in the rational minimal models [21, 7]) the correlation function is invariant under the action of $sl(2, \mathbb{C})$,

$$\sum_{i=1}^N \partial_{z_i} \langle \Phi_1(z_1) \cdots \Phi_N(z_N) \rangle = 0 \quad (10)$$

$$\sum_{i=1}^N (z_i \partial_{z_i} + \Delta_i) \langle \Phi_1(z_1) \cdots \Phi_N(z_N) \rangle = 0 \quad (11)$$

$$\sum_{i=1}^N (z_i^2 \partial_{z_i} + 2z_i \Delta_i) \langle \Phi_1(z_1) \cdots \Phi_N(z_N) \rangle = 0 \quad (12)$$

Indeed according to [21, 23] the operators

$$l_- = \partial_z, \quad l_0 = z\partial_z + \Delta, \quad l_+ = z^2\partial_z + 2z\Delta \quad (13)$$

are the differential realization of $sl(2, \mathbb{C})$, with $\partial_z = \frac{\partial}{\partial z}$. When there are logarithmic operators in CFT theories, the ordinary correlation functions, for instance ordinary two-point function, should be replaced by eq. (8). According to [1], in the logarithmic conformal field theory, l_0 is given by the following modified representation:

$$[L_0, A(z)] = (z\partial_z + \Delta_A + \mathcal{D})A(z), \quad (14)$$

where the operator \mathcal{D} is such that $\mathcal{D}\Phi(z) \equiv 0$ and $\mathcal{D}\Psi(z) \equiv \Phi$. This can be constructed by taking the OPE of $T(z)$ with $\Phi(0)$ and with $\Psi(0)$. That is to say

$$T(z)\Psi(0) = \frac{\Delta}{z^2}\Psi(0) + \frac{\partial_z \Psi(0)}{z} + \frac{\Phi(0)}{z^2} + \cdots, \quad (15)$$

$$T(z)\Phi(0) = \frac{\Delta}{z^2}\Phi(0) + \frac{\partial_z \Phi(0)}{z} + \cdots. \quad (16)$$

If we expand $T(z)$ by Laurant series we can write [1],

$$[l_n, \Psi(z)] = (z^{n+1}\partial_z + \Delta(n+1)z^n)\Psi(z) + (n+1)z^n\Phi(z), \quad (17)$$

$$[l_n, \Phi(z)] = (z^{n+1}\partial_z + \Delta(n+1)z^n)\Phi(z) \quad (18)$$

which means that we can redefine l_n to L_n such that:

$$[L_n, A(z)] = (z^{n+1}\partial_z + \Delta_A(n+1)z^n + (n+1)z^n\mathcal{D})A(z) \quad (19)$$

with the property $\mathcal{D}\Psi \equiv \Phi$ and $\mathcal{D}\Phi \equiv 0$. For example, one can show that eq.(8) is invariant under modified L_0 (i.e. non-diagonal L_0).

$$\begin{aligned} L_0 \langle \Psi(z)\Psi(0) \rangle &= \langle [L_0, \Psi(z)\Psi(0)] \rangle \\ &= \langle [(z\partial_z + \Delta_\Psi)\Psi(z) + \Phi(z)]\Psi(0) \rangle + \langle \Psi(z)\Delta_\Psi\Psi(0) \rangle + \langle \Psi(z)\Phi(0) \rangle \\ &= (z\partial_z + 2\Delta_\Psi) \langle \Psi(z)\Psi(0) \rangle + \langle \Psi(z)\Phi(0) \rangle + \langle \Phi(z)\Psi(0) \rangle . \end{aligned} \quad (20)$$

By using eqs.(8) and (9) we find that:

$$L_0 \langle \Psi(z)\Psi(0) \rangle = 0. \quad (21)$$

Conversely, the eq.(19) allows us to write Ward identities (for $n=-1, 0, 1$) for the calculation of the correlation functions. For instance, for two-point functions we have:

$$L_{-1} \langle \Psi(z)\Psi(0) \rangle = L_{-1} \langle \Psi(z)\Phi(0) \rangle = L_{-1} \langle \Phi(z)\Phi(0) \rangle = 0 \quad (22)$$

$$L_0 \langle \Psi(z)\Psi(0) \rangle = L_0 \langle \Psi(z)\Phi(0) \rangle = L_0 \langle \Phi(z)\Phi(0) \rangle = 0 \quad (23)$$

$$L_+ \langle \Psi(z)\Psi(0) \rangle = L_+ \langle \Psi(z)\Phi(0) \rangle = L_+ \langle \Phi(z)\Phi(0) \rangle = 0. \quad (24)$$

Using eq.(23) we find,

$$(z\partial_z + 2\Delta_\Psi) \langle \Psi(z)\Psi(0) \rangle + 2 \langle \Psi(z)\Phi(0) \rangle = 0, \quad (25)$$

$$(z\partial_z + 2\Delta_\Psi) \langle \Psi(z)\Phi(0) \rangle + \langle \Psi(z)\Phi(0) \rangle = 0, \quad (26)$$

$$(z\partial_z + 2\Delta_\Psi) \langle \Phi(z)\Phi(0) \rangle = 0, \quad (27)$$

which can be solved yeilding:

$$\langle \Phi(z)\Phi(0) \rangle = -\frac{a}{z^{2\Delta_\Psi}}, \quad (28)$$

$$\langle \Psi(z)\Phi(0) \rangle = \frac{1}{z^{2\Delta_\Psi}} [2a \log z + a'], \quad (29)$$

$$\langle \Psi(z)\Psi(0) \rangle = \frac{1}{z^{2\Delta_\Psi}} [2a \log^2 z - 2a' \log z + a''] \quad (30)$$

where so far a, a' and a'' are arbitrary parameters. One can show that eq.(24) leads to the following relation:

$$\langle \Phi(z)\Phi(0) \rangle = 0 \quad (31)$$

or $a = 0$ which has already been pointed out in [15]. This means that if we insist on having conformal invariance, the two-point function of ordinary fields which have logarithmic partner must be zero.

For $a \neq 0$ we don't have the conformal invariance, however the correlation functions are invariant under L_+ and L_- , which is the subalgebra of $sl(2, \mathbb{C})$. The condition (31) seems to be very restrictive on two-point correlation functions. For example, in $c = -2$ theory, which belongs to LCFT's of $c_{p,1}$ series (with $p=2$) [20], the requirement of the conformal invariance forced sets the two-point functions of fields with logarithmic partner to zero. In other words, the ordinary fields with logarithmic partners do not propagate.

Here, we introduce another method for finding certain differential equations which can be used to obtain the correlation functions by means of ordinary l_n 's. This approach is based on the simple observation that in applying the differential operator $(z\partial_z + 2\Delta_\Psi)$ (the differential representation of l_0) on eq.(8), we obtain:

$$l_0 \langle \Psi(z)\Psi(0) \rangle = \langle [l_0, \Psi(z)\Psi(0)] \rangle \equiv (z\partial_z + 2\Delta_\Psi) \langle \Psi(z)\Psi(0) \rangle = -2z^{-2\Delta_\Psi}. \quad (32)$$

which behaves as eq.(9). Similarly the action of ordinary l_\pm are as follows:

$$\langle [l_{-1}, \Psi(z)\Psi(0)] \rangle \equiv (\partial_z) \langle \Psi(z)\Psi(0) \rangle = 0 \quad (33)$$

$$\langle [l_+, \Psi(z)\Psi(0)] \rangle \equiv (z^2\partial_z + 2z\Delta_\Psi) \langle \Psi(z)\Psi(0) \rangle = -2z^{-2\Delta_\Psi+1}. \quad (34)$$

Now we consider the action of l_0 and l_+ on the correlation of $\langle \Psi(z)\Psi(0) \rangle$. This can be written in the following form:

$$\langle [l_0, [l_+, \Psi(z)\Psi(0)]] \rangle = (z^2\partial_z + 2z\Delta_\Psi)(z\partial_z + 2\Delta_\Psi) \langle \Psi(z)\Psi(0) \rangle =$$

$$(z^2\partial_z + 2z\Delta_\Psi)(z\partial_z + 2\Delta_\Psi)\left(-\frac{2}{z^{2\Delta_\Psi}}[\log z + \lambda']\right) = 0 \quad (35)$$

Indeed, this is the action of differential representation of l_0 and l_+ on the logarithmic two-point function.

At first glance, it seems that the logarithmic correlation function, $\langle \Psi(z_1)\Psi(z_2) \rangle$ is invariant under the set of

$$\{l_-, l_0^2, l_-l_0, l_+l_0, l_0l_-l_+, l_-^2l_+, l_+l_-l_+\}$$

where the last three of them are observed using the fact that l_-l_+ times such correlation functions behave as ordinary correlation functions in CFT. Using the commutation relation of l_0, l_+ and l_- ,

$$[l_0, l_\pm] = \pm l_\pm \quad [l_+, l_-] = -2l_0, \quad (36)$$

the last three members of Eq. (25) can be written in terms of the first four which we call (A, B, C, D) respectively.

Simple calculations show that, the following algebraic relations hold:

$$[A, B] = -2C - A, \quad [B, C] = 2AB + C, \quad [C, D] = -2l_0^3 - 2l_0 \dots \quad (37)$$

Explicit calculation shows that this algebra is not closed. In each step, we need to add new operators to the algebra.

On the other hand, the action of l_0 on logarithmic correlation functions behave as expected. However the two-point function is the exception. In the case of three-point and four-point correlation functions, the action of differential representation of l_0 on correlation functions which have logarithmic term, behave like ordinary three-point and four-point functions of CFT. We will come to this point in the next section.

In the rest of this section we determine the connection between the above algebra and the infinite dimensional \mathcal{W}_∞ -algebra. The algebraic commutation relation of V^i 's, generators of spin- $(i+2)$ \mathcal{W}_∞ -algebra, are as follow [24]:

$$[V_m^i, V_n^j] = \sum_{l \geq 0} g_{2l}^{ij}(m, n) V_{m+n}^{i+j-2l} + c_i(m) \delta^{i,j} \delta_{m+n,0} \quad (38)$$

where $g_{2l}^{ij}(m, n)$ are structure constants of the algebra and $c_i(m)$ are central terms. V_m^i denotes the m th Laurent mode of spin- $(i+2)$ current $V^i(z)$. In our case, this current can be

constructed from the logarithmic operator Ψ [14]. For \mathcal{W}_∞ -algebra with spins $2 \leq s < \infty$, the indices i and j range from 0 to ∞ . The structure constants and central terms can be determined by demanding that eq. (38) be consistent with the Jacobi identities [24, 25],

$$c_i(m) = m(m^2 - 1)(m^2 - 4) \cdots (m^2 - (i + 2)^2)c_i \quad (39)$$

where c_i 's are central charges, and the structure constants take the form

$$g_l^{ij}(m, n) = \frac{1}{2(l+1)!} \Phi_l^{ij} N_l^{ij}(m, n) \quad (40)$$

where $N_l^{ij}(m, n)$ are given by

$$N_l^{ij}(m, n) = \sum_{k=0}^{l+1} (-1)^k \binom{l+1}{k} [i+1+m]_{l+1-k} [i+1-m]_k [j+1+n]_k [j+1-n]_{l-1-k} \quad (41)$$

and $[a]_n = a(a-1) \cdots (a-n+1) = \frac{a!}{(a-n)!}$. The functions Φ_l^{ij} are given by

$$\Phi_l^{ij} = {}_4F_3 \left[\begin{matrix} -\frac{1}{2}, \frac{3}{2}, -\frac{l}{2} - \frac{1}{2}, -\frac{l}{2} \\ -i - \frac{1}{2}, -j - \frac{1}{2}, i + j - l + \frac{5}{2} \end{matrix} ; 1 \right] \quad (42)$$

where ${}_4F_3(1)$ is a generalized hypergeometric function [25]. Now consider a "wedge" of generators V_m^i for which $|m| \leq (i+1)$. It can be easily shown that the commutator of any two generators within a wedge only involves generators within the same wedge. The resulting wedge subalgebra of \mathcal{W}_∞ which is known as \mathcal{W}_\wedge , has the feature that it is anomaly free, the central terms vanish for all commutators. The algebra \mathcal{W}_\wedge can be realized by generalization of $sl(2)$, subalgebra of the Virasoro algebra, generated by l_+ , l_- and l_0 . Let us consider the tensor-operator algebra of $sl(2, \mathbb{C})$. The tensor operator algebra may be constructed in the following way. The $sl(2, \mathbb{C})$ generators satisfy the commutation relations of eqs. (36). The Casimir operator may be written as

$$Q = l_0^2 - \frac{1}{2}(l_+ l_- + l_- l_+). \quad (43)$$

The three generators, l_- , l_+ and l_0 transform as the $\mathbf{3}$ representations of $sl(2, \mathbb{C})$. Higher-tensor operators, T_m^l with $(-l \leq m \leq l)$, transforming in general as the $(2l+1)$ -dimensional representation of $sl(2, \mathbb{C})$, are constructed from appropriate polynomials of degree l in the

l 's. According to [25], we start with the highest-weight state $T_l^l = (l_+)^l$, and then construct the lower-weight states in the usual way by $(l - m)$ successive application of L_- ,

$$T_m^l = \frac{1}{(-2l)_{l-m}} [l_-, [l_-, \dots [l_-, (l_+)^l] \dots]] \quad (44)$$

where $(a)_n = a(a+1)\dots(a+n-1) = \frac{(a+n-1)!}{(a-1)!}$. Subtleties arise when we determine the exact relation of the above construction with the \mathcal{W}_∞ -algebra. Acting upon a Hilbert space in which the quadratic Casimir takes on a definite value $Q = \mu$, the operators T_m^l close into an infinite-dimensional algebra which is parametrized by μ , and known as $\mathcal{T}(\mu)$. Therefore, we would expect that the above construction for \mathcal{W}_\wedge -algebra should coincide with $\mathcal{T}(\mu)$ for some specific value of μ . It has been shown in ref. [24], that the wedge subalgebra \mathcal{W}_\wedge contained in \mathcal{W}_∞ -algebra is the $sl(2, \mathbb{C})$ operator algebra $\mathcal{T}(0)$, specified by the value $\mu = 0$ for the quadratic Casimir. However for any value of the parameter μ , we can allow the representation to be infinite dimensional [24]. It can be easily checked that in our case the logarithmic correlation functions are not invariant under all of T_m^l 's, but they are invariant under T_m^l with $-l \leq m \leq 0$. Therefore the important result is that the logarithmic correlation functions are invariant under the Borel subalgebra of \mathcal{W}_\wedge -algebra.

3 Correlation Functions with Logarithmic Behavior

In this section, we calculate the correlation functions that have a logarithmic term. A straightforward way for calculating the correlation functions is to use Ward identities [26]. For simplicity, first we consider the correlation functions of conformal fields where only two of them have two fields of equal dimension in their OPE. This can be done by using the corresponding symmetry algebra, which was found in the last section. It can be shown that in this case all correlation functions behave as follows:

$$l_0 \langle A(z_1)B(z_2)\mathcal{O}(z_3)\dots\mathcal{O}(z_n) \rangle = k_n \langle \mathcal{O}(z_1)\dots\mathcal{O}(z_n) \rangle \quad (45)$$

where the OPE of $A(z_{n-1})B(z_n)$ is given by eq. (1) and $\mathcal{O}(z_i)$ are ordinary operators. For the calculation of the two point function of such a conformal field theory, we have

$$l_0 \langle \Psi(z_1)\Psi(z_2) \rangle = k_2 \langle \mathcal{O}(z_1)\mathcal{O}(z_2) \rangle = k_2 \frac{1}{z_{12}^{2\Delta_\Psi}} \quad (46)$$

and the following Ward identities:

$$l_0 l_0 \langle \Psi(z_1) \Psi(z_2) \rangle = 0, \quad l_- \langle \Psi(z_1) \Psi(z_2) \rangle = 0, \quad l_+ l_0 \langle \Psi(z_1) \Psi(z_2) \rangle = 0, \quad (47)$$

and so on, where $z_{ij} = z_i - z_j$. Now we substitute the differential realization of l_0 and l_{\pm} from (13) into the above equations by means of the "co-product",

$$\begin{aligned} [z_1 \partial_{z_1} + z_2 \partial_{z_2} + 2\Delta_{\Psi}] \langle \Psi(z_1) \Psi(z_2) \rangle &= k_2 \langle \mathcal{O}(z_1) \mathcal{O}(z_2) \rangle \\ [(z_1 \partial_{z_1} + z_2 \partial_{z_2} + 2\Delta_{\Psi})(z_1 \partial_{z_1} + z_2 \partial_{z_2} + 2\Delta_{\Psi})] \langle \Psi(z_1) \Psi(z_2) \rangle &= 0 \\ [\partial_{z_1} + \partial_{z_2}] \langle \Psi(z_1) \Psi(z_2) \rangle &= 0 \\ [z_1^2 \partial_{z_1} + z_2^2 \partial_{z_2} + 2(z_1 + z_2) \Delta_{\Psi}] \langle \Psi(z_1) \Psi(z_2) \rangle &= 0 \end{aligned} \quad (48)$$

and so on. Solving the above equations for $\langle \Psi(z_1) \Psi(z_2) \rangle$, we have

$$\langle \Psi(z_1) \Psi(z_2) \rangle = k_2 \langle \mathcal{O}(z_1) \mathcal{O}(z_2) \rangle [\log z_{12} + \lambda'] \quad (49)$$

where k_2 is -2 and λ' is a constant.

A similar methods can be used for solving the three point correlation function. Instead of (46) we have

$$l_0 \langle A(z_1) B(z_2) \Psi(z_3) \rangle = k_3 \langle A(z_1) B(z_2) \Phi(z_3) \rangle = \frac{k_3}{z_{12}^{\Delta_A + \Delta_B - \Delta_{\Psi}} z_{13}^{\Delta_A - \Delta_B + \Delta_{\Psi}} z_{23}^{-\Delta_A + \Delta_B + \Delta_{\Psi}}} \quad (50)$$

and similar Ward identities for the three-point function. Solving these equations gives:

$$\langle A(z_1) B(z_2) \Psi(z_3) \rangle = k_3 \langle A(z_1) B(z_2) \Phi(z_3) \rangle (a \log z_{12} + b \log z_{13} + c \log z_{23} + \lambda) \quad (51)$$

where k_3, a, b and c must satisfy the following relation:

$$k_3 + a + b + c = 0. \quad (52)$$

For the four-point correlation function, $\langle A(z_1) B(z_2) \mathcal{O}(z_3) \mathcal{O}(z_4) \rangle$, we have

$$\begin{aligned} L_0 \langle A(z_1) B(z_2) \mathcal{O}_1(z_3) \mathcal{O}_2(z_4) \rangle &= k_4 \langle \mathcal{O}(z_1) \mathcal{O}(z_2) \mathcal{O}(z_3) \mathcal{O}(z_4) \rangle \\ &= \frac{k_4}{z_{12}^{\Delta_A + \Delta_B - \Delta_{\mathcal{O}_1} + \Delta_{\mathcal{O}_2}} z_{13}^{\Delta_A - \Delta_B + \Delta_{\mathcal{O}_1} - \Delta_{\mathcal{O}_2}} z_{23}^{-\Delta_A + \Delta_B + \Delta_{\mathcal{O}_1} - \Delta_{\mathcal{O}_2}} z_{34}^{2\Delta_{\mathcal{O}_2}}} F(x). \end{aligned} \quad (53)$$

Solving the relevant equations by a similar method, we find

$$\begin{aligned} \langle A(z_1) B(z_2) \mathcal{O}(z_3) \mathcal{O}(z_4) \rangle &= k_4 \langle \mathcal{O}(z_1) \mathcal{O}(z_2) \mathcal{O}(z_3) \mathcal{O}(z_4) \rangle \\ &\times [a \log z_{12} + b \log z_{13} + c \log z_{14} + d \log z_{23} + e \log z_{24} + f \log z_{34} + \lambda"] \end{aligned} \quad (54)$$

where λ'' is a constant and k_4, a, b, c, d, e and f must satisfy:

$$k_4 + a + b + c + d + e + f = 0. \quad (55)$$

It can be seen that eq. (54), is valid for four-point function of $\langle A(z_1)B(z_2)\mathcal{O}(z_3)\Psi(z_4) \rangle$ type. However correlation function of $\langle \Psi(z_1)\Psi(z_2)\Psi(z_3) \rangle$ type cannot be calculated by this method [27].

4 OPE Coefficients

4.1 OPE of Ordinary Conformal Fields

The most general expression for the operator product expansion of ordinary conformal fields is [21, 7]:

$$\Phi_n(z, \bar{z})\Phi_m(0, 0) = \sum_p \sum_k z^{\Delta_p - \Delta_n - \Delta_m + \sum k_i} \bar{z}^{\bar{\Delta}_p - \bar{\Delta}_n - \bar{\Delta}_m + \sum \bar{k}_i} C_{nm}^{p, \{k\}, \{\bar{k}\}} \Phi_p^{\{k\}, \{\bar{k}\}}(0, 0) \quad (56)$$

where the coefficients are

$$C_{nm}^{p, \{k\}, \{\bar{k}\}} = C_{nm}^p \beta_{nm}^{p, \{k\}} \bar{\beta}_{nm}^{p, \{\bar{k}\}}, \quad \{k\} = \{k_1, k_2, \dots, k_n\}. \quad (57)$$

This OPE is the first regular hypergeometric function $F(x)$. Note that we have $\Phi_i(0) | 0 \rangle = |\Delta_i \rangle$. Acting on $| 0 \rangle$, by eq. (56) we obtain:

$$\Phi_n(z) | \Delta_m \rangle = \sum_p C_{nm}^p z^{\Delta_p - \Delta_n - \Delta_m} \phi_p(z) | \Delta_p \rangle \quad (58)$$

$$\begin{aligned} \phi_p(z) &= \sum_k z^{\sum k_i} \beta_{nm}^{p, \{k\}} L_{-k_1} \cdots L_{-k_n} \\ | z, \Delta_p \rangle &= \phi_p(z) | \Delta_p \rangle. \end{aligned} \quad (59)$$

Expanding $| z, \Delta_p \rangle$ in terms of the complete basis $| N, \Delta_p \rangle$, i.e.,

$$| z, \Delta_p \rangle = \sum_N z^N | N, \Delta_p \rangle, \quad (60)$$

and applying L_j on eq. (58), we obtain:

$$L_j | N + j, \Delta_p \rangle = (\Delta_p - \Delta_m + j\Delta_n + N) | N, \Delta_p \rangle. \quad (61)$$

We also had that

$$L_0 | N, \Delta_p \rangle = (\Delta_p + N) | N, \Delta_p \rangle. \quad (62)$$

By solving the recursion relations we can find $\beta_{nm}^{p,\{k\}}$.

For the first level we have

$$L_1 | 1, \Delta_p \rangle = (\Delta_p - \Delta_m - \Delta_n) | \Delta_p \rangle \quad (63)$$

which results in

$$| 1, \Delta_p \rangle = \alpha_1 L_{-1} | \Delta_p \rangle; \quad \alpha_1 = \frac{\Delta_p - \Delta_m + \Delta_n}{2\Delta_p}. \quad (64)$$

For the second level by means of eqs. (60) and (64) we have

$$\begin{aligned} L_1 | 2, \Delta_p \rangle &= \hat{A} \alpha_1 L_{-1} | \Delta_p \rangle \\ L_2 | 2, \Delta_p \rangle &= \hat{B} | \Delta_p \rangle \end{aligned} \quad (65)$$

which results in:

$$| 2, \Delta_p \rangle = (\alpha_2 L_{-1}^2 + \alpha_3 L_{-2}) | \Delta_p \rangle \quad (66)$$

with α_2 and α_3 satisfying the following system of equations:

$${}_2\mathcal{N}_{ij} \alpha_j = {}_2\mathcal{A}_i, \quad i, j = 2, 3$$

where

$${}_2\mathcal{N} = \begin{bmatrix} 4\Delta_p + 2 & 3 \\ 6\Delta_p & 4\Delta_p + \frac{c}{2} \end{bmatrix}, \quad {}_2\mathcal{A} = \begin{bmatrix} \hat{A} \alpha_1 \\ \hat{B} \end{bmatrix}. \quad (67)$$

and index "2" refers to the level. This system can now be solved to give:

$$\begin{bmatrix} \alpha_2 \\ \alpha_3 \end{bmatrix} = \frac{1}{\Delta_1} \begin{bmatrix} \hat{C} - 3\hat{B} \\ -6\Delta_p \hat{A} \alpha_1 + \hat{D} \hat{B} \end{bmatrix} \quad (68)$$

where

$$\begin{aligned} \hat{A} &= \Delta_p - \Delta_m + \Delta_n + 1 \\ \hat{B} &= \Delta_p - \Delta_m + 2\Delta_n \\ \hat{C} &= 4\Delta_p + \frac{c}{2} \\ \hat{D} &= 4\Delta_p + 2 \end{aligned}$$

$$\Delta_1 = c(2\Delta_p + 1) + 2\Delta_p(8\Delta_p - 5). \quad (69)$$

A similar method will work for higher levels. For the level N, we have in place of eq. (66), an expansion corresponding to the partition of N. We then find a system of equations by successively applying L_j , and finally the coefficients are derived. For the third level, using Eq. (61), we have:

$$\begin{aligned} L_1 | 3, \Delta_p \rangle &= (\hat{A} + 1)(\alpha_2 L_{-2} + \alpha_3 L_{-1}^2) | \Delta_p \rangle \\ L_2 | 3, \Delta_p \rangle &= (\hat{B} + 1)\alpha_1 L_{-1} | \Delta_p \rangle \\ L_3 | 3, \Delta_p \rangle &= (\hat{B} + \Delta_n) | \Delta_p \rangle \end{aligned} \quad (70)$$

and $| 3, \Delta_p \rangle$ is given by

$$| 3, \Delta_p \rangle = (\alpha_4 L_{-1}^3 + \alpha_5 L_{-1} L_{-2} + \alpha_6 L_{-3}) | \Delta_p \rangle \quad (71)$$

where α_4 , α_5 and α_6 satisfy the following system of equations:

$${}_3\mathcal{N}_{ij}\alpha_j = {}_3\mathcal{A}_i, \quad i, j = 4, 5, 6$$

with

$${}_3\mathcal{N} = \begin{bmatrix} 18\Delta_p + 6 & 4\Delta_p + \frac{c}{2} + 9 & 5 \\ 24\Delta_p & 16\Delta_p + 2c & 6\Delta_p + 2c \\ 6\Delta_p + 6 & 2\Delta_p + 7 & 4 \end{bmatrix} \quad {}_3\mathcal{A} = \begin{bmatrix} (\hat{B} + 1)\alpha_1 \\ \hat{B} + \Delta_n \\ (\hat{A} + 1)(\alpha_2 + \alpha_3) \end{bmatrix}. \quad (72)$$

Solving of the above system we obtain:

$$\alpha_4 = \frac{-1}{6\Delta_2} \left[2\alpha_1(\hat{B} + 1)(6\Delta_p^2 + 6c\Delta_p - 11\Delta_p + 3c) + (\hat{B} + \Delta_n)(6\Delta_p + 2c + 1) - (\hat{A} + 1)(\alpha_2 + \alpha_3)(24\Delta_p^2 + 11c\Delta_p - 26\Delta_p + c^2 + 8c) \right] \quad (73)$$

$$\alpha_5 = \frac{1}{\Delta_2} \left[2\alpha_1(\hat{B} + 1)(3\Delta_p^2 + c\Delta_p - 5\Delta_p + c) + (\hat{B} + \Delta_n)(7\Delta_p - 1) - 2(\hat{A} + 1)(\alpha_2 + \alpha_3)(9\Delta_p^2 + 3c\Delta_p - 7\Delta_p + c) \right] \quad (74)$$

$$\alpha_6 = \frac{1}{2\Delta_2} \left[-4(\hat{B} + 1)\alpha_1(4\Delta_p^2 + c\Delta_p - \Delta_p + c) + (\hat{B} + \Delta_n)(-4\Delta_p^2 + c\Delta_p - 20\Delta_p + c + 4) + 4(\hat{A} + 1)(\alpha_2 + \alpha_3)(16\Delta_p^2 + 2c\Delta_p - 10\Delta_p + c) \right] \quad (75)$$

where

$$\Delta_2 = c^2(\Delta_p + 1) - c(\Delta_p^2 + 11\Delta_p - 2) - 4\Delta_p(3\Delta_p^2 - 7\Delta_p + 2).$$

4.2 OPE of Logarithmic Conformal Fields

The second most general solution of $F(x)$, which implies new OPE for two conformal fields which have two fields ϕ and ψ , of equal dimension in their fusion rule, eq. (1), is:

$$\begin{aligned} \Phi_n(z, \bar{z})\Phi_m(0, 0) &= \sum_p \sum_k z^{\Delta_p - \Delta_n - \Delta_m + \sum k_i} \bar{z}^{\bar{\Delta}_p - \bar{\Delta}_n - \bar{\Delta}_m + \sum \bar{k}_i} \\ &\times \left[C_{nm}^{p, \{k\}, \{\bar{k}\}} \log |z|^2 \phi_p^{\{k\}, \{\bar{k}\}}(0, 0) + C_{nm}'^{p, \{k\}, \{\bar{k}\}} \psi_p^{\{k\}, \{\bar{k}\}}(0, 0) \right] \end{aligned} \quad (76)$$

where $\phi_p^{\{k\}, \{\bar{k}\}}$ is an ordinary conformal fields, which has been discussed in the last subsection and $\psi_p^{\{k\}, \{\bar{k}\}}$ denote the new pseudo-operator with unusual properties.

The coefficients $C_{nm}^{p, \{k\}, \{\bar{k}\}}$ are the same as eq.(57) and

$$C_{nm}'^{p, \{k\}, \{\bar{k}\}} = C_{nm}^{p, \{k\}, \{\bar{k}\}} \beta_{nm}^{p, \{k\}} \bar{\beta}_{nm}^{p, \{\bar{k}\}} \quad \{k\} = \{k_1, k_2, \dots, k_n\}. \quad (77)$$

Similar to the expressions for $\phi_p^{\{k\}, \{\bar{k}\}}$, we have the following for $\psi_p^{\{k\}, \{\bar{k}\}}$:

$$\psi_i(0) | 0 \rangle = | \Delta'_p \rangle, \quad | z, \Delta'_p \rangle = \psi_p(z) | \Delta'_p \rangle \quad (78)$$

and instead of eq. (58), we have

$$\phi_n(z) | \Delta_m \rangle = \sum_p z^{\Delta_p - \Delta_n - \Delta_m} [\log z C_{nm}^p \phi_p(z) + C_{nm}'^p \psi_p(z)] | \Delta_p \rangle \quad (79)$$

where

$$| z, \Delta'_p \rangle = \psi_p(z) | \Delta_p \rangle \quad (80)$$

and its expansion in terms of z^N is

$$| z, \Delta'_p \rangle = \sum_N z^N | N, \Delta'_p \rangle. \quad (81)$$

Collecting the above expressions together, we obtain the following relations [1, 6, 21]

$$L_j | N + j, \Delta_p \rangle = (\Delta_p - \Delta_m + j\Delta_n + N) | N, \Delta_p \rangle \quad (82)$$

$$L_j | N + j, \Delta'_p \rangle = | N, \Delta_p \rangle + (\Delta_p - \Delta_m + j\Delta_n + N) | N, \Delta'_p \rangle \quad (83)$$

$$L_0 | N, \Delta_p \rangle = (N + \Delta_p) | N, \Delta_p \rangle. \quad (84)$$

$$L_0 | N, \Delta'_p \rangle = | N, \Delta_p \rangle + (N + \Delta_p) | N, \Delta'_p \rangle. \quad (85)$$

Now by using (82-85), we can calculate the OPE coefficients of two conformal operators, which have at least two operators with equal dimension in their OPE.

In the same way as in the ordinary case, for the first level we have

$$\begin{aligned}
& |1, \Delta'_p \rangle = \alpha'_1 L_{-1} | \Delta_p \rangle + \beta_1 L_{-1} | \Delta'_p \rangle \\
L_1 |1, \Delta'_p \rangle &= (2\Delta_p \alpha'_1 + 2\beta_1) | \Delta_p \rangle + 2\Delta_p \beta_1 | \Delta'_p \rangle \\
&= | \Delta_p \rangle + (\Delta_p - \Delta_m + \Delta_n) | \Delta'_p \rangle
\end{aligned} \tag{86}$$

with α'_1 and β_1 satisfying the following system of equations;

$${}_1\mathcal{M}_{ij}\gamma_j = {}_1\mathcal{B}_i; \quad i, j = 1, 2; \quad \gamma_1 = \beta_1, \quad \gamma_2 = \alpha'_1. \tag{87}$$

where

$${}_1\mathcal{M} = \begin{bmatrix} 2\Delta_p & 2 \\ 0 & 2\Delta_p \end{bmatrix} \quad {}_1\mathcal{B} = \begin{bmatrix} 1 \\ (\hat{A} - 1) \end{bmatrix} \tag{88}$$

which results in:

$$\begin{bmatrix} \beta_1 \\ \alpha'_1 \end{bmatrix} = \frac{1}{2\Delta_p^2} \begin{bmatrix} \Delta_m - \Delta_n \\ \Delta_p(\hat{A} - 1) \end{bmatrix}. \tag{89}$$

It is not surprising that $\alpha'_1 = \alpha_1$.

For the second level we have

$$|2, \Delta'_p \rangle = (\alpha'_2 L_{-1}^2 + \alpha'_3 L_{-2}) | \Delta'_p \rangle + (\beta_2 L_{-1}^2 + \beta_3 L_{-2}) | \Delta_p \rangle. \tag{90}$$

Using (82-85), we see that $\alpha'_2, \alpha'_3, \beta_2$ and β_3 satisfy the following system of equations:

$${}_2\mathcal{M}_{ij}\gamma_j = {}_2\mathcal{B}_i; \quad i, j = 1, 2, 3, 4; \quad \gamma_1 = \beta_2, \quad \gamma_2 = \beta_3, \quad \gamma_3 = \alpha'_2, \quad \gamma_4 = \alpha'_3 \tag{91}$$

where

$${}_2\mathcal{M} = \begin{bmatrix} 4\Delta_p + 2 & 3 & 4 & 0 \\ 6\Delta_p & 4\Delta_p + \frac{c}{2} & 6 & 4 \\ 0 & 0 & 4\Delta_p + 2 & 3 \\ 0 & 0 & 6\Delta_p & 4\Delta_p + \frac{c}{2} \end{bmatrix} \quad {}_2\mathcal{B} = \begin{bmatrix} \hat{A}\beta_1 + \alpha_1 \\ 1 \\ \hat{A}\alpha_1 \\ \hat{B} \end{bmatrix}. \tag{92}$$

These result in:

$$\begin{bmatrix} \beta_2 \\ \beta_3 \\ \alpha'_2 \\ \alpha'_3 \end{bmatrix} = \frac{1}{\Delta_1^2} \begin{bmatrix} \hat{C}[\hat{A}(\Delta_1\beta_1 - 4\hat{C}\alpha_1) + \Delta_1\alpha_1 - 24\hat{B}] + 3[c(3\hat{A}\alpha_1 + 2\hat{B}) + 10\hat{B} - \Delta_1] \\ \Delta_1(\hat{D} - 6\Delta_p(\hat{A}\beta_1 + \alpha_1)) + 6\hat{A}\alpha_1(16\Delta_p^2 - c) - 4\hat{B}(\hat{D} - 9) \\ \hat{A}\alpha_1\Delta_1\hat{C} - 3\hat{B}\Delta_1 \\ -6\hat{A}\alpha_1\Delta_1\Delta_p + \hat{B}\Delta_1(4\Delta_p + 2) \end{bmatrix}. \quad (93)$$

For the third level, using eqs. (82-85), we have:

$$|3, \Delta_p\rangle = (\beta_4 L_{-1}^3 + \beta_5 L_{-1} L_{-2} + \beta_6 L_{-3}) | \Delta_p \rangle + (\alpha'_4 L_{-1}^3 + \alpha'_5 L_{-1} L_{-2} + \alpha'_6 L_{-3}) | \Delta'_p \rangle \quad (94)$$

where α'_i and β_i ($i = 4, 5, 6$) satisfy the following system of equations:

$${}_3\mathcal{M}_{ij}\gamma_j = {}_3\mathcal{B}_i, \quad i, j = 1, \dots, 6, \quad (95)$$

γ_i 's ($i = 1, 2, 3$) denote β_i 's ($i = 4, 5, 6$) and γ_i 's ($i = 4, 5, 6$) denote α'_i 's ($i = 4, 5, 6$), respectively. The exact expression for ${}_3\mathcal{M}_i$ and ${}_3\mathcal{B}_i$ are:

$${}_3\mathcal{M} = \begin{bmatrix} 18\Delta_p + 6 & 4\Delta_p + 9 + \frac{c}{2} & 5 & 18 & 4 & 0 \\ 24\Delta_p & 16\Delta_p + 2c & 6\Delta_p + 2c & 24 & 16 & 6 \\ 6\Delta_p + 6 & 2\Delta_p + 7 & 4 & 6 & 2 & 0 \\ 0 & 0 & 0 & 18\Delta_p + 6 & 4\Delta_p + 9 + \frac{c}{2} & 5 \\ 0 & 0 & 0 & 24\Delta_p & 16\Delta_p + 2c & 6\Delta_p + 2c \\ 0 & 0 & 0 & 6\Delta_p + 6 & 2\Delta_p + 7 & 4 \end{bmatrix} \quad (96)$$

$${}_3\mathcal{B} = \begin{bmatrix} (\hat{B} + 1)\beta_1 + \alpha_1 \\ 1 \\ (\hat{A} + 1)(\beta_2 + \beta_3) + (\alpha_2 + \alpha_3) \\ (\hat{B} + 1)\alpha'_1 \\ (\hat{B} + \Delta_n) \\ (\hat{A} + 1)(\alpha'_2 + \alpha'_3) \end{bmatrix}. \quad (97)$$

In the above results, three significant remarks are in order:

1- In the diagonal blocks of ${}_i\mathcal{M}$, there are two copies of relevant ordinary ${}_i\mathcal{N}$.

2- The upper off diagonal block of ${}_i\mathcal{M}$ is the derivative of the diagonal block with respect to Δ_p and the lower off-diagonal block is zero.

This is another evidence insupport of the fact that logarithmic operator together with ordinary operator form the basis of Jordan cell for L_0 .

Hence,

$${}_i\mathcal{M} = \begin{bmatrix} {}_i\mathcal{N} & \partial_{\Delta_p} {}_i\mathcal{N} \\ 0 & {}_i\mathcal{N} \end{bmatrix} \quad {}_i\mathcal{B} = \begin{bmatrix} {}_i\mathcal{D} \\ {}_i\mathcal{A} \end{bmatrix} \quad (98)$$

where $\partial_{\Delta_p} = \frac{\partial}{\partial \Delta_p}$, ${}_i\mathcal{A}$ is the same as in the last subsection and ${}_i\mathcal{D}$ is a column matrix with elements which resemble the first half of ${}_i\mathcal{B}$ in eqs. (88,92).

Finally, β_i s and α'_i s are as follows:

$$\begin{aligned} [\alpha'_i] &= [\alpha_i] = {}_i\mathcal{N}^{-1} {}_i\mathcal{A} \\ [\beta_i] &= {}_i\mathcal{N}^{-1} {}_i\mathcal{D} + (\partial_{\Delta_p} {}_i\mathcal{N}^{-1}) {}_i\mathcal{A}. \end{aligned} \quad (99)$$

3- Most significant remark is that in the expressions (89, 92, 97), first half of elements is partial derivative of second half of elements with respect to Δ_p . This means that ${}_i\mathcal{D} = \partial_{\Delta_p}({}_i\mathcal{A})$.

Finally we can conclude as follows:

$$|n, \Delta' \rangle = \alpha_{k_1, k_2, \dots} L_{-1}^{k_1} L_{-2}^{k_2} L_{-3}^{k_3} \dots | \Delta_p \rangle + (\partial_{\Delta_p} \alpha_{k_1, k_2, k_3 \dots}) L_{-1}^{k_1} L_{-2}^{k_2} L_{-3}^{k_3} \dots | \Delta'_p \rangle \quad (100)$$

where $\sum m k_m = n$. It seems that appearance of ∂_{Δ_p} has more important rule in logarithmic operator.

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