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DISORDERED SYSTEMS AND LOGARITHMIC CONFORMAL FIELD THEORY

— Dedicated to my sister Mali —

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We review a recent development in theoretical understanding of the quenched averaged correlation functions of disordered systems and the logarithmic conformal field theory (LCFT) in d-dimensions. The logarithmic conformal field theory is the generalization of the conformal field theory when the dilatation operator is not diagonal and has the Jordan form. It is discussed that at the random fixed point the disordered systems such as random-bond Ising model, Polymer chain, etc. are described by LCFT and their correlation functions have logarithmic singularities. As an example we discuss in detail the application of LCFT to the problem of random-bond Ising model in $2 \leq d \leq 4$.

Keywords: conformal field theory, logarithmic conformal field theory, disordered systems, random-bond Ising model.

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1. Introduction

Random systems represent the spatial inhomogeneity where scale invariance is only preserved on average but not for specific disorder realization. The understanding of the role played by quenched impurities of the nature of phase transition is one of the significant subjects in statistical physics and has attracted a great deal of attention.[1] According to the Harris criterion,[2] quenched randomness is a relevant perturbation at the second-order critical point for systems of dimension d, when its specific heat exponent α, of the pure system is positive. Concerning the effect of randomness on the correlation functions, it is known that the presence of randomness induces a logarithmic factor to the correlation functions of pure system.[3,4] Theoretical treatment of the quenched disordered systems is a non-trivial task in view of the fact that, one has to average the logarithm of the partition function over various realization of the disorder in the statistical ensemble and therefore find physical quantities.[1] There are two standard methods to perform this averaging, the supersymmetry (SUSY) approach, and the well-known replica approach. Recently using the replica approach it has been shown by Cardy,[5] that the logarithmic factor multiplying power law behavior are to be expected in the scaling behavior near non-mean field critical points (see also Ref. 54). It is shown also that the results are valid for systems with short-range interactions and in an arbitrary number of dimensions. He concludes that in the limit of $n \to 0$ of replicas the theory possess of a set of fields which are degenerate (they have the same scaling dimensions) and finds a pair of fields which form a Jordan cell structure for dilatation operator and derives logarithmic operator in such disordered systems. It is proved that the quenched disordered theory with $Z = 1$ can be described by logarithmic conformal field theory as well. The logarithmic conformal field theories (LCFT)[6,7] are extensions of conventional conformal field theories,[8–10] which have emerged in recent years in a number of interesting physical problems of WZNW models,[11–15] supergroups

and super-symmetric field theories,[16−22] Haldane-Rezzayi state in the fractional quantum Hall effect,[23−27] multi-fractality,[28] two-dimensional turbulence,[29−31] gravitaitionally dressed theories,[32] Polymer and abelian sandpiles,[33−35,5] String theory and D-brane recoil,[36−44] Ads/CFT correspondence,[45−52] Seiberg-Witten solution to SUSY Yang-Mills theory,[53] disordered systems.[54−63] Also the material such as Null vectors, Characters, partition functions, fusion rules, Modular Invariance, C-theorem, LCFT's with boundary and operator product expansions have been discussed in.[64−84]

The LCFT are characterized by the fact that their dilatation operator L_0 are not diagonalized and admit a Jordan cell structure. The non-trivial mixing between these operators leads to logarithmic singularities in their correlation functions. It has been shown[6] that the correlator of two fields in such field theories, has a logarithmic singularity as follows,

$$< \psi(r)\psi(r^{'}) > \sim |r - r^{'}|^{-2\Delta_{\psi}} \log |r - r^{'}| + \dots \tag{1}$$

In this article we review the conformal field theory (CFT) and logarithmic conformal field theory (LCFT), which have appeared in the last decade as a powerful tool for the description of the correlation functions of second-order phase transition of pure and disordered critical systems near their fixed points. In section 2 and 3 we present a brief and self-content review of the CFT and LCFT and their basic tools in d-dimensions. In section 4 using the replica method we show that the disordered systems near their fixed point can be described by LCFT. As an example we discuss in details the correlation functions of the random-bond Ising model and its connection to LCFT. We give the explicit expression of the various types of quenched averaged 2, 3 and 4-point correlation functions of the local energy density. We also show that the ratios of these correlation functions to the connected ones have specific universal asymptotic and write down these universal functions explicitly.

2. Conformal Field Theory

In the following sub-sections we introduce the necessary techniques and the basic definition such as conformal transformation, conformal group, its representation, correlation functions, Ward-identities, Virasoro algebra and its representation in 2d conformal field theories, etc.

2.1. *Conformal Transformation & Conformal Group*

Let us start with definition of conformal transformation.

Definition 1: A transformation of coordinates $\mathbf{x}' \to \mathbf{x}$ is called conformal if it leaves the d-dimensional metric $g_{\mu,\nu}$ unchanged up to a scalar factor $\Lambda(x)$, i.e.

$$g'_{\mu\nu}(x) = \Lambda(x) g_{\mu\nu}(x) \tag{2}$$

consider an infinitesimal transformation

$$x^\mu \to x'^\mu = x^\mu + \epsilon^\mu(x) \tag{3}$$

to order (ϵ):

$$g_{\mu\nu} \to g_{\mu\nu} - (\epsilon_{\mu;\nu} + \epsilon_{\nu;\mu}) \tag{4}$$

where $\epsilon_{\alpha;\beta}$ is the covariant derivative of ϵ_α. The condition that the transformation be conformal gives:

$$\epsilon_{\mu;\nu} + \epsilon_{\nu;\mu} = f(x) g_{\mu\nu} \tag{5}$$

where

$$f(x) = \Lambda(x) - 1$$

Now we can eliminate $f(x)$. Let us suppose that $g_{\mu\nu} = \eta_{\mu\nu} = Diag(1, -1, \cdots)$. Contract eq. (5) with $\eta_{\mu\nu}$ to get:

$$f(x) = \frac{2}{d}(\partial_\nu \epsilon_\nu) = \frac{2}{d}(\partial \cdot \epsilon) \tag{6}$$

This gives us the first relation between $f(x)$ and ϵ. Differentiate eq. (5) w.r.t. x^α and after reordering of indices:

$$2\partial_\mu \partial_\nu \epsilon_\alpha = \eta_{\mu\alpha}\partial_\nu f + \eta_{\nu\alpha}\partial_\mu f - \eta_{\mu\nu}\partial_\alpha f \tag{7}$$

Now contract (7) with $\eta^{\mu\nu}$ gives:

$$2\partial^2 \epsilon_\mu = (2 - d)\partial_\mu f \tag{8}$$

To find final equation for f apply ∂_μ to eq.(8) and ∂^2 to eq.(5) and find:

$$(2 - d)\partial_\mu \partial_\nu f = \eta_{\mu\nu}\partial^2 f \tag{9}$$

now contract above eq. with $\eta^{\mu\nu}$ (using $\eta^{\mu\nu}\eta_{\mu\nu} = d$), therefore we find that the f satisfies the following equation :

$$2(1 - d)\partial^2 f = 0 \tag{10}$$

2.2. *Conformal Group with $d > 2$*

For $d > 2$ eq. (10) reduces to following simple equation,

$$\partial^2 f = 0 \quad \Rightarrow \quad f(x) = A + B_\mu x^\mu \tag{11}$$

Using the relation $f(x) = \frac{2}{d}\partial_\alpha \epsilon^\alpha$:

$$\epsilon_\mu = a_\mu + b_{\mu\nu}x^\nu + c_{\mu\nu\alpha}x^\nu x^\alpha \tag{12}$$

The case $b \equiv c \equiv 0 \quad \Rightarrow$ infinitesimal translation. Now suppose $a \equiv c \equiv 0$. Substitute $\epsilon_\mu = b_{\mu\nu}x^\nu$ in eq. (5) gives:

$$b_{\mu\nu} + b_{\nu\mu} = \frac{2}{d} b^\alpha_\alpha \eta_{\mu\nu} \tag{13}$$

Therefore $b_{\mu\nu}$ has two part which are antisymmetric and proportional to $\eta_{\mu\nu}$, i.e.

$$b_{\mu\nu} = \lambda\eta_{\mu\nu} + m_{\mu\nu} \qquad m_{\mu\nu} = -m_{\nu\mu} \tag{14}$$

where λ is scaling factor. The term which is proportional to $\eta_{\mu\nu}$ is scaling transformation and $m_{\mu\nu}$ part is an infinitesimal rotation. To understand the meaning of the part $c_{\mu\nu\alpha}x^\nu x^\alpha$ we start with eq.(9) for $d > 2$:

$$(2-d)\partial_\mu\partial_\nu f = \eta_{\mu\nu}\partial^2 f = 0 \Rightarrow \partial_\mu\partial_\nu f = 0 \tag{15}$$

or:

$$\partial_\mu\partial_\nu\partial\cdot\epsilon = 0 \tag{16}$$

$$\partial_\nu\partial\cdot\epsilon = -2b_\nu \Rightarrow \epsilon^\mu = b^\mu\mathbf{x}\cdot\mathbf{x} - 2x^\mu\mathbf{b}\cdot\mathbf{x} \tag{17}$$

This is known as special conformal transformation which has combination of inversion and translation. For finite transformation:

$$\begin{aligned} x'^\mu &= x^\mu + a^\mu \\ x'^\mu &= \lambda x^\mu \\ x'^\mu &= m^\mu_\nu x^\nu \\ x'^\mu &= \frac{x^\mu - b^\mu x^2}{1 - 2\mathbf{b}\cdot\mathbf{x} + b^2x^2} \end{aligned} \tag{18}$$

The generators of conformal group are: (for $d > 2$)

$$\begin{aligned} p_\mu &= -i\partial_\mu \\ D &= -ix^\mu\partial_\mu \\ L_{\mu\nu} &= i(x_\mu\partial_\nu - x_\nu\partial_\mu) \\ K_\mu &= -i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu) \end{aligned} \tag{19}$$

and the algebra is

$$\begin{aligned} [D, P_\mu] &= iP_\mu \\ [D, K_\mu] &= -iK_\mu \\ [K_\mu, P_\nu] &= 2i(\eta_{\mu\nu}D - L_{\mu\nu}) \\ [K_\rho, L_{\mu\nu}] &= i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu) \\ [P_\rho, L_{\mu\nu}] &= i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu) \\ [L_{\mu\nu}, L_{\rho\sigma}] &= i(\eta_{\nu\rho}L_{\mu\sigma} + \eta_{\mu\sigma}L_{\nu\rho} - \eta_{\mu\rho}L_{\nu\sigma} - \eta_{\nu\sigma}L_{\mu\rho}) \end{aligned} \tag{20}$$

Define new generators as:

$$\begin{aligned} J_{\mu\nu} &= L_{\mu\nu} \\ J_{-1,\mu} &= \frac{1}{2}(P_\mu - K_\mu) \\ J_{-1,0} &= D \\ J_{0,\mu} &= \frac{1}{2}(P_\mu + K_\mu) \end{aligned} \tag{21}$$

Note that $J_{a,b} = -J_{b,a}$ and $a, b \in \{-1, 0, 1, \cdots, d\}$. $J_{a,b}$ satisfy the $SO(d+1,1)$ algebra. Number of its parameters is $\frac{1}{2}(d+2)(d+1)$. $J_{a,b}$ satisfy:

$$[J_{a,b}, J_{c,d}] = i(\eta_{ad}J_{bc} + \eta_{bc}J_{ad} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}) \tag{22}$$

2.3. *Representation of Conformal Group*

Consider infinitesimal transformation with parameters ω^g. We would like to find the representation of T_g so that:

$$\Phi'(x') = (1 - i\omega^g T_g)\Phi(x) \tag{23}$$

Now define:

$$L_{\mu\nu}\Phi(0) = S_{\mu\nu}\Phi(0) \tag{24}$$

Near the origin:

$$\begin{aligned} &\exp(ix^\alpha P_\alpha)L_{\mu\nu}\exp(-ix^\alpha P_\alpha) \\ &= S_{\mu\nu} - x_\mu P_\nu + x_\nu P_\mu \end{aligned} \tag{25}$$

using:

$$P_\mu\Phi(x) = -i\partial_\mu\Phi(x) \tag{26}$$

one finds,

$$L_{\mu\nu}\Phi(x) = i(x_\mu\partial_\nu - x_\nu\partial_\mu)\Phi(x) + S_{\mu\nu}\Phi(x) \tag{27}$$

Similarly define the effect of D and K_μ on the origin as Δ and k_μ respectively, leads to:

$$\begin{aligned} &K_\mu\Phi(x) = \\ &(k_\mu + 2x_\mu\Delta - x^\nu S_{\mu\nu} - 2ix_\mu x^\nu\partial_\nu + ix^2\partial_\mu)\Phi(x) \\ &D\Phi(x) = (\Delta - ix^\nu\partial_\nu)\Phi(x) \end{aligned} \tag{28}$$

Now for finite transformation we can define the quasi-primary field as following.

Definition 2: The field $\Phi(x)$ which under conformal transformation transforms as:

$$\Phi(x) \to \Phi'(x') = \Omega(x)^{\Delta_i/2}\Phi(x) \tag{29}$$

is called *quasi-primary filed* with scaling dimension $\Delta_i/2$. The relation between $\Omega(x)$ and the Jacobian is, $|\frac{\partial x'}{\partial x}| = \Omega^{-d/2}$.

2.4. *Correlation Functions*

Correlation functions of the conformal field theory should transform as following:

$$\begin{aligned} &< \Phi'(x'_1)\cdots\Phi'(x'_N) >= \\ &\Omega(x_1)^{\Delta_1/2}\cdots\Omega(x_N)^{\Delta_N/2} < \Phi(x_1)\cdots\Phi(x_N) > \end{aligned} \tag{30}$$

The conformal structure implies strong constraints on the correlation functions of the theory. These constrains can be found by using the infinitesimal transformation and the above transformation.

One can show that the two, three and four point correlation functions of $\Phi(x)$ are given by:

$$< \Phi_1(x_1)\Phi_2(x_2) > = \begin{cases} \frac{c_{1,2}}{|x_1-x_2|^{2\Delta_1}}, & \Delta_1 = \Delta_2 \\ 0, & \Delta_1 \neq \Delta_2 \end{cases} \tag{31}$$

$$< \Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3) > = \frac{c_{1,2,3}}{x_{1,2}^{\Delta_1+\Delta_2-\Delta_3} x_{2,3}^{\Delta_2+\Delta_3-\Delta_1} x_{1,3}^{\Delta_1+\Delta_3-\Delta_2}} \tag{32}$$

$$< \Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3)\Phi_4(x_4) > = f(\frac{x_{1,2}x_{3,4}}{x_{1,3}x_{2,4}}, \frac{x_{1,2}x_{3,4}}{x_{2,3}x_{1,4}}, \cdots)\Pi_{i<j}^4 x_{i,j}^{\Delta/3-\Delta_i-\Delta_j} \tag{33}$$

where $x_{i,j} = |x_i - x_j|$ and $\Delta = \Sigma_{i=1}^4 \Delta_i$.

2.5. *The Ward Identities*

Consider an infinitesimal transformation as:

$$\Phi'(x') = (1 - i\eta^a(x)G_a)\Phi(x) \tag{34}$$

where G_a's are the generators of group and $\eta^a(x)$'s are infinitesimal functions. Under this transformation action changes as:

$$\delta S = -\int d^d x \partial_\mu (j_a^\mu \eta^a(x)) \tag{35}$$

where j_a^μ is conserved current corresponding to the transformation (34). On the other hand we can find the change of N-point correlation functions under this transformation. Defining $\Phi(N) = \Phi(x_1)\cdots\Phi(x_N)$, it can be shown that to order $\eta(x)$:

$$< \delta\Phi(N) >= -\int d^d x \partial_\mu < j_a^\mu \Phi(N) > \eta^a(x). \tag{36}$$

Also using explicit expression of transformation (i.e. eq. (34)):

$$\begin{aligned} \delta\Phi(N) &= -i\sum_{i=1}^N (\Phi(x_1)\cdots G_a\Phi(x_i)\cdots\Phi(x_N))\eta^a(x_i) \\ &= -i\int d^d x \eta^a(x) \sum_{i=1}^N \{\Phi(x_1)\cdots G_a\Phi(x)\cdots\Phi(x_N)\}\delta(x-x_i) \end{aligned} \tag{37}$$

Therefore for given (small) η:

$$\begin{aligned} &\frac{\partial}{\partial x^\mu} < j_a^\mu(x)\Phi(x_1)\cdots\Phi(x_N) > \\ &= i\sum_{i=1}^N \delta(x-x_i) < \Phi(x_1)\cdots G_a\Phi(x)\cdots\Phi(x_N) > \end{aligned} \tag{38}$$

This is the Ward identity corresponding to current $j_a^\mu(x)$.

2.6. *Ward identity corresponding to the conformal invariance*

We know that the Stress-Tensor ($T_{\mu\nu}$) is conserved current due to the invariance of S under transformation $x'^{\mu} = x^{\mu} + \epsilon^{\mu}$ with constant ϵ's. It's properties are: 1) $T_{\mu\nu} = T_{\nu\mu}$ and 2) $\partial_{\mu} T^{\mu\nu} = 0$. Conserved charges are $P^{\nu} = \int d^{d-1}x T^{0\nu}$. P^{ν} as an operator in Hilbert space acting as : $[P_{\nu}, \Phi] = -i\partial_{\nu}\Phi$. More generally $[Q_a, \Phi] = -iG_a\Phi$. There is another definition of $T^{\mu\nu}$. Consider the changes in metric as $g_{\mu\nu} \to g'_{\mu\nu} = g_{\mu\nu} + \delta g_{\mu\nu}$, under this transformation action S transforms as :

$$\delta S = \int d^d x \sqrt{g} T^{\mu\nu} \delta g_{\mu\nu} \tag{39}$$

This enable us to find more restrictions of $T^{\mu\nu}$ for conformal transformation. Suppose that theory possess Wyle symmetry so that:

$$g_{\mu\nu}(x) \to \Lambda(x) g_{\mu\nu}(x) \tag{40}$$

For infinitesimal transformation, $g_{\mu\nu} \to g_{\mu\nu} + \omega(x) g_{\mu\nu}$, or $\delta g_{\mu\nu} = \omega(x) g_{\mu\nu}$. Substitute this result in eq.(39):

$$\begin{aligned} \delta S &= \int d^d x \sqrt{g} T^{\mu\nu} \omega(x) g_{\mu\nu} \\ \int d^d x \sqrt{g} T^{\mu}_{\mu} \omega(x) &= 0 \\ &\Rightarrow T^{\mu}_{\mu} = 0 \end{aligned} \tag{41}$$

This means that the stress-tensor is traceless.

Now we can write the Ward identities (WI) due to conformal invariance. For invariance under translation:

$$\partial_{\mu} < T^{\mu}_{\nu} \Phi(N) > = \sum_{i=1}^{N} \delta(x - x_i) \frac{\partial}{\partial x_i^{\nu}} < \Phi(N) > \tag{42}$$

for Lorantz invariance (its current is $j^{\mu\nu\alpha} = T^{\mu\nu} x^{\alpha} - T^{\mu\alpha} x^{\nu}$ and its generators is $L_{\mu\nu} = S_{\mu\nu} + i(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu})$):

$$\begin{aligned} &\partial_{\mu} < (T^{\mu\nu} x^{\alpha} - T^{\mu\alpha} x^{\nu}) \Phi(N) > = \\ &- \sum_{i=1}^{N} \delta(x - x_i) \{ x_i^{\nu} \partial_i^{\alpha} - x_i^{\alpha} \partial_i^{\nu} - i S_i^{\nu\alpha} \} < \Phi(N) > \end{aligned} \tag{43}$$

Using the first identity:

$$< (T^{\mu\nu} - T^{\nu\mu}) \Phi(N) > = i \sum_{i=1}^{N} \delta(x - x_i) S_i^{\mu\nu} < \Phi(N) > . \tag{44}$$

WI corresponds to the scaling invariance

$$\partial_{\mu} < T^{\mu}_{\nu} x^{\nu} \Phi(N) > = \sum_{i=1}^{N} \delta(x - x_i) \{ (x_i^{\nu} \frac{\partial}{\partial x_i^{\nu}} + \Delta_i) < \Phi(N) > \} \tag{45}$$

Also using the first identity:

$$< T^{\mu}_{\mu}\Phi(N) > = \sum_{i=1}^{N} \delta(x - x_i)\Delta_i < \Phi(N) > . \tag{46}$$

2.7. *Conformal invariance in 2- dimensions*

Suppose $\mathbf{x} = (x^0, x^1)$ and $g_{\mu\nu} = Diag(1, 1)$. Using eq.(5) we have:

$$\partial_\mu \epsilon_\nu + \partial_\nu \epsilon_\mu = \frac{2}{d} g_{\mu\nu}(\partial \cdot \epsilon) \tag{47}$$

For $\nu \neq \mu$ and $\mu = \nu = 1$ we have the following equations

$$\begin{aligned} \partial_1 \epsilon_2 &= -\partial_2 \epsilon_1 \\ \partial_1 \epsilon_1 &= \partial_2 \epsilon_2 \end{aligned} \tag{48}$$

Define $z = x^0 + ix^1$, $\bar{z} = x^0 - ix^1$, and $d^2 s = dz d\bar{z}$ so that :

$$\begin{aligned} g_{zz} &= g_{\bar{z}\bar{z}} = 0 \\ g_{z\bar{z}} &= g_{\bar{z}z} = \frac{1}{2} \end{aligned} \tag{49}$$

now for $w = \epsilon_1 + i\epsilon_2$, the eqs. (48), reduce to the condition $\partial_{\bar{z}} w(z, \bar{z}) = 0$. This shows that the group of conformal transformations in two dimensions is isomorphic to the (infinite-dimensional) group of arbitrary analytic coordinate transformation $z \to w(z)$ and $\bar{z} \to \bar{w}(\bar{z})$.
The Mobius transformation is a subset of holomorphic transformation which has the following expression:

$$z \to f(z) = \frac{az + b}{cz + d} \qquad ad - bc = 1. \tag{50}$$

where a, b, c and d are complex numbers. One can show that the two transformations f_1 and f_2 gives, $f_1 f_2 = f$, so that the parameters of f_1 and f_2 be a_1, b_1, c_1, d_1 and a_2, b_2, c_2, d_2, respectively, the parameters of f_3 will be $a_3 = a_1 a_2 + b_1 c_2, b_3 = a_1 b_2 + b_1 a_2, c_3 = a_1 a_2 + a_1 c_2, d_3 = c_1 b_2 + a_1 a_2$.

2.8. *Holomorphic form of conformal ward identity*

In 2D we use the two-dimensional expression for generator of spin of i-th field as $S_{i\mu\nu} = s_i \epsilon_{\mu\nu}$ where $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$. Therefore conformal Ward Identities reduces to the following form:

$$\frac{\partial}{\partial x^\mu} < T^{\mu}_{\nu}\Phi(N) > = -\sum_{i=1}^{N} \delta(x - x_i) \frac{\partial}{\partial x_i^\nu} < \Phi(N) >$$

$$\epsilon_{\mu\nu} < T^{\mu\nu}\Phi(N) > = -\sum_{i=1}^{N} \delta(x - x_i) < \Phi(N) >$$

$$< T^{\mu}_{\mu}(x)\Phi(N) > = -\sum_{i=1}^{N} \delta(x - x_i)\Delta_i < \Phi(N) > \tag{51}$$

Now using the following identity:

$$\delta^2(x) = \frac{1}{\pi}\partial_{\bar{z}}\frac{1}{z} = \frac{1}{\pi}\partial_z\frac{1}{\bar{z}} \tag{52}$$

we can write the conformal WI as:

$$2\pi\partial_z < T_{z\bar{z}}\Phi(N) > +2\pi\partial_{\bar{z}} < T_{zz}\Phi(N) > = -\sum_{i=1}^{N} \partial_{\bar{z}}(\frac{1}{z - w_i})\partial_{w_i} < \Phi(N) >,$$

$$2\pi\partial_z < T_{\bar{z}\bar{z}}\Phi(N) > +2\pi\partial_{\bar{z}} < T_{z\bar{z}}\Phi(N) > = -\sum_{i=1}^{N} \partial_{\bar{z}}(\frac{1}{z - w_i})\partial_{\bar{w}_i} < \Phi(N) >,$$

$$2 < T_{z\bar{z}}\Phi(N) > +2 < T_{\bar{z}z}\Phi(N) > = -\sum_{i=1}^{N} \delta(x - x_i)\Delta_i < \Phi(N) >,$$

$$-2 < T_{z\bar{z}}\Phi(N) > +2 < T_{\bar{z}z}\Phi(N) > = -\sum_{i=1}^{N} \delta(x - x_i)s_i < \Phi(N) >, \tag{53}$$

Using the last equations:

$$2\pi < T_{\bar{z}z}\Phi(N) > = -\sum_{i=1}^{N} \partial_{\bar{z}}(\frac{1}{z - w_i})h_i < \Phi(N) >,$$

$$2\pi < T_{z\bar{z}}\Phi(N) > = -\sum_{i=1}^{N} \partial_{z}(\frac{1}{\bar{z} - \bar{w}_i})\bar{h}_i < \Phi(N) >, \tag{54}$$

where $h = \frac{1}{2}(\Delta + s)$ and $\bar{h} = \frac{1}{2}(\Delta - s)$. Using the above equation we can rewrite the eq.(53) as:

$$\partial_{\bar{z}} < T\Phi(N) > = \partial_{\bar{z}}\{\sum_{i=1}^{N} \frac{1}{z - w_i}\partial_{w_i} < \Phi(N) > +\frac{h_i}{(z - w_i)^2} < \Phi(N) >\}$$

$$\partial_{z} < \bar{T}\Phi(N) > = \partial_{z}\{\sum_{i=1}^{N} \frac{1}{\bar{z} - \bar{w}_i}\partial_{w_i} < \Phi(N) > +\frac{\bar{h}_i}{(\bar{z} - \bar{w}_i)^2} < \Phi(N) >\} \tag{55}$$

where $T = -2\pi T_{zz}$ and $\bar{T} = -2\pi T_{\bar{z}\bar{z}}$. Therefore:

$$< T(z)\Phi(N) > = \sum_{i=1}^{N}\{\frac{1}{z - w_i}\partial_{w_i} < \Phi(N) > +\frac{h_i}{(z - w_i)^2} < \Phi(N) >\} + \cdots \tag{56}$$

Also same equation holds for $\bar{T}$ (replace z with $\bar{z}$). This gives us the *Operator Product* of T and Φ. For example consider $N = 1$ and find that:

$$T(z)\Phi(w) = \frac{1}{z - w}\partial_w\Phi(w) + \frac{h}{(z - w)^2}\Phi + \cdots \tag{57}$$

In the next section we will start from the OPE of T and Φ and introduce the LCFT.

2.9. *Correlation Functions in Two Dimensions*

Let us now find $\delta_\epsilon \Phi(x)$. Using the eq.(36):

$$\delta_\epsilon < \Phi(N) > = \int_M d^2x \partial_\mu < T^{\mu\nu}(x)\epsilon_\nu(x)\Phi(N) > \\ \frac{i}{2}\oint_c \{-dz < T^{\bar{z}\bar{z}}\epsilon_{\bar{z}}\Phi(N) > + d\bar{z} < T^{zz}\epsilon_z \Phi(N) >\} \tag{58}$$

or

$$\delta_{\epsilon\bar{\epsilon}} < \Phi(N) > = -\frac{1}{2\pi i}\oint_c dz < T(z)\Phi(N) > + C.C \tag{59}$$

Using the operator product of $T(z)$ and $\Phi(w)$:

$$\delta_\epsilon < \Phi(N) > = -\sum_{i=1}^{N}(\epsilon(w_i)\partial_{w_i} + \partial_{w_i}\epsilon(w_i)h_i) < \Phi(N) > \tag{60}$$

Therefore for holomorphic part:

$$\delta_\epsilon \Phi(z) = -\epsilon\partial_z\Phi(z) - h\Phi(z)\partial_z\epsilon \tag{61}$$

For infinitesimal transformation $\epsilon(z) = a + bz + cz^2$:

$$\sum_{i=1}^{N}\partial_{w_i} < \Phi(N) > = 0$$

$$\sum_{i=1}^{N}(w_i\partial_{w_i} + h_i) < \Phi(N) > = 0$$

$$\sum_{i=1}^{N}(w_i^2\partial_{w_i} + 2w_i h_i) < \Phi(N) > = 0 \tag{62}$$

One can solve the above equation and find that:

$$< \Phi_1(z_1,\bar{z}_1)\Phi_2(z_2,\bar{z}_2) > = \delta_{h_1,h_2}\frac{c_{1,2}}{(z_1-z_2)^{2h}(\bar{z}_1-\bar{z}_2)^{2\bar{h}}} \tag{63}$$

$$< \Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3) > = \frac{c_{1,2,3}}{x_{1,2}^{h_1+h_2-h_3}x_{2,3}^{h_2+h_3-h_1}x_{1,3}^{h_1+h_3-h_2}} \tag{64}$$

$$< \Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3)\Phi_4(x_4) > = f(\frac{x_{1,2}x_{3,4}}{x_{1,3}x_{2,4}}, \frac{x_{1,2}x_{3,4}}{x_{2,3}x_{1,4}}, \cdots)\Pi_{i<j}^4 x_{i,j}^{h/3-h_i-h_j} \tag{65}$$

where $h = \sum_{i=1}^4 h_i$. Note that in four point function unknown function f has only one crossing ratio, because, if one define $\eta = \frac{z_{1,2}z_{3,4}}{z_{2,3}z_{1,4}}$, therefore will find that, $\frac{z_{1,2}z_{3,4}}{z_{1,4}z_{2,3}} = \frac{\eta}{1-\eta}$ and $\frac{z_{1,4}z_{2,3}}{z_{1,3}z_{2,4}} = 1-\eta$. So f is function of η and $\bar{\eta}$.

In the end of this sub-section let us introduce the central charge c via the OPE of T and T. For instance consider the following actions in two dimensions;

$$S_1 = \frac{1}{2}g\int d^2x \partial_\mu\phi\partial^\mu\phi$$

$$S_2 = \frac{1}{2}g\int d^2x \psi^+\gamma^0\gamma^\mu\partial_\mu\psi \tag{66}$$

it can be shown that :

$$T(z)T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \cdots \tag{67}$$

where $c = 1$ and $c = 1/2$ for S_1 and S_2, respectively. This is just the definition of central charge c.

2.10. *Transformation of Stress-Tensor*

According to definition of stress-tensor:

$$\delta_\epsilon \Phi = -\frac{1}{2\pi i}\oint \epsilon(z)dzT(z)\Phi \tag{68}$$

and for $T(w)$:

$$\begin{aligned}\delta_\epsilon T(w) &= -\frac{1}{2\pi i}\oint \epsilon(z)dzT(z)T(w) \\ &= -\frac{1}{12}c\partial_w^3\epsilon(w) - 2T(w)\partial_w\epsilon(w) - \epsilon(w)\partial_w T(w)\end{aligned} \tag{69}$$

This gives the transformation of T under infinitesimal transformation $z \to z+\epsilon(z)$. For finite transformation ($z \to W(z)$):

$$T'(W) = (\frac{dW}{dz})^{-2}[T(z) - \frac{c}{12}\{W;z\}] \tag{70}$$

where $\{W;z\} = \frac{d^3W/dz^3}{dW/dz} - 3/2(\frac{d^2W/dz^2}{dW/dz})^2$, which is called the Schwartzian derivative.

As an example one can check that for Mobius transformation $W = \frac{az+b}{cz+d}$ (with $ad-bc=1$) we have $\{W;z\} = 0$.
Also the map $W(z) = L/2\pi \ln z$ maps the plane to the strip with periodic boundary conditions. One can show that 1) $< T_{plane} >= 0$ and 2) $< T_{cyl} >= \frac{-c\pi^2}{6L^2}$.

2.11. *The Virasoro Algebra & Its Representation*

Expanding the stress-tensor in terms of a Laurent series as:

$$T(z) = \sum_n z^{-n-2}L_n \tag{71}$$

with

$$L_n = \frac{1}{2\pi i}\oint dz z^{n+1}T(z) \tag{72}$$

Using the above expansion and operator product expansion (OPE) of TT:

$$\begin{aligned}[L_n, L_m] &= \oint \frac{dw}{2\pi i}\oint \frac{dz}{2\pi i} z^{n+1}w^{m+1}(\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w}) \\ &= \oint \frac{dw}{2\pi i}(\frac{c}{12}n(n^2-1)w^{n+m+1} + 2(n+1)w^{n+m+1}T(w) + w^{n+m+2}\partial T(w)) \\ [L_n, L_m] &= \frac{c}{12}n(n^2-1)\delta_{n+m,0} + (n-m)L_{n+m}\end{aligned} \tag{73}$$

These commutation relations are known as Virasoro Algebra. We intend to create a representation of this algebra in terms of states, so let $|0>$ be a vacuum state in our field theory such that $L_n|0>=0$ for $n=0,1,-1$. Define the state $|h>=\Phi(0)|0>$ for any primary field Φ of conformal weight h. It can be shown that $L_0|h>=h|h>$, so that $|h>$ are eigenvectors of L_0. Also we have $L_n|h>=0$ for $n>0$. We call these states **highest − weight** states in the representation. We can obtain all states in the representation by applying a sequence of L_{-n} operators to a highest weight state. Alternatively we can define fields:

$$\Phi^{-k_1,\cdots,-k_n}(z)=L_{-k_1}\cdots L_{-k_n}\Phi(z) \tag{74}$$

with

$$L_{-k}\Phi(w)=\oint\frac{dz}{2\pi i}\frac{T(z)\Phi(w)}{(z-w)^{k-1}}. \tag{75}$$

where $k_1+\cdots+k_n=N$. The fields $\Phi^{-k_1,\cdots,-k_n}(z)$ are known as the descendent of field $\Phi(z)$ at level N.

For example one can check that the stress-tensor is the descendent of identity operator I, i.e. $L_{-2}I=T(0)$.

Now define null vector $|\xi>$ as being a linear combination of states of the same level. Similar to primary state it is it must be stable under the operation of L_n and L_0.

$$\begin{aligned} L_n|\xi>&=0 \qquad n>0\\ L_0|\xi>&=(h+N)|\xi> \end{aligned} \tag{76}$$

For instance we can show that the state $(L_{-2}+aL_{-1}^2)\Phi$ with $a=-\frac{3}{2(2h+1)}$ is a null state in the level $N=2$. The existence of null vectors gives us additional differential equation for determining of unknown function $f(\eta)$ in four-point correlation functions which contains the field Φ.

3. Logarithmic Conformal Field Theory

In an ordinary conformal field theory primary fields are the highest weights of the representations of the Virasoro algebra. The operator product expansion that defines a primary field $\Phi(w,\bar{w})$ is

$$T(z)\Phi_i(w,\bar{w})=\frac{\Delta_i}{(z-w)^2}\Phi_i(w,\bar{w})+\frac{1}{(z-w)}\partial_w\Phi_i(w,\bar{w}) \tag{77}$$

$$T(\bar{z})\Phi_i(w,\bar{w})=\frac{\bar{\Delta}_i}{(\bar{z}-\bar{w})}\Phi_i(w,\bar{w})+\frac{1}{(\bar{z}-\bar{w})}\partial_{\bar{w}}\Phi_i(w,\bar{w}) \tag{78}$$

where $T(z):=T_{zz}(z)$ and $\bar{T}(\bar{z}):=T_{\bar{z}\bar{z}}(\bar{z})$.

The primary fields are those which transform under $z\to f(z)$ and $\bar{z}\to\bar{f}(\bar{z})$ as:

$$\Phi_i(z,\bar{z})\to\Phi_i(z,\bar{z})'=(\frac{\partial f^{-1}}{\partial z})^{\Delta_i}(\frac{\partial\bar{f}^{-1}}{\partial\bar{z}})^{\bar{\Delta}_i}\Phi_i(f^{-1}(z),\bar{f}^{-1}(\bar{z})) \tag{79}$$

It can be shown that the OPE of T and Φ is equivalent to the commutation relation between L_n's and Φ. Let us introduce the radial quantization which is defined as:

$$R(\Phi_1(z)\Phi_2(w)) = \begin{cases} \Phi(z)\Phi(w) & |w| < |z| \\ \Phi(w)\Phi(z), & |w| > |z| \end{cases} \tag{80}$$

Now suppose we have two function $a(z)$ and $b(z)$ (are holomorphic) and consider the integral $\oint_w dz a(z)b(z)$. To have a well defined expression for this integral we should consider the radial ordered of fields. We consider two contours c_1 and c_2 so that the c_1 is the circle path (anti-clockwise) with center in the origin and its radius is $r = |w| + \epsilon$ and c_2 is similar path as c_1 (but clockwise) with $r = |w| - \epsilon$. Therefore,

$$\oint_c dz a(z)b(z) = \oint_{c_1} dz a(z)b(w) - \oint_{c_2} dz b(w)a(z) \tag{81}$$

Define $A = \oint dz a(z)$, therefore,

$$\oint_c dz a(z)b(z) = [A, b(w)] \tag{82}$$

From eq. (72) one finds,

$$[L_n, \Phi(w)] = \frac{1}{2\pi i} \oint dz z^{n+1} T(z)\Phi(w) \tag{83}$$

Using the OPE of T and Φ

$$[L_n, \Phi_i(z)] = z^{n+1}\partial_z \Phi_i + (n+1)z^n \Delta_i \Phi_i \tag{84}$$

One can regard Δ_i's as the diagonal elements of a diagonal matrix D,

$$[L_n, \Phi_i(z)] = z^{n+1}\partial_z \Phi_i + (n+1)z^n D_i^j \Phi_j \tag{85}$$

One can however, extend the above relation for any matrix D, which is not necessarily diagonal. This new representation of L_n also satisfies the Virasoro algebra for any arbitrary matrix D. Because we have not altered the first term in the right hand side of the equation (85), this is still a conformal transformation.

By a suitable change of basis, one can make D diagonal or Jordanian. If it becomes diagonal, the field theory is nothing but the ordinary conformal field theory. The general case is that there are some Jordanian blocks in the matrix D. The latter is the case of a LCFT.[7]

Here, there arise some other fields which do not transform like ordinary primary fields, and are called logarithmic operators. For the simplest case, consider a two-dimensional Jordan cell.

The fields Φ and Ψ satisfy

$$[L_n, \Phi(z)] = z^{n+1}\partial_z \Phi + (n+1)z^n \Delta \Phi \tag{86}$$

and

$$[L_n, \Psi(z)] = z^{n+1}\partial_z \Psi + (n+1)z^n \Delta \Psi + (n+1)z^n \Phi, \tag{87}$$

and they transform as below

$$\Phi(z) \to (\frac{\partial f^{-1}}{\partial z})^{\Delta}\Phi(f^{-1}(z)) \tag{88}$$

$$\Psi(z) \to (\frac{\partial f^{-1}}{\partial z})^{\Delta}[\Psi(f^{-1}(z)) + \log(\frac{\partial f^{-1}(z)}{\partial z})\Phi(f^{-1}(z)] \tag{89}$$

Note that we have considered only the chiral fields. The logarithmic fields, however cannot be factorized to the left- and right-handed fields. For simplicity we derive the results for chiral fields. The corresponding results for full fields are simply obtained by changing

$$z^{\Delta} \to z^{\Delta}\bar{z}^{\bar{\Delta}} \tag{90}$$

and

$$\log z \to \log |z|^2 \tag{91}$$

Now compare at the relations (86,87) and (88,89); one can assume the field Ψ as the derivation of the field Φ with respect to its conformal weight, Δ. Now let us consider the action of Möbius generators $(L_0, L_{\pm})$ on the correlation functions. Whenever the field Ψ is absent, the *form* of the correlators is the same as ordinary conformal field theory. By the term *form* we mean that some of the constants which cannot be determined in the ordinary conformal field theory may be fixed in the latter case. Now we want to compute correlators containing the field Ψ. At first we should compute the two-point functions. The two-point functions of the field Φ is as below

$$< \Phi(z)\Phi(w) >= \frac{c}{(z-w)^{2\Delta}} \tag{92}$$

In the ordinary conformal field theory the constant c cannot be determined only with assuming conformal invariance; to obtain it, one should know for example the stress-energy tensor, although for $c \neq 0$ one can set it equal to one by renormalizing the field.

Assuming the conformal invariance of the two-point function $< \Psi(z)\Phi(w) >$, means that acting the set $\{L_0, L_{\pm 1}\}$ on the correlator yields zero. Action of L_{-1} ensures that the correlator depends only on the $z - w$. the relations for L_{+1} and L_0 are as below

$$[z^2\partial_z + w^2\partial_w + 2\Delta(z+w)] < \Psi(z)\Phi(w) > +2z < \Phi(z)\Phi(w) > = 0 \tag{93}$$

$$[z\partial_z + w\partial_w + 2\Delta] < \Psi(z)\Phi(w) > + < \Phi(z)\Phi(w) > = 0 \tag{94}$$

Consistency of these two equations for any z and w, fixes c to be zero. Then, solving the above equation for $< \Psi(z)\Phi(w) >$ leads to

$$\begin{aligned} &< \Phi(z)\Phi(w) > = 0, \\ &< \Phi(z)\Psi(w) > = \frac{a}{(z-w)^{2\Delta}} \end{aligned} \tag{95}$$

Now assuming the conformal invariance of the two-point function $< \Psi(z)\Psi(w) >$, gives us a set of partial differential equation. Solving them, we obtain

$$< \Psi(z)\Psi(w) >= \frac{1}{(z-w)^{2\Delta}}[b - 2a\log(z-w)] \tag{96}$$

Now we extend the above results to the case where Jordanian block is $n+1$-dimensional. So there is $n+1$ fields with the same weight Δ.

$$[L_n, \Phi_i(z)] = z^{n+1}\partial_z\Phi_i + (n+1)z^n\Delta\Phi_i + (n+1)z^n\Phi_{i-1}, \tag{97}$$

where $\Phi_{-1} = 0$.

All we use is the conformal invariance of the theory. From the above fields, only Φ_0 is primary. Acting L_{-1} on any two-point function of these fields, shows that

$$< \Phi_i(z)\Phi_j(w) >= f_{ij}(z-w). \tag{98}$$

Acting L_0 and L_{+1}, leads to

$$\begin{aligned} < [L_0, \Phi_i(z)\Phi_j(0)] > &= (z\partial_z + 2\Delta) < \Phi_i(z)\Phi_j(0) > \\ &\quad + < \Phi_{i-1}(z)\Phi_j(0) > + < \Phi_i(z)\Phi_{j-1}(0) > = \ 0 \end{aligned} \tag{99}$$

$$\begin{aligned} < [L_{+1}, \Phi_i(z)\Phi_j(0)] > &= (z^2\partial_z + 2z\Delta) < \Phi_i(z)\Phi_j(0) > \\ &\quad + 2z < \Phi_{i-1}(z)\Phi_j(0) > = \ 0. \end{aligned} \tag{100}$$

Then it is easy to see that

$$< \Phi_{i-1}(z)\Phi_j(0) >=< \Phi_i(z)\Phi_{j-1}(0) > . \tag{101}$$

Using $\Phi_{-1} = 0$ and the above equation, gives us the following two-point functions.

$$< \Phi_i(z)\Phi_j(w) >= 0 \qquad \text{for} \qquad i + j < n \tag{102}$$

Now solving the Ward identities for $< \Phi_0(z)\Phi_n(w) >$ among with the relation (101), leads to

$$< \Phi_i(z)\Phi_{n-i}(w) >=< \Phi_0(z)\Phi_n(w) >= a_0(z-w)^{-2\Delta}. \tag{103}$$

The form of the correlation function $< \Phi_1(z)\Phi_n(w) >$ is as below

$$< \Phi_1(z)\Phi_n(w) >= (z-w)^{-2\Delta}[a_1 + b_1\log(z-w)], \tag{104}$$

but the conformal invariance fixes b_1 to be equal to $-2a_0$. So

$$\begin{aligned} < \Phi_i(z)\Phi_{n+1-i}(w) > &= < \Phi_1(z)\Phi_n(w) > \\ &= (z-w)^{-2\Delta}[a_1 - 2a_0\log(z-w)] \quad \text{for} \quad i > 0 \end{aligned} \tag{105}$$

Repeating this procedure for the two-point functions of the other fields Φ_i with Φ_n, and knowing that they are in the following form

$$< \Phi_i(z)\Phi_n(w) >= (z-w)^{-2\Delta}\sum_{j=0}^{i} a_{ij}(\log(z-w))^j, \tag{106}$$

gives

$$\sum_{j=1}^{i} j a_{ij} (\log(z-w))^{j-1} + 2 \sum_{j=0}^{i-1} a_{i-1,j} (\log(z-w))^{j} = 0 \tag{107}$$

or

$$(j+1) a_{i,j+1} + 2 a_{i-1,j} = 0$$

So

$$a_{i,j+1} = \frac{-2}{j+1} a_{i-1,j} = \cdots = \frac{(-2)^{j+1}}{(j+1)!} a_{i-j-1,0} =: \frac{(-2)^{j+1}}{(j+1)!} a_{i-j-1} \tag{108}$$

or

$$< \Phi_i(z) \Phi_n(w) > = \ (z-w)^{-2\Delta} \sum_{j=0}^{i} \frac{(-2)^j}{j!} a_{i-j} (\log(z-w))^j, \tag{109}$$

and also we have

$$< \Phi_i(z) \Phi_k(w) > = < \Phi_{i+k-n}(z) \Phi_n(w) > \qquad \text{for} \qquad i+k \geq n. \tag{110}$$

So for the case of n logarithmic field, we found all the two point functions. The interesting points are

i) some of the two-point functions become zero.

ii) some of the two-point functions are logarithmic, and the highest power of the logarithm, which occurs in the $< \Phi_n \Phi_n >$, is n.

The most general case is the case where there is more than one Jordanian block in the matrix D ,or in other words, there is more than one set of logarithmic operators. The dimension of these blocks may be equal or not equal. Using the same procedure, one can find that

$$\begin{aligned} &< \Phi_i^I(z) \Phi_j^J(w) > \\ &\quad = \begin{cases} (z-w)^{-2\Delta} \sum_{k=0}^{i+j-n} \frac{(-2)^k}{k!} a_{n-k} [\log(z-w)]^k, & i+j \geq n \\ 0, & i+j < n \end{cases} \end{aligned} \tag{111}$$

where I and J label the Jordan cells, $n = \max\{n_I, n_J\}$ and n_I and n_J are the dimensions of the corresponding Jordan cells.

Also note that the conformal dimensions of the cells I and J must be equal, otherwise the two-point functions are trivially zero.

Now we want to consider the three-point functions of logarithmic fields. The simplest case is the case where, besides Φ, only one extra logarithmic field Ψ exists in the theory. The three-point functions of the fields Φ are the same as ordinary conformal field theory.

$$A(z_1, z_2, z_3) :=< \Phi(z_1) \Phi(z_2) \Phi(z_3) > = \frac{a}{(\xi_1 \xi_2 \xi_3)^{\Delta}} =: a f(\xi_1, \xi_2, \xi_3), \tag{112}$$

where

$$\xi_i = \frac{1}{2} \sum_{j,k} \epsilon_{ijk} (z_j - z_k).$$

If one acts the set $\{L_0, L_{\pm 1}\}$ on the three-point function $< \Psi(z_1)\Phi(z_2)\Phi(z_3) >:= B(z_1, z_2, z_3)$, the result is an inhomogeneous partial differential equation for the function $B(z_1, z_2, z_3)$ where the inhomogeneous part is $A(z_1, z_2, z_3)$. So the form of $B(z_1, z_2, z_3)$ should be as below,

$$B(z_1, z_2, z_3) = [b + \sum b_i \log \xi_i] f(\xi_1, \xi_2, \xi_3). \tag{113}$$

Solving the above mentioned differential equations, we find the parameters b_i to be

$$b_1 = -b_2 = -b_3 = a. \tag{114}$$

The final result is

$$< \Psi(z_1)\Phi(z_2)\Phi(z_3) >= [b + a \log \frac{\xi_1}{\xi_2 \xi_3}] f(\xi_1, \xi_2, \xi_3) \tag{115}$$

If there are two fields Ψ in the three-point function, one can write it in the following form

$$< \Psi(z_1)\Psi(z_2)\Phi(z_3) > = [c + \sum c_i \log \xi_i + \sum_{ij} c_{ij} \log \xi_i \log \xi_j] f(\xi_1, \xi_2, \xi_3). \tag{116}$$

Again the Ward identities can be used to determine the parameters c_i and c_{ij},

$$\begin{aligned} < \Psi(z_1)\Psi(z_2)\Phi(z_3) > = \\ [c - 2b \log \xi_3 + a[(-\frac{\log \xi_1}{\log \xi_2})^2 + (\log \xi_3)^2] f(\xi_1, \xi_2, \xi_3). \end{aligned} \tag{117}$$

Finally, for the correlator of three Ψ's we use

$$\begin{aligned} < \Psi(z_1)\Psi(z_2)\Psi(z_3) > = [d + d_1 D_1 + d_2 D_2 + d_2' D_1^2 \\ + d_3 D_3 + d_3' D_1 D_2 + d_3' D_1^3] f(\xi_1, \xi_2, \xi_3) \end{aligned} \tag{118}$$

where

$$D_1 := \log(\xi_1 \xi_2 \xi_3) \tag{119}$$

$$D_2 := \log \xi_1 \log \xi_2 + \log \xi_2 \log \xi_3 + \log \xi_1 \log \xi_3 \tag{120}$$

$$D_3 := \log \xi_1 \ \log \xi_2 \ \log \xi_3. \tag{121}$$

This is the most general symmetric up to third power logarithmic function of the relative positions. Using the Ward identities, this three-point function is calculated to be

$$\begin{aligned} < \Psi(z_1)\Psi(z_2)\Psi(z_3) > = [d - cD_1 + 4bD_2 - bD_1^2 + 8aD_3 \\ - 4aD_1 D_2 + aD_1^3] f(\xi_1, \xi_2, \xi_3) \end{aligned} \tag{122}$$

Now there is a simple way to obtain these correlators. Remember of the relation between $\Phi(z)$ and $\Psi(z)$

$$\Psi(z) = \frac{\partial}{\partial \Delta} \Phi(z). \tag{123}$$

Consider any three-point function which contains the field Ψ. This correlator is related to another correlator which has a Φ instead od Ψ according to

$$< \Psi(z_1)A(z_2)B(z_3) > = \frac{\partial}{\partial\Delta} < \Phi(z_1)A(z_2)B(z_3) >, \tag{124}$$

To be more exact, the left hand side satisfies the Ward identities if the right hand side does so.

But the three-point function for ordinary fields are known. So it is enough to differentiate it with respect to the weight Δ. Obviously, a logarithmic term appears in the result. In this way one can easily obtain the above three-point functions. In fact instead of solving certain partial differential equations, one can easily differentiate with respect to the conformal weight. This method can also be used when there are n logarithmic fields. To obtain the three- point function containing the field Φ_i, one should write the three-point function, which contains the field Φ_0, and then differentiate it i times with respect to Δ.

Note that in the first three-point function, there may be more than one field with the same conformal weight Δ. Then one must treat the conformal weights to be independent variables, differentiate with respect to one of them, and finally put them equal to their appropriate value. Second, there are some constants, or unknown functions in the case of more than three-point functions, in any correlator. In differentiation with respect to a conformal weight, one must treat these *formally* as functions of the conformal weight as well.

As an example consider

$$< \Phi(z_1)\Phi(z_2)\Phi(z_3) > = \frac{a}{(\xi_1)^{\Delta_2+\Delta_3-\Delta_1}(\xi_2)^{\Delta_3+\Delta_1-\Delta_2}(\xi_3)^{\Delta_2+\Delta_1-\Delta_3}}. \tag{125}$$

Differentiate with respect to Δ_1, and then put $\Delta_1 = \Delta_2 = \Delta_3$, and $\frac{\partial a}{\partial\Delta_1} = b$. This is (115). This method can be used for any n-point function:

$$< \Phi_i(z_1)\cdots A(z_{n-1})B(z_n) > = \frac{\partial^i}{\partial\Delta^i} < \Phi_0(z_1)\cdots A(z_{n-1})B(z_n) >, \tag{126}$$

provided one treats the constants and functions of the correlator as functions of the conformal weight.

3.1. *Logarithmic Conformal Field Theory With Continuous Weights*

In this subsection we study the logarithmic conformal field theories in which conformal weights are continuous subset of real numbers. A general relation between the correlators consisting of logarithmic fields and those consisting of ordinary conformal fields is investigated. As an example the correlators of the Coulomb-gas model are explicitly studied.

In the previous subsection assuming conformal invariance we have explicitly calculated two- and three-point functions for the case of more than one set of logarithmic fields when their conformal weights belong to a discrete set. Regarding

logarithmic fields *formally* as derivations of ordinary fields with respect to their conformal dimension, we have calculated n-point functions containing logarithmic fields in terms of those of ordinary fields. Here, we want to consider logarithmic conformal field theories with continuous weights.[72] The simplest example of such theories is the free field theory.

In the last part it is shown that if there are quasi-primary fields in a conformal field theory, it causes logarithmic terms in the correlators of the theory. By quasi-primary fields, it is meant a family of operators satisfying

$$[L_n, \Phi^{(j)}(z)] = z^{n+1}\partial_z\Phi^{(j)}(z) + (n+1)z^n\Delta\Phi^{(j)}(z) + (n+1)z^n\Delta\Phi^{(j-1)}(z), \quad (127)$$

where Δ is the conformal weight of the family. Among the fields $\Phi^{(j)}$, the field $\Phi^{(0)}$ is primary. It was shown that one can interpret the fields $\Phi^{(j)}$, *formally*, as the j-th derivative of a field with respect to the conformal weight:

$$\Phi^{(j)}(z) = \frac{1}{j!}\frac{\partial^j}{\partial\Delta^j}\Phi^{(0)}(z), \quad (128)$$

and use this to calculate the correlators containing $\Phi^{(j)}$ in terms of those containing $\Phi^{(0)}$ only. The transformation relation (79), and the symmetry of the theory under the transformations generated by $L_{\pm1}$ and L_0, were also exploited to obtain two-point functions for the case where conformal weights belong to a discrete set. There were two features in two point functions. First, for two families Φ_1 and Φ_2, consisting of n_1+1 and n_2+1 members, respectively, it was shown that the correlator $<\Phi_1^{(i)}\Phi_2^{(j)}>$ is zero unless $i+j \geq \max(n_1, n_2)$. (It is understood that the conformal weights of these two families are equal. Otherwise, the above correlators are zero.) Another point was that one could not use the derivation process with respect to the conformal weights to obtain the two-point functions of these families from $<\Phi_1^{(0)}\Phi_2^{(0)}>$, since the correlators contain a multiplicative term $\delta_{\Delta_1,\Delta_2}$, which can not be differentiated with respect to the conformal weight.

Now, suppose that the set of conformal weights of the theory is a continuous subset of the real numbers. First, reconsider the arguments resulted to the fact that $<\Phi_1^{(i)}\Phi_2^{(j)}>$ is equal to zero for $i+j \geq \max(n_1, n_2)$. These came from the symmetry of the theory under the action of $L_{\pm1}$ and L_0. Symmetry under the action of L_{-1} results in

$$<\Phi_1^{(i)}(z)\Phi_2^{(j)}(w)> = <\Phi_1^{(i)}(z-w)\Phi_2^{(j)}(0)> =: A^{ij}(z-w). \quad (129)$$

We also have

$$\begin{aligned} <[L_0, \Phi_1^{(i)}(z)\Phi_2^{(j)}(0)]> &= (z\partial + \Delta_1 + \Delta_2)A^{ij}(z) + A^{i-1,j}(z) + A^{i,j-1}(z) \\ &= 0, \end{aligned} \quad (130)$$

and

$$<[L_1, \Phi_1^{(i)}(z)\Phi_2^{(j)}(0)]> = (z^2\partial + 2z\Delta_1)A^{ij}(z) + 2zA^{i-1,j}(z) = 0. \quad (131)$$

These show that

$$(\Delta_1 - \Delta_2)A^{ij}(z) + A^{i-1,j}(z) - A^{i,j-1}(z) = 0. \quad (132)$$

If $\Delta_1 \neq \Delta_2$, it is easily seen, through a recursive calculation, that A^{ij}'s are all equal to zero. This shows that the support of these correlators, as distribution of Δ_1 and Δ_2, is $\Delta_1 - \Delta_2 = 0$. So, one can use the ansatz,

$$A^{ij}(z) = \sum_{k\geq 0} A^{ij}_k(z)\delta^{(k)}(\Delta_1 - \Delta_2). \tag{133}$$

Inserting this in (132), and using $x\delta^{(k+1)}(x) = -(k+1)\delta^{(k)}(x)$, it is seen that

$$\sum_{k\geq 0}[-(k+1)A^{ij}_{k+1}(z) + A^{i-1,j}_k(z) - A^{i,j-1}_k(z)]\delta^{(k)}(\Delta_1 - \Delta_2) = 0, \tag{134}$$

or

$$(k+1)A^{ij}_{k+1}(z) = A^{i-1,j}_k(z) - A^{i,j-1}_k(z), \qquad k \geq 0 \tag{135}$$

This equation is readily solved:

$$A^{ij}_k(z) = \frac{1}{k!}\sum_{l=0}^{k}\binom{k}{l}A^{i-k+l,j-l}_0(z), \tag{136}$$

where A^{ij}_0's remain arbitrary. Also note that A^{ij}_k's with a negative index are zero. We now put (133) in (130). This gives

$$(z\partial + \Delta_1 + \Delta_2)A^{ij}_k(z) + A^{i-1,j}_k(z) + A^{i,j-1}_k(z) = 0, \tag{137}$$

Using (136), it is readily seen that it is sufficient to write (137) only for $k = 0$. This gives

$$(z\partial + \Delta_1 + \Delta_2)A^{ij}_0(z) + A^{i-1,j}_0(z) + A^{i,j-1}_0(z) = 0. \tag{138}$$

Putting the ansatz

$$A^{ij}_0(z) = z^{-(\Delta_1+\Delta_2)}\sum_{m=0}^{i+j}\alpha^{ij}_m(\ln z)^m \tag{139}$$

in (138), one arrives at

$$(m+1)\alpha^{ij}_{m+1} + \alpha^{i-1,j}_m + \alpha^{i,j-1}_m = 0, \tag{140}$$

the solution to which is

$$\alpha^{ij}_m = \frac{(-1)^m}{m!}\sum_{s=0}^{m}\binom{m}{s}\alpha_0^{i-m+s,j-s}. \tag{141}$$

From this

$$A^{ij}_0(z) = z^{-(\Delta_1+\Delta_2)}\sum_{m=0}^{i+j}(\ln z)^m\frac{(-1)^m}{m!}\sum_{s=0}^{m}\binom{m}{s}\alpha_0^{i-m+s,j-s}, \tag{142}$$

and

$$A^{ij}_k(z) = \\ [\frac{1}{k!}\sum_{l=0}^{k}(-1)^l\binom{k}{l}\sum_{m=0}^{i+j-k}(\ln z)^m\frac{(-1)^m}{m!}\sum_{s=0}^{m}\binom{m}{s}\alpha_0^{i-k-m+l+s,j-l-s}]z^{-(\Delta_1+\Delta_2)}. \tag{143}$$

So we have

$$A^{ij}(z) = z^{-(\Delta_1+\Delta_2)} \sum_{k\geq 0} \delta^{(k)}(\Delta_1-\Delta_2)[\frac{1}{k!}\sum_{l=0}^{k}(-1)^l\binom{k}{l} \sum_{m=0}^{i+j-k}(\ln z)^m \frac{(-1)^m}{m!}\sum_{s=0}^{m}\binom{m}{s}\alpha_0^{i-k-m+l+s,j-l-s}], \tag{144}$$

or

$$A^{ij}(z) = z^{-(\Delta_1+\Delta_2)} \sum_{p,q,r,s\geq 0} \frac{(-1)^{q+r+s}}{p!q!r!s!}\alpha^{i-p-r,j-q-s}(\ln z)^{r+s}\delta^{(p+q)}(\Delta_1-\Delta_2), \tag{145}$$

where

$$\alpha^{ij} := \alpha_0^{ij}. \tag{146}$$

These constants, defined for nonnegative values of i and j, are arbitrary and not determined from the conformal invariance only.

Now differentiate (145) formally with respect to Δ_1. In this process, α^{ij}'s are also assumed to be functions of Δ_1 and Δ_2. This leads to

$$\frac{\partial A^{ij}(z)}{\partial\Delta_1} = z^{-(\Delta_1+\Delta_2)} \sum_{p,q,r,s} \frac{(-1)^{q+r+s}}{p!q!r!s!}\{\frac{\partial\alpha^{i-p-r,j-q-s}}{\partial\Delta_1}(\ln z)^{r+s}\delta^{(p+q)}(\Delta_1-\Delta_2) + \alpha^{i-p-r,j-q-s}[(\ln z)^{r+s}\delta^{(p+q+1)}(\Delta_1-\Delta_2) - (\ln z)^{r+s+1}\delta^{(p+q)}(\Delta_1-\Delta_2)]\}, \tag{147}$$

or

$$\frac{\partial A^{ij}(z)}{\partial\Delta_1} = z^{-(\Delta_1+\Delta_2)} \sum_{p,q,r,s} \frac{(-1)^{q+r+s}}{p!q!r!s!}(\ln z)^{r+s}\delta^{(p+q)}(\Delta_1-\Delta_2) \times [(p+r)\alpha^{i-p-r,j-q-s} + \frac{\partial\alpha^{i-p-r,j-q-s}}{\partial\Delta_1}]. \tag{148}$$

Comparing this with $A^{i+1,j}$, it is easily seen that

$$A^{i+1,j} = \frac{1}{i+1}\frac{\partial A^{ij}}{\partial\Delta_1}, \tag{149}$$

provided

$$\frac{\partial\alpha^{i-p-r,j-q-s}}{\partial\Delta_1} = (i+1-p-r)\alpha^{i+1-p-r,j-q-s}. \tag{150}$$

Note, however, that the left hand side of (150) is just a *formal differentiation*. That is, the functional dependence of α^{ij}'s on Δ_1 and Δ_2 is not known, and their derivative is just another constant. Repeating this procedure for Δ_2, we finally arrive at

$$\alpha^{ij} = \frac{1}{i!j!}\frac{\partial^i}{\partial\Delta_1^i}\frac{\partial^j}{\partial\Delta_2^j}\alpha^{00}, \tag{151}$$

and

$$A^{ij} = \frac{1}{i!j!}\frac{\partial^i}{\partial\Delta_1^i}\frac{\partial^j}{\partial\Delta_2^j}A^{00}. \tag{152}$$

These relations mean that one can start from A^{00}, which is simply

$$A^{00}(z) = z^{-(\Delta_1+\Delta_2)}\delta(\Delta_1-\Delta_2)\alpha^{00}, \tag{153}$$

and differentiate it with respect to Δ_1 and Δ_2, to obtain A^{ij}. In each differentiation, some new constants appear, which are denoted by α^{ij}'s but with higher indices. Note also that the definition is self-consistent. So that this formal differentiation process is well-defined.

One can use this two-point functions to calculate the one-point functions of the theory. We simply put $\Phi_2^{(0)} = 1$. So, $\Delta_2 = 0$,

$$< \Phi^{(0)}(z) > = \ \beta^0\delta(\Delta), \tag{154}$$

and

$$< \Phi^{(i)}(z) > = \ \sum_{k=0}^{i}\frac{\beta^{n-k}}{k!}\delta^k(\Delta), \tag{155}$$

where

$$\beta^i := \frac{1}{i!}\frac{\partial^i\beta^0}{\partial\Delta^i}. \tag{156}$$

The more than two-point function are calculated exactly the same as in previous subsection.

3.2. *The Coulomb–gas model as an example of LCFT's*

As an explicit example of the general formulation of the previous section, consider the Coulomb-gas model characterized by the action[8,9]

$$S = \frac{1}{4\pi}\int\partial^2x\sqrt{g}[-g^{\mu\nu}(\partial_\mu\Phi)(\partial_\nu\Phi) + iQR\Phi], \tag{157}$$

where Φ is a real scalar field, Q is the charge of the theory, R is the scalar curvature of the surface and the surface itself is of a spherical topology, and is everywhere flat except at a single point.

Defining the stress tensor as

$$T^{\mu\nu} := -\frac{4\pi}{\sqrt{g}}\frac{\delta S}{\delta g_{\mu\nu}}, \tag{158}$$

it is readily seen that

$$T^{\mu\nu} = -(\partial^\mu\Phi)(\partial^\nu\Phi) + \frac{1}{2}g^{\mu\nu}g^{\alpha\beta}(\partial_\alpha\Phi)(\partial_\beta\Phi) - iQ[\phi^{i\mu\nu} - g^{\mu\nu}\nabla^2\Phi], \tag{159}$$

and

$$T(z) := T_{zz}(z) = -(\partial\phi)^2 - iQ\partial^2\phi, \tag{160}$$

where in the last relation the equation of motion has been used to write

$$\Phi(z,\bar{z}) = \phi(z) + \bar{\phi}(\bar{z}). \tag{161}$$

It is well known that this theory is conformal, with the central charge

$$c = 1 - 6Q^2 \tag{162}$$

There are, however, some features which need more care in our later calculations. First, this theory can not be normalized so that the expectation value of the unit operator become unity. In fact, using e^S as the integration measure, it is seen that

$$< 1 > \propto \ \delta(Q) \tag{163}$$

one can, at most, normalize this so that

$$< 1 > = \ \delta(Q) \tag{164}$$

Second, ϕ has a z-independent part, which we denote it by ϕ_0. The expectation value of ϕ_0 is not zero. In fact, from the action (157),

$$< \phi > = < \phi_0 > = \ \frac{1}{N(Q)} \int \partial\phi_0 \phi_0 \exp(2iQ\phi_0), \tag{165}$$

where N is determined from (164) and

$$< 1 >= \frac{1}{N} \int \partial\phi_0 \exp(2iQ\phi_0). \tag{166}$$

This shows that $N(0) = \pi$, and

$$< \phi_0 >= \frac{1}{2i}[\delta'(Q) + \frac{N'(0)}{N(0)}\delta(Q)] \tag{167}$$

More generally

$$< f(\phi_0) > = \ \frac{1}{N} f(\frac{1}{2i}\frac{\partial}{\partial Q})(N < 1 >) \ = \ \frac{1}{N} f(\frac{1}{2i}\frac{\partial}{\partial Q})[N\delta(Q)]. \tag{168}$$

Third, the normal ordering procedure is defined as following. One can write

$$\phi(z) = \phi_0 + \phi_+(z) + \phi_-(z), \tag{169}$$

where $< 0|\phi_-(z) = 0, \ \ \phi_+(z)|0 >= 0$, and

$$[\phi_0, \phi_\pm] = 0. \tag{170}$$

The normal ordering is so that one puts all '-' parts at the left of all '+' parts. It is then seen that

$$<: f[\phi] :> = < f(\phi_0) > \tag{171}$$

Here, the dependence of f on ϕ in the left hand side may be quite complicated; even f can depend on the values of ϕ at different points. In the right hand side, however, one simply changes $\phi(z) \to \phi_0$.

Now consider the two point function. From the equation of motion, we have

$$< \phi(z)\phi(w) > = \ -\frac{1}{2}\ln(z-w) < 1 > +b \tag{172}$$

we also have

$$<: \phi(z)\phi(w) :> = < \phi_0^2 > = \ -\frac{1}{4N}\frac{\partial^2}{\partial Q^2}[N\delta(Q)] \tag{173}$$

Note that there is an arbitrary term in (172), due to the ultraviolet divergence of the theory. One can use this arbitrariness, combined with the arbitrariness in $N(Q)$, to redefine the theory as

$$\phi(z)\phi(w) \ = : -\frac{1}{2}\ln(z-w) + : \phi(z)\phi(w) :, \tag{174}$$

and

$$< f(\phi_0) >: = \ f(\frac{1}{2i}\frac{\partial}{\partial Q})\delta(Q) \tag{175}$$

these relations, combined with (171) are sufficient to obtain all of the correlators. One can, in addition, use (160) (in normal ordered form) to arrive at

$$T(z)\phi(w) = \frac{\partial_w \phi}{z-w} - \frac{iQ/2}{(z-w)^2} + \text{r.t.}, \tag{176}$$

and

$$T(z)T(w) = \frac{\partial_w T}{z-w} - \frac{2T(w)}{(z-w)^2} + \frac{(1-6Q^2)/2}{(z-w)^4}. \tag{177}$$

Eq. (176) can be written in the form

$$[L_n, \phi(z)] = z^{n+1}\partial\phi - \frac{iQ}{2}(n+1)z^n. \tag{178}$$

This shows that the operators ϕ and 1 are a pair of logarithmic operators with $\Delta = 0$ (in the sense of first part of section 3). Note that $< 1 >$ is equal to $\delta(Q)$. One can easily show that

$$T(z) : \mathrm{e}^{i\alpha\phi(w)} : = \frac{\partial_w : \mathrm{e}^{i\alpha\phi(w)} :}{z-w} - \frac{\alpha(\alpha+2Q)/4}{(z-w)^2} : \mathrm{e}^{i\alpha\phi(w)} : + \text{r.t.}, \tag{179}$$

which shows that $: \mathrm{e}^{i\alpha\phi} :$ is a primary field with

$$\Delta_\alpha = \frac{\alpha(\alpha+2Q)}{4} \tag{180}$$

To this field, however, there corresponds a quasi conformal family (pre–logarithmic operators, see Ref. 55), whose members are obtained by explicit differentiation with respect to α (α is not the conformal weight but is a function of it):

$$W_\alpha^{(n)} : = : \phi^n \mathrm{e}^{i\alpha\phi} : = (-i)^n \frac{\partial}{\partial \alpha^n} : \mathrm{e}^{i\alpha\phi} : . \tag{181}$$

To calculate the correlators of W's, it is sufficient to calculate $< W^{(0)}_{\alpha_1} \cdots W^{(0)}_{\alpha_k} >$. One has, using Wick's theorem and (174),

$$\Pi_{j=1}^{k} : \mathrm{e}^{i\alpha_j \phi(z_j)} : = \mathrm{e}^{1/2 \sum_{1\le i<j\le k} \alpha_i \alpha_j \ln(z_i - z_j)} : \mathrm{e}^{i \sum_{j=1}^{k} \alpha_j \phi(z_j)} : \tag{182}$$

From this using (171) and (174), we have

$$\begin{aligned} < \Pi_{j=1}^{k} W^{(0)}_{\alpha_j}(z_j) > &= [\Pi_{1\le i<j\le k}(z_i - z_j)^{\frac{\alpha_i \alpha_j}{2}}] \mathrm{e}^{1/2 \sum_{j=1}^{k} \alpha_j \frac{\partial}{\partial Q}} \delta(Q) \\ &= [\Pi_{1\le i<j\le k}(z_i - z_j)^{\frac{\alpha_i \alpha_j}{2}}] \delta(Q + \frac{1}{2} \sum_{j=1}^{k} \alpha_j). \end{aligned} \tag{183}$$

Obviously, differentiating with respect to any α_i, leads to logarithmic terms for the correlators consisting of logarithmic fields $W^{(n)}_{\alpha}$. The power of logarithmic terms is equal to the sum of superscripts of the fields $W^{(n)}_{\alpha}$.

3.3. *Logarithmic Conformal Field Theory in d dimensions*

In the previous subsection we discussed the LCFT in 2-dimensions. Generalization for arbitrary dimension d has been given in Ref. 68. We have dealt with two dimensional conformal field theory relying heavily on the underlying Virasoro algebra, and have described how the appearance of logarithmic singularities is related to the modification of the representation of the Virasoro algebra. In this subsection we try to understand LCFT's in the context of d-dimensional conformal invariance. As is well known, one of the basic assumptions of conformal field theory is the existence of a family of operators, called scaling fields, which transform under scaling $S : x \to x' = \lambda x$, simply as follows:

$$\phi(x) \to \phi'(x') \ = \ \lambda^{-\Delta} \phi(x), \tag{184}$$

where Δ is the scaling weight of $\phi(x)$. It is also assumed that under the conformal group, such fields transform as,

$$\phi(x) \ \to \ \phi'(x') \ = \| \ \frac{\phi x'}{\phi x} \ \|^{\frac{-\Delta}{d}} \ \phi(x), \tag{185}$$

where d is the dimension of space and $\| \ \frac{\phi x'}{\phi x} \ \|$ is the Jacobian of the transformation. Equation (185) which encompasses eq. (184) defines the transformation of the quasi-primary fields. For future use we note that the Jacobian equals λ^d for scaling transformation and $\| \ x \ \|^{-2d}$ for the Inversion transformation $I : x \to x' = \frac{x}{\|x\|^2}$, being unity for the other elements of the conformal group. Combination of (185) with the definition of symmetry of the correlation functions, i.e.:

$$< \phi'_1(x'_1) \cdots \phi'_N(x'_N) > = < \phi_1(x'_1) \cdots \phi_N(x'_N) >, \tag{186}$$

allows one to determine the two and the three point functions up to a constant and the four point function up to a function of the cross ratio.

It's precisely the assumption that scaling fields constitute irreducible representations

of the scaling transformation, which imposes power law singularity on the correlation functions.

As we will see, relaxing this assumption, one naturally arrives at logarithmic singularities. It also leads to many other peculiarities, in the relation between correlation functions. To begin with, we consider a multiplet of fields,

$$\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_n \end{pmatrix}, \tag{187}$$

and note that under scaling $x \to \lambda x$, the most general form of the transformation of Φ is,

$$\Phi(x) \to \Phi'(x') = \lambda^{T'} \Phi(x) \tag{188}$$

where T' is an arbitrary matrix. More generally, we replace (188) by,

$$\Phi(x) \to \Phi'(x') = \| \frac{\phi x'}{\phi x} \|^T \Phi(x). \tag{189}$$

where T is an $n \times n$ arbitrary matrix. When T is diagonalizable, one arrives at ordinary scaling fields by redefining Φ, so that all the fields transform as 1-dimensional representation. Otherwise, following Ref. 68 we assume that T has Jordan form,

$$T = \begin{pmatrix} \frac{-\Delta}{d} & 0 & \cdots & 0 \\ 1 & \frac{-\Delta}{d} & \cdots & 0 \\ & & & \\ 0 & 1 & \ddots & 0 \\ 0 & \cdots & 1 & \frac{-\Delta}{d} \end{pmatrix}. \tag{190}$$

Rewriting T as $\frac{-\Delta}{d} 1 + J$, where $J_{ij} = \delta_{i-1,j}$, eq. (189) can be written in the form,

$$\Phi(x) \to \Phi'(x') = \| \frac{\phi x'}{\phi x} \|^{\frac{-\Delta}{d}} \Lambda_x \Phi(x), \tag{191}$$

where $\Lambda_x = \| \frac{\phi x'}{\phi x} \|^J$ is a lower triangular matrix of the form,

$$(\Lambda_x)_{ij} = \frac{\{\ln \| \frac{\phi x'}{\phi x} \|\}^{i-j}}{(i-j)!}, \qquad (\Lambda_x)_{ii} = 1, \tag{192}$$

i. e. for $N = 2$ we have,

$$\begin{aligned} \phi_1'(x') &= \| \frac{\phi x'}{\phi x} \|^{\frac{-\Delta}{d}} \phi_1(x), \\ \phi_2'(x') &= \| \frac{\phi x'}{\phi x} \|^{\frac{-\Delta}{d}} \big(\ln \| \frac{\phi x'}{\phi x} \| \, \phi_1(x) + \phi_2(x) \big). \end{aligned} \tag{193}$$

An important point is that the top field $\phi_1(x)$ *always* transform as an ordinary quasi-primary field. A most curious property of the transformation (191) is that each field ϕ_{k+1} transforms as if it is a formal derivative of ϕ_k with respect to $\frac{-\Delta}{d}$,

$$\phi_{k+1}(x) = \frac{1}{k}\frac{\phi}{\phi(\frac{-\Delta}{d})}\phi_k(x). \tag{194}$$

This formal relation which determines the transformation of all the fields of a Jordan cell from that of the top field ϕ_1, essentially means that with due care, one can determine the correlation functions of the lower fields from those of the ordinary top fields simply by formal differentiation. Also one can find the two-point, three-point correlation functions of fields for jordan -cell of rank r. For example we already know by standard arguments that the two point function of the top fields ϕ_α and ϕ_β belonging to two different Jordan cells (Δ_α, n) and (Δ_β, m) vanishes, i.e.:

$$< \phi_\alpha(x)\phi_\beta(y) >= \frac{A\delta_{\Delta_\alpha,\Delta_\beta}}{\parallel x-y \parallel^{2\Delta_\alpha}}. \tag{195}$$

Due to the observation (194), it follows that the two point function of all the fields of two different Jordan cells with respect to each other vanish. Therefore we can calculate the two point function of the fields within the same Jordan cell. As we will see logarithmic conformal symmetry gives many interesting and unexpected results in this case. Let's denote the matrix of two point functions $< \phi_i(x)\phi_j(y) >$ for all $\phi_i, \phi_j \in (\Delta, n)$ by $G(x,y)$, then from rotation and translation symmetries, this matrix should depends only on $\parallel x-y \parallel$. From scaling symmetry and using (185) and (186), we have,

$$\Lambda G(\parallel x-y \parallel)\Lambda^t = \lambda^{2\Delta} G(\lambda \parallel x-y \parallel), \tag{196}$$

where $\Lambda = \lambda^{dJ}$, and from inversion symmetry, we have,

$$\Lambda_x G(\parallel x-y \parallel)\Lambda_y^t = \parallel x-y \parallel^{-2\Delta_\alpha} G(\frac{\parallel x-y \parallel}{\parallel x \parallel\parallel y \parallel}), \tag{197}$$

where

$$\begin{aligned} \Lambda_x &= \parallel x \parallel^{-2dJ}, \\ \Lambda_y &= \parallel y \parallel^{-2dJ}. \end{aligned} \tag{198}$$

Defining the matrix F as $G(\parallel x-y \parallel) = \frac{F(\|x-y\|)}{\|x-y\|^{2\Delta}}$, we will have from (196) and (197),

$$\Lambda F(\parallel x-y \parallel)\Lambda^t = F(\lambda \parallel x-y \parallel), \tag{199}$$

and

$$\Lambda_x F(\parallel x-y \parallel)\Lambda_y^t = F(\frac{\parallel x-y \parallel}{\parallel x \parallel \parallel y \parallel}). \tag{200}$$

For every arbitrary λ, we now choose the points x and y such that $\parallel x \parallel = \lambda^{\frac{-1}{4}}$ and $\parallel y \parallel = \lambda^{\frac{-3}{4}}$. *It should be noted that in this way by varying λ, we can span all*

the points of space. From (198), we will have $\Lambda_x = \Lambda^{\frac{1}{2}}$ and $\Lambda_y = \Lambda^{\frac{3}{2}}$, therefore eq. (199) turns into,

$$\Lambda^{\frac{1}{2}} F(\| x-y \|)\big(\Lambda^{\frac{3}{2}}\big)^t = F(\lambda \| x-y \|). \tag{201}$$

Combining (199) and (201) and using invertibility of Λ, we arrive at,

$$F = \Lambda^{\frac{1}{2}} F\big(\Lambda^t\big)^{\frac{-1}{2}}, \tag{202}$$

by iterating (202), we will have $F = \Lambda F \Lambda^t$, and by rearranging, we have,

$$F\Lambda^t = \Lambda F. \tag{203}$$

Expanding Λ in terms of power of $\ln\lambda$ as $\Lambda = 1 + (d\ln\lambda)J + \frac{(d\ln\lambda)^2}{2!}J^2 + \cdots$ and comparing both sides, we arrive at

$$F\big(J^t\big)^k = \big(J\big)^k F, \qquad k = 1, 2, \cdots, n-1. \tag{204}$$

Since $\big(J^k\big)_{ij} = \delta_{i,j+k}$, we will have from (204),

$$F_{i,j-k} = F_{i-k,j}, \tag{205}$$

which means that on each opposite diagonal of the matrix F, all the correlations are equal. Moreover from $FJ = JF$, one obtains,

$$\sum_{l=1}^{n} F_{il}\delta_{j,l+1} = \sum_{l=1}^{n} \delta_{i,l+1}F_{lj}, \tag{206}$$

which means that if $j = 1$ and $1 < i \leq n$, then $F_{i-1,j} = 0$, or

$$F_{ij} = 0 \qquad \text{for} \qquad j = 1 \qquad \text{and} \qquad 1 \leq i \leq n-1. \tag{207}$$

Combining this with (205), we find that all the correlations above the opposite diagonal are zero. In order to find the final form of F, we use eq. (199) again, this time in infinitesimal form, let $\Lambda = 1 + \alpha J + o(\alpha^2)$ where $\alpha = d\ln\lambda$, then from $\Lambda F(x)\Lambda^t = F(\lambda x)$, we have,

$$d\big(JF + FJ^t\big) = x\frac{dF}{dx}. \tag{208}$$

Due to the property (205) only the last column of F should be found, therefore from (208) we obtain,

$$\begin{aligned} x\frac{dF_{1,n}}{dx} &= dF_{1,n-1} \equiv 0, \\ x\frac{dF_{i,n}}{dx} &= 2dF_{i,n-1}, \qquad \text{if} \quad i > 1 \end{aligned} \tag{209}$$

which upon introducing the new variable $y = 2d\ln x$ gives,

$$F_{1,n} = c_1, \quad F_{2,n} = c_1 y + c_2, \quad F_{3,n} = \frac{1}{2}c_1 y^2 + c_2 y + c_3, \quad \text{etc.} \tag{210}$$

with the recursion relations,

$$\frac{dF_{i,n}}{dy} = F_{i-1,n}. \tag{211}$$

Thus we have arrived at the final form of the matrix F, which is as follows:

$$F = \begin{pmatrix} 0 & \cdots & 0 & 0 & g_0 \\ 0 & \cdots & 0 & g_0 & g_1 \\ 0 & \cdots & g_0 & g_1 & g_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ g_0 & \cdots & g_{n-2} & g_{n-1} & g_n \end{pmatrix}, \tag{212}$$

where each g_i is a polynomial of degree i in y, and $g_i = \frac{dg_{i+1}}{dy}$. All the correlations depend on the n constants $c_1, \cdots, c_n$, which remain undetermined. We have checked (as the reader can check for the single $n = 2$ case) that inversion symmetry puts no further restrictions on the constants c_i.

The observation that the transformation properties of the members of a Jordan cell are as of the formal derivative of the top field in the cell, allows one to determine the correlation functions of all the fields within a single or different Jordan cells, once the correlation function of the top fields are determined. As an example, from ordinary CFT, we know that conformal symmetry completely determines the three point function up to a constant. Let ϕ_α, ϕ_β and ϕ_γ be the top fields of three Jordan cells (Δ_α, l), (Δ_β, m) and (Δ_γ, n) respectively. Therefore we know that,

$$< \phi_\alpha(x)\phi_\beta(y)\phi_\gamma(z) >= \frac{A_{\alpha\beta\gamma}}{\| x-y \|^{\Delta_\alpha+\Delta_\beta-\Delta_\gamma} \| x-z \|^{\Delta_\alpha+\Delta_\gamma-\Delta_\beta} \| y-z \|^{\Delta_\beta+\Delta_\gamma-\Delta_\alpha}}, \tag{213}$$

where the constant $A_{\alpha\beta\gamma}$ in principle depends on the weights $\Delta_\alpha, \Delta_\beta$ and Δ_γ. Denoting the second field of the cell (Δ_α, l) by $\phi_{\alpha 1}$,[a] we will readily find from eq.(213) that,

$$\begin{aligned} < \phi_{\alpha 1}(x)\phi_\beta(y)\phi_\gamma(z) > &= -d\frac{\partial}{\partial\Delta_\alpha} < \phi_\alpha(x)\phi_\beta(y)\phi_\gamma(z) > \\ &= \frac{A'_{\alpha\beta\gamma}}{\| x-y \|^{\Delta_\alpha+\Delta_\beta-\Delta_\gamma} \| x-z \|^{\Delta_\alpha+\Delta_\gamma-\Delta_\beta} \| y-z \|^{\Delta_\beta+\Delta_\gamma-\Delta_\alpha}} \\ &+ \frac{dA_{\alpha\beta\gamma}}{\| x-y \|^{\Delta_\alpha+\Delta_\beta-\Delta_\gamma} \| x-z \|^{\Delta_\alpha+\Delta_\gamma-\Delta_\beta} \| y-z \|^{\Delta_\beta+\Delta_\gamma-\Delta_\alpha}} \\ &\ln\big(\frac{\| y-z \|}{(\| x-y \|)(\| x-z \|)}\big), \end{aligned} \tag{214}$$

where $A'_{\alpha\beta\gamma} = -d\frac{\partial}{\partial\Delta_\alpha}A_{\alpha\beta\gamma}$ is a new undetermined constant. For the correlation functions of fields within a single cell, one should then take the limit $\beta, \gamma \to \alpha$

[a]For simplicity, we have denoted the top field by ϕ and the second field by ϕ_1, instead of ϕ_1 and ϕ_2 respectively.

in the above formula. It's not difficult to check that this formula satisfies all the requirements demanded by conformal symmetry.

4. Disordered Systems & Logarithmic Conformal Field Theory

4.1. *Introduction*

Consider a renormalization group transformation acting on the space of all possible couplings of a model, $\{k\}$, see Ref. 1. The transformation has the form $\{k'\} = R\{k\}$ where R depends, in general, on the specific transformation chosen, and in particular, on the length rescaling parameter b. Suppose there is a fixed point at $\{k\} = \{k^*\}$ the renormalization group equations, linearized about the fixed point, are

$$k'_a - k^*_a \sim \sum_b T_{ab}(k_b - k^*_b) \tag{215}$$

where $T_{ab} = \frac{\partial k'_a}{\partial k_b}|_{k=k^*}$. Denote the eigenvalues of the matrix T by λ^i and its left eigenvectors by $\{\phi^i\}$ so that

$$\sum_a \phi^i_a T_{ab} = \lambda^i \phi^i_b \tag{216}$$

Now we define scaling variables $u_i \equiv \sum_a \phi^i_a(k_a - k^*_a)$, which are linear combinations of the deviations $k_a - k^*_a$ from the fixed point which transform multiplicatively near the fixed point

$$\begin{aligned} u_i &= \sum_a \phi^i_a(k'_a - k^*_a) = \sum_{a,b} \phi^i_a T_{ab}(k_a - k^*_a) \\ &= \sum_b \lambda^i \phi^i_b(k_b - k^*_b) = \lambda^i u_i \end{aligned} \tag{217}$$

It is convenient to define the quantities y_i by $\lambda^i = b^{y_i}$, the y_i's are called RG eigenvalues. Now we can distinguish three cases,
(1) If $y_i > 0$, u_i is relevant,
(2) If $y_i < 0$, u_i is irrelevant,
(3) If $y_i = 0$, u_i is marginal.

4.2. *Rule of the rescaling factor b*

Consider an infinitesimal transformation, that $b = 1 + \delta l$, with $\delta l << 1$. In this case the coupling transforms infinitesimally

$$k_a \to k_a + (\frac{dk_a}{dl})\delta l + O((\delta l)^2) \tag{218}$$

and the RG equation has the differential form

$$\frac{dk_a}{dl} = -\beta_a(\{k\}) \tag{219}$$

where the function β_a are called the RG beta-function. Also the matrix of derivatives at the fixed point is now $T_{ab} = \delta_{ab} + (\frac{\partial \beta_a}{\partial k_b})\delta l$, with eigenvalues $(1+\delta l)^{y_i} \sim 1 + y_i \delta l$. Hence the y_i's are simply the eigenvalues of the matrix $-\frac{\partial \beta_a}{\partial k_b}$ evaluated at the zero of the beta-functions (because at the fixed points we have $k_a \to k_a$ and beta-function must be zero).

4.3. *The perturbative RG*

When two fixed points are sufficiently close, it is then possible to deduce universal properties at one fixed point in terms of those at the other. Such an analysis is the basis of the ϵ-expansion and many other similar techniques. It also allows us to describe the properties of fixed points with exactly marginal scaling variables.

Now consider a fixed point Hamiltonian H^* which is perturbed by a number of scaling fields, so that the partition function is:[1]

$$Z = Tr \exp\{-H^* - \sum_i g_i \sum_r a^{x_i} \phi_i(r)\} \tag{220}$$

Let us expand this in powers of couplings g_i,

$$\begin{aligned} Z = Z^* \{ 1 &- \sum_i g_i \int < \phi_i(r) > \frac{d^d r}{a^{d-x_i}} \\ &+ 1/2 \sum_{ij} g_i g_j \int < \phi_i(r_1)\phi_j(r_2) > \frac{d^d r_1 d^d r_2}{a^{2d-x_i-x_j}} \\ &- 1/3! \sum_{ijk} g_i g_j g_k \int < \phi_i(r_1)\phi_j(r_2)\phi_k(r_3) > \frac{d^d r_1 d^d r_2 d^d r_3}{a^{3d-x_i-x_j-x_k}} + \cdots \} \end{aligned} \tag{221}$$

where all correlation functions are to be evaluated with respect to the fixed point hamiltonian H^*. We now implement the RG by changing the microscopic cut-off $a \to ba$, with $b = 1 + \delta l$, and asking now the couplings g_i should be changed in order to preserve the partition function Z. The length a appears in three ways in eq.(221),
(1) Explicitly , through the divisors a^{d-x_i}
(2) Implicity , we must restrict all integrals to $|r_i - r_j| > a$.
(3) Through the dependence on the system size L in the dimensionless ratio L/a.
For the *explicit* dependence we have:

$$g_i \to (1+\delta l)^{d-x_i} g_i \sim g_i + (d - x_i) g_i \delta l \tag{222}$$

The *implicit* dependence will appear in the second order term. After changing $a \to a(1+\delta l)$ we may break up the integrals,

$$\int_{|r_1-r_2|>a(1+\delta l)} = \int_{|r_1-r_2|>a} - \int_{a(1+\delta l)>|r_1-r_2|>a} \tag{223}$$

The first term simply gives back the original contribution to Z and the second term may be expressed using the operator product expansion

$$\phi_i(r_1)\phi_j(r_2) = \sum_k C_{ijk}(r_1 - r_2)\phi_k((r_1 + r_2)/2) \tag{224}$$

where $C_{ijk}(r_1 - r_2) = \frac{c_{ijk}}{|r_1 - r_2|^{x_i + x_j - x_k}}$, ($c_{ijk}$ are known as OPE coefficients) as:

$$\frac{1}{2}\sum_{i,j} c_{ijk} a^{x_k - x_i - x_j} \int_{a(1+\delta l) > |r_1 - r_2| > a} < \phi_k((r_1 + r_2)/2) > \frac{d^d r_1 d^d r_2}{a^{2d - x_i - x_j}} \tag{225}$$

The integral gives a factor $S_d a^d \delta l$, where $S_d = 2\pi^{d/2}/\Gamma(d/2)$ (the area of d-dimensional sphere). This term may then be compensated by making the change

$$g_k \to g_k - 1/2 S_d \sum_{i,j} c_{ijk} g_i g_j \delta l \tag{226}$$

Finally by changing the variable $g_k \to (2/S_d) g_k$ we find the beta-function for coupling g_k as (with $y_k = d - x_k$):

$$\frac{dg_k}{dl} = y_k g_k - \sum_{i,j} c_{ijk} g_i g_j + \cdots \tag{227}$$

4.4. *Quenched Random Ferromagnets*

We consider the random bond Ising model and suppose that positions of the impurities are fixed, and tracing over the only magnetic degrees of freedom.[5,60] Let us describe these disordered systems in the continuum limit by the following Hamiltonian,

$$H = H^0 + \int j(r) E(r) d^d r \tag{228}$$

where H^0 is the Hamiltonian of the renormalization group at fixed point describing the pure Ising model. The field $j(r)$ is a quenched random variable coupled to the local energy density $E(r)$. For simplicity we assume that the random variable $j(r)$ is a gaussian variable which is characterized with its two moments $< j(r) >= 0$ and $< j(r) j(r') >= g\delta(r - r')$. The explicit distribution of $j(r)$ is,

$$P(j(r)) = N \exp(-\frac{1}{2g} \int j(r)^2 d^d r) \tag{229}$$

where N is normalization constant.

We are interested in computing the average of the free energy, or quenched averaged correlation functions. Since the free energy is proportional to logarithm of the partition functions $Z[j]$, we have to compute the average of $\log Z[j]$. The replica method is based on the identity

$$\log Z[j] = \lim_{n \to 0} \frac{Z^n[j] - 1}{n} = \frac{d}{dn} Z^n[j]|_{n=0} \tag{230}$$

The standard procedure of averaging over disorder is to introduce replicas , i.e. , n identical copies of the same model

$$Z^n = Tr \exp\{-\sum_{a=1}^{n} H_a^0 - \int d^d r j(r) \sum_{a=1}^{n} E_a(r)\}. \tag{231}$$

Averaging over the disorder gives rise to the following effective replica Hamiltonian:

$$H_R = \sum_{a=1}^{n} H_a^0 - g \int \sum_{a\neq b}^{n} E_a(r) E_b(r) d^d r. \tag{232}$$

We keep only the non-diagonal terms , since using the operator algebra of the pure system one can absorb the diagonal terms into H_a^0. The replicas are now coupled via the disorder. The scaling dimension of coupling g is $y_g = d - 2x_E^0$ and it is relevant at the pure fixed point if $d > 2x_E^0$. If y_g is small , it is possible to develop perturbative RG in powers of these variable and we can find a random fixed point with perturbative RG.

To find the renormalization group equation for g we need the operator product expansion of $(\sum_{a\neq b} E_a E_b)$ with itself. Since the replicas are decoupled in H^0 , we may evaluate this using the operator product expansion of E_a with itself, the first few terms of which have the form

$$E_a \cdot E_b \sim \delta_{ab} + c\delta_{ab} E_a + \cdots, \tag{233}$$

where c is a coefficient whose value is fixed and universal. The coefficient c vanishes when $d = 2$. This is a consequence of the self-duality if Ising model in two dimensions.[1] Now we have

$$(\sum_{a\neq b} E_a E_b) \cdot (\sum_{c\neq d} E_c E_d) = \sum_{a\neq b, c\neq d} E_a E_b E_c E_d. \tag{234}$$

The OPE of $E_a E_b$ is zero (see eq.(238)) and the OPE of $E_c E_d$'s is also zero, so we have two ways of writing the OPE. So

$$\sum_{a\neq b, c\neq d} E_a E_b E_c E_d = 2 \sum_{a\neq b, c\neq d} (\delta_{bd} + c\delta_{bd} E_b) E_a E_c. \tag{235}$$

The first term is

$$2 \sum_{a\neq b, c\neq d} \delta_{bd} E_a E_c 2\{(n-1)\sum_{a=1} E_a^2 + (n-2)\sum_{a\neq b} E_a E_b\}, \tag{236}$$

the second term can be written as;

$$\begin{aligned} c \sum_{a\neq b, c\neq d} \delta_{bd} E_a E_c E_b &= 2c \sum_{a\neq b, c\neq d} \delta_{bd}(\delta_{ac} + c\delta_{ac} E_a) E_b \\ &= 2c \sum_{a\neq b, c\neq d} \delta_{bd}\delta_{ac} E_b + 2c^2 \sum_{a\neq b, c\neq d} \delta_{bd}\delta_{ac} E_a E_b \\ &= 2c(n-1)\sum_{a=1} E_a + 2c^2 \sum_{a\neq b, c\neq d} \delta_{bd}\delta_{ac} E_a E_b \\ &= 2c(n-1)\sum_{a=1} E_a + 2c^2 \sum_{a\neq b} E_a E_b \end{aligned} \tag{237}$$

Therefore:

$$(\sum_{a\neq b} E_a E_b)\cdot(\sum_{c\neq d} E_c E_d) = (2(n-2)+2c^2)\sum_{a\neq b} E_a E_b + 2(n-1)\sum_{a=1} E_a^2 + 2c(n-1)\sum_{a=1} E_a \qquad (238)$$

The renormalization group equation for g is thus

$$\frac{dg}{dl} = y_g^0 g - (2(n-2)+2c^2)g^2 + O(g^3,\cdots). \qquad (239)$$

Therefore if one denote $\sum_{a\neq b} E_a E_b$ with Φ so the coefficient in OPE $\Phi\cdot\Phi = 1 + b\Phi + \cdots$, is $2(n-2)+2c^2$.

Now Consider a scaling operator ϕ with a scaling dimension x_ϕ so that $\phi\Phi = b_\phi\phi + \cdots$ and denote coupling of ϕ with t. Now we have

$$\begin{aligned}\beta_g &= y_g^0 g - bg^2 + \cdots \\ \beta_\phi &= y_\phi^0 t - 2b_\phi t g + \cdots\end{aligned} \qquad (240)$$

From the above equations,

$$g^* = \frac{y_g}{b} \qquad (241)$$

Therefore using eq.(245) we obtain:

$$y_\phi = \frac{\partial\beta_\phi}{\partial t}|_0 = y_\phi^0 - 2b_\phi\frac{y_g^0}{b}, \qquad (242)$$

and then

$$d - x_\phi = (d - x_\phi^0) - \frac{2b_\phi y_g^0}{b}, \qquad (243)$$

so

$$x_\phi = x_\phi^0 + \frac{2b_\phi y_g^0}{b}. \qquad (244)$$

Now consider the case where ϕ is $E_t = \sum_a E_a$ and $\tilde{E}_a = E_a - (\frac{1}{n})\sum_a E_a$ with $\sum_a \tilde{E}_a = 0$. The combination $E_t = \sum_{a=1}^n E_a$ is a singlet (symmetric under the permutation or the replica group) and the $\tilde{E}_a = E_a - \frac{1}{n}\sum_{b=1}^n E_b$ transforms according to an $(n-1)$-dimensional representation of S_n. The fields E_t and $\tilde{E}_a$ have proper scaling dimensions. To find the scaling dimensions of new fields we should find the OPE coefficients $E_t\Phi$ and $\tilde{E}_a\Phi$,

$$\begin{aligned}E_t\Phi &= (\sum_{a=1}^n E_a)(\sum_{b\neq c}^n E_b E_c) \\ &= \sum_{a,b\neq c}^n E_a E_b E_c\end{aligned}$$

$$= 2\sum_{a,b\neq c}^{n}(\delta_{ab}+c\delta_{ab}E_a)E_c$$
$$= 2\sum_{a,b\neq c}^{n}\delta_{ab}E_c + 2c\sum_{a,b\neq c}^{n}(\delta_{ac}+c\delta_{ac}E_a)\delta_{ab}$$
$$= 2\sum_{b\neq c}^{n}\delta_{bb}E_c + O(c) = 2(n-1)\sum_{c}E_c, \tag{245}$$

so $b_{E_t} = 2(n-1)$.

$$\tilde{E}_a\Phi = (E_a - \frac{1}{n}\sum_{b=1}^{n}E_b)(\sum_{c\neq d=1}^{n}E_cE_d)$$
$$= \sum_{c\neq a}^{n}E_aE_cE_d - \frac{1}{n}(\sum_{b=1}^{n}E_b)(\sum_{c\neq d=1}^{n}E_cE_d)$$
$$= 2\sum_{c\neq d}^{n}(\delta_{ac}+c\delta_{ac}E_a)E_d - \frac{1}{n}(\sum_{b=1}^{n}E_b)(\sum_{c\neq d=1}^{n}E_cE_d)$$
$$= 2\sum_{d\neq a}^{n}E_d - \frac{1}{n}(2n-2)(\sum_{b=1}^{n}E_b) + \cdots$$
$$= -2(E_a - \frac{1}{n}\sum_{b=1}^{n}E_b) + \cdots \tag{246}$$

so $b_{\tilde{E}_a} = -2$. Now we can obtain x_{E_t} , $x_{\tilde{E}_a}$,

$$x_{E_t} = x^0_{E_t} + \frac{2b_E}{b}y_g \tag{247}$$
$$x_{\tilde{E}_a} = x^0_{\tilde{E}_a} + \frac{2b_{\tilde{E}_a}}{b}y_g, \tag{248}$$

we have neglected $O(c)$ and $O(c^2\cdots)$ so $b = 2(n-2)$ and in the limit of $n \to 0$,

$$x_{E_t} = x^0_E + (1+n/2)y_g \tag{249}$$
$$x_{\tilde{E}_a} = x^0_{\tilde{E}_a} + (1-n/2)y_g, \tag{250}$$

Now define $x_{E_t} = 2\Delta_{E_t}$ and $x_{\tilde{E}_a} = 2\Delta_{\tilde{E}_a}$.

The important observation is that the fields E_t and $\tilde{E}_a$ have the proper scaling dimensions close to $n \to 0$ as $\Delta_{E_t} = \Delta^{(0)}_{E_a} + y_g/2 + O(y_g^2)$ and $\Delta_{\tilde{E}} = \Delta^{(0)}_{E} + y_g/2 + O(y_g^2)$ respectively. It is clear that the singlet field E_t becomes degenerate with the $(n-1)$ operators $\tilde{E}_a$. However they do not form the basis of the Jordan cell for the dilatation operator. Starting from the following replicated Hamiltonian one can show that correlation functions calculated against this effective Hamiltonian will correspond to correlator averaged against the initial Hamiltonian with quenched

disorder,

$$
\begin{aligned}
H_R \;=\; \sum_{a=1}^{n} H_a + t\int d^dx \sum_{a=1}^{n} E_a(r) \;-\; g\int d^dr \sum_{a\neq b} E_a(r)E_b(r) \\
\overline{< E(r) >_H} \leftrightarrow \lim_{n\to 0} < E_a(r) >_{repl} \\
\overline{< E(0)E(r) >_H} \leftrightarrow \lim_{n\to 0} < E_a(0)E_a(r) >_{repl} \\
\overline{< E(0) >_H < E(r) >_H} \leftrightarrow \lim_{n\to 0} < E_a(0)E_b(r) >_{repl} \qquad a\neq b \\
\overline{< E(r_1)E(r_2)E(r_3) >_H} \leftrightarrow \lim_{n\to 0} < E_1(1)E_1(2)E_1(3) >_{repl} \\
\overline{< E(1)E(2) >_H < E(3) >_H} \leftrightarrow \lim_{n\to 0} < E_1(1)E_1(2)E_2(3) >_{repl} \\
\overline{< E(1) >_H < E(2) >_H < E(3) >_H} \leftrightarrow \lim_{n\to 0} < E_1(1)E_2(2)E_3(3) >_{repl} \qquad (251)
\end{aligned}
$$

etc.

To find the logarithmic pair consider:

$$
\begin{aligned}
< E_t(0)E_t(r) > &= n(< E_1(0)E_1(r) > -(n-1) < E_1(0)E_2(r) >) \\
&\equiv nA(n)r^{-2\Delta_E(n)} \\
< \tilde{E}_a(0)\tilde{E}_a(r) > &= (1-\frac{1}{n})(< E_1(0)E_1(r) > - < E_1(0)E_2(r) >) \\
&\equiv (1-\frac{1}{n})B(n)r^{-2\tilde{\Delta}_E(n)} \qquad (252)
\end{aligned}
$$

The above equations enable us to write the quenched averaged connected two-point correlation functions of energy density operator in terms of $< E_1(0)E_1(r) >$ and $< E_1(0)E_1(r) >$ in the limit of $n \to 0$ as: $\overline{< E(0)E(r) >}_c =< E_1(0)E_1(r) > - < E_1(0)E_2(r) >$ which is equal $B(0)r^{-2\Delta_E}$ and it has a pure scaling behavior. However the correlation functions $< E_1(0)E_1(r) >$ and $< E_1(0)E_2(r) >$ have the logarithmic singularities and behave as:

$$
\begin{aligned}
< E_1(0)E_1(r) > &= (A'(0) - B'(0) + B(0) - B(0)\frac{y_g}{2}\ln r)r^{-2\Delta_E} \\
< E_1(0)E_2(r) > &= (A'(0) - B'(0) - A(0)\frac{y_g}{2}\ln r)r^{-2\Delta_E} \qquad (253)
\end{aligned}
$$

where $A(0) = B(0)$. The prime sign in the eq. (258) means differentiating with respect to n.

This means that in the limit $n \to 0$ the field E_t and E_a form a basis of Jordan cell, i.e. their two point correlation functions behave as: $< E_t(0)E_t(r) >= 0$, further $< E_t(0)E_a(r) >= a_1 r^{-2\Delta_E}$, and $< E_a(0)E_b(r) >= (-2a_1 \ln r + D_{a,b})r^{-2\Delta}$, where a_1 and $D_{a,b}$ are some constants.

Also the ratio of quenched averaged two-point correlators of the energy density operator to connected one has a universal r-dependence as:

$$
\frac{\overline{< E(0)E(r) >}}{\overline{< E(0)E(r) >}_c} \sim \frac{\overline{< E(0) >< E(r) >}}{\overline{< E(0)E(r) >}_c} \sim \ln r \qquad (254)
$$

We note that in 2D we have dealt with two-dimensional conformal field theory, relying heavily on the underlying Virasoro algebra. For an extension to d dimensions one has to modify the representation of the Virasoro algebra to higher dimensions. We consider a doublet of fields (Jordan cell)

$$\Phi = \begin{pmatrix} E_t \\ E_a \end{pmatrix}, \tag{255}$$

and note that under D-dimensional conformal transformation $\mathbf{x} \to \mathbf{x}'$, we have, $\Phi(\mathbf{x}) \to \Phi'(\mathbf{x}') = \mathbf{G^T}\Phi(\mathbf{x})$ where T is a two dimensional matrix which has Jordan form and $G = ||\frac{\partial x'}{\partial x}||$ is the Jacobian. The fields E_t and E_a, transforms as:

$$E_t(\mathbf{x}') = G^{-\frac{2\Delta_E}{D}} E_t(\mathbf{x})$$

$$E_a(\mathbf{x}') = G^{-\frac{2\Delta_E}{D}} (\ln(G) E_t(x) + E_a(x)) \tag{256}$$

This expresses that the top-field E_t always transforms as an ordinary scaling operator. It can be verified that the correlation functions of fields E_t and E_a have the standard d- dimensional logarithmic conformal field theory structures.

Let us consider the case that $d = 2$. Therefore we can write:

$$\begin{aligned} L_0 E_t &= \Delta_E E_t \\ L_0 E_a &= \Delta_E E_a + E_t \\ L_0 E_b &= \Delta_E E_b + E_t \\ L_0 E_c &= \Delta_E E_c + E_t \end{aligned}$$

where we have used the replica symmetry. Using the above equations, it is evident that the dimension of difference-fields $E_a - E_b$ with $a \neq b$ is Δ_E and it transforms as an ordinary operator under the scaling transformation. The important observation is that the individual logarithmic operator E_a do not contribute to the *connected* quenched averaged correlation functions. Instead the connected averaged correlation functions depend on the difference fields $E_a - E_b$ only. For instance in the following we write the connected quenched averaged 2,3 and 4-point functions of local energy density in terms of the difference operators explicitly,

$$\overline{<E(1)E(2)>}_c = \frac{1}{2} <(E_a - E_b)_{(1)}(E_a - E_b)_{(2)}> \tag{257}$$

$$\overline{<E(1)E(2)E(3)>}_c = <(E_a - E_b)_{(1)}(E_a - E_c)_{(2)}(E_a - E_b)_{(3)}> \tag{258}$$

$$\begin{aligned} \overline{<E(1)E(2)E(3)E(4)>}_c = & \qquad (259) \\ & <(E_a - E_b)_{(1)}(E_a - E_c)_{(2)}(E_a - E_d)_{(3)}(E_a - E_b)_{(4)}> \\ -\frac{1}{2} & <(E_a - E_b)_{(1)}(E_c - E_d)_{(2)}(E_c - E_d)_{(3)}(E_a - E_b)_{(4)}> \\ -\frac{1}{4} & <(E_a - E_b)_{(1)}(E_c - E_d)_{(2)}(E_a - E_b)_{(3)}(E_c - E_d)_{(4)}> \\ -\frac{1}{4} & <(E_a - E_b)_{(1)}(E_a - E_b)_{(2)}(E_c - E_d)_{(3)}(E_c - E_d)_{(4)}> \end{aligned}$$

where the last equation has only 15 independent terms. This shows that quenched averaged connected correlation functions have a pure scaling behavior. Let us verify this results from direct calculation of quenched averaged connected 3-point correlation function of energy density.

We are interested in the exact derivation of the various 3-point quenched averaged functions as $\overline{< E(1)E(2)E(3) >}$, also $\overline{< E(1)E(2) >< E(3) >}$, and finally $\overline{< E(1) >< E(2) >< E(3) >}$, which can be written in terms of the replica correlation functions $< E_1(1)E_1(2)E_1(3) > = a$ $< E_1(1)E_1(2)E_2(3) > = b$ and $< E_1(1)E_2(2)E_3(3) > = c$, respectively. One can derive the correlation functions a, b and c by means of the 3-point functions of E_t and $\tilde{E}_a$ as:

$$\begin{aligned} < E_t(1)E_t(2)E_t(3) > &= na + 3n(n-1)b + n(n-1)(n-2)c \\ &\equiv nA_1 \qquad (260) \\ < \tilde{E}_a(1)\tilde{E}_a(2)E_t(3) > &= n_1 a + (n_1^2(n-1) \\ &\quad -4n_1^2 + \frac{1}{n^2}(n-1)^2 + \frac{2}{n^2}(n-1)(n-2))b \\ &\quad +(-\frac{2}{n}n_1(n-1)(n-2) + \frac{1}{n^2}(n-2)^2(n-1))c \\ &\equiv (1-\frac{1}{n})B_1 \qquad (261) \end{aligned}$$

and finally,

$$\begin{aligned} < \tilde{E}_a(1)\tilde{E}_a(2)\tilde{E}_a(3) > &= (n_1^2 - \frac{n-1}{n^3})a \\ &\quad +(-3n_1^2\frac{n-1}{n} - \frac{3}{n^3}(n-1)(n-2) + \frac{3}{n^2}n_1(n-1))b \\ &\quad +(\frac{3}{n^2}n_1(n-1)(n-2) - \frac{1}{n^3}(n-1)(n-2)(n-3))c \\ &\equiv (1-\frac{1}{n})(1-\frac{2}{n})C_1 \qquad (262) \end{aligned}$$

where $n_1 = (1-\frac{1}{n})$. To derive the above equations we use the replica symmetry and symmetries of the various type of 3-point correlation functions under interchanging of positions. We note that replica symmetry leads to have $< \tilde{E}_a(1)E_t(2)E_t(3) >= 0$ and therefore, dose not give any new relationship between a, b and c. Using the above equations, it can be found that the correlation functions a, b and c are as following:

$$\begin{aligned} a &= \frac{3nB_1 - 3nC_1 + n^2C_1 + A_1 - 3B_1 + 2C_1}{n^2} \\ b &= \frac{nB_1 - nC_1 + A_1 - 3B_1 + 2C_1}{n^2} \\ c &= \frac{A_1 - 3B_1 + 2C_1}{n^2} \qquad (263) \end{aligned}$$

Where A_1 , B_1 and C_1 are pure scaling functions of variables $r_{i,j}$. Using the above equations we can show that the connected quenched averaged 3-point function

behaves as:

$$\overline{< E(1)E(2)E(3) >_c} = 2c + a - 3b = C_1 \tag{264}$$

which is a scaling function and confirms the observation that the logarithmic operators (individually) do not play any role in the connected quenched averaged correlation functions. In addition one can derive the correlation functions $< E_i(1)E_j(2)E_k(3) >$ for given i, j and k in the limit of $n \to 0$ and show that they have the following form:

$$< E_i(1)E_j(2)E_k(3) > = [\alpha_{ijk} - \beta_{ijk}D_1 + \gamma_{ijk}(4D_2 - D_1^2)]f(1,2,3) \tag{265}$$

where $f(1,2,3) = (r_{12}r_{13}r_{23})^{-2\Delta_E}$, $D_1 = \ln(r_{12}r_{13}r_{23})$ and $D_2 = \ln r_{23}\ln r_{13} + \ln r_{13}\ln r_{12} + \ln r_{23}\ln r_{12}$. It can also be shown that the ratio of various symmetrized 3-point functions to the connected one behaves asymptotically as a *universal* function

$$\frac{1}{3}(4D_2 - D_1^2). \tag{266}$$

We generalize the above calculations to derive the various type of 4-point correlation functions and show that the ratio of the various disconnected to the connected one has the following universal asymptotic:

$$\sim \frac{1}{36}[O_1^3 - 6O_2 - 3O_3 - 12O_4 - 18O_5] \tag{267}$$

where $O_1 = \ln(r_{12}r_{13}r_{14}r_{23}r_{24}r_{34})$, $O_2 = (\ln r_{ij}\ln r_{kl}^2 + \cdots)$ with $i \neq j \neq k \neq l$, $O_3 = (\ln r_{ij}\ln r_{ik}^2 + \cdots)$ with $i \neq j \neq k$, $O_4 = (\ln r_{ij}\ln r_{kl}\ln r_{lj} + \cdots)$ with $i \neq j \neq k \neq l$, and finally $O_5 = (\ln r_{ij}\ln r_{ik}\ln r_{il} + \cdots)$ with $i \neq j \neq k \neq l$

One can check directly that these different types of the 3 and 4-point correlation functions have the general property of a logarithmic conformal field theory that the logarithmic partner can be regarded as the formal derivative of the ordinary fields (top field) with respect to their conformal weight. In this case, one can consider the E_a fields as the derivative of E_t with respect to n . We emphasis that the derivative with respect to scaling weight can be written in terms of the derivative with respect to n. These properties enable us to calculate any N-point correlation function containing the logarithmic field E_a in terms of the correlation functions of the top-fields. We have shown that the individual logarithmic operators E_a do not have any contribution to the quenched averaged connected correlation functions of the energy density. We also obtain that the connected correlation functions can be written in terms of the difference fields which transform as an ordinary scaling operator. However they will play a crucial role to the disconnected averaged correlation functions. Also we find that the ratio of the various types of 3 and 4-point quenched averaged correlation functions to the connected ones have a universal asymptotic behavior and give their explicit form. Our analysis are valid in all dimensions as long as the dimension is below the upper critical dimensions. To derive the above results we have used the replica symmetry. Any attempt towards the breaking of

this symmetry will change the above picture and may produces more than one logarithmic fields in the block and produce higher order logarithmic singularities. These results can be easily generalized to other problem such as polymer statistics, percolation and random phase sine-Gordon model etc.

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