Differential Equations, From Stochastic to Deterministic; An Example

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Abstract

ABSTRACT. A mathematical discussion is made on obtaining the classical Black-Scholes equation of computational finance using a martingale-based analysis of arbitrage-free market.

1 Introduction

First we recall the definition of martingale. For $t \geq 0$ let I_t be the set of all information available up to time t, so $I_t \subset I_s$ for $t \leq s$. Let the random process $\{ X_t : t \geq 0 \}$ satisfy the following:

(1) $\{ X_t : t \geq 0 \}$ is I_t*-adapted*; that is, for every $t \geq 0$, given I_t, the random variable X_t is completely known. In other words, for every $t \geq 0$, the conditional variable $(X_t | I_t)$ is *deterministic*.

(2) For every $t \geq 0$, the expected value of $|X_t|$ is finite:

$$E \ |X_t| < \infty.$$

Then, *with respect to the probability measure* P, the random process $\{ X_t : t \geq 0 \}$ is called a

- **martingale** if $E_P (X_{t+u} | I_t) = X_t$ for all $t \geq 0$ and all $u \geq 0$; in other words, if given all information about present value and past history of X_t, we still cannot predict how X_t will change in the future.
- **submartingale** if $E_P (X_{t+u} | I_t) \geq X_t$ for all $t \geq 0$ and all $u \geq 0$; in other words, if we "expect" X_t to increase as time passes.
- **supermartingale** if $E_P (X_{t+u} | I_t) \leq X_t$ for all $t \geq 0$ and all $u \geq 0$; in other words, if we "expect" X_t to decrease as time passes.

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In the above, the notation E_P stands for expectation computed with respect to the probability measure P:

$$E_P Y := \int_{-\infty}^{+\infty} y \; dF_{Y,P}(y);$$

with $F_{Y,P}$ the cumulative distribution function of the random variable Y under probability measure P; that is, for every $y \in \mathbf{R}$ ($\mathbf{R}$ stands for the set of real numbers), $F_{Y,P}(y)$ is the probability of the event $Y \le y$ *under probability measure* P.

2 Arbitrage-Free Markets

A fundamental assumption made in financial mathematics is 'arbitrage-free markets". It is the hypothesis that no opportunities for making a risk-free profit exist in the markets. This is a reasonable assumption since any such opportunities become quickly known and are grasped by the market players, resulting in sharp increase in demand for such opportunities and causing the price of such opportunities to rise, thereby making them cease to be opportunities, and moving the market back to the balanced "arbitrage-free" status.

In addition, many financial theories assume that asset prices are martingales; that is, future movements of the asset prices are completely unpredictable. This, of course, does not represent the real world accurately. In the real world, people who invests in the market usually do so with the expectation that prices of financial instruments will rise to bring profit. In other words, with S_t the asset price at time t, and I_t the set of all market information available up to time t, the market is expected to behave in such a way that for all $t \ge 0$ and all $u \ge 0$

$$E_P \left(e^{-r(t+u)} S_{t+u} | I_t \right) \;\ge\; e^{-rt} S_t.$$

We have multiplied the asset price S_t by the discount factor e^{-rt} to get the present value of the asset price (value at time $t = 0$). Here, P is the true probability measure of the states of the world.

In other words, investors invest in the market with the belief that the discounted asset price behaves as a submartingale *under the true probabilities of the states of the world.* It would be desirable to work with a martingale process so that one can use the rich results and techniques of martingale theory. Fortunately, this is possible by a very elegant result of financial mathematics:

The Arbitrage Theorem. *If the probability measure P of the true states of the world is strictly positive; that is, if we exclude all impossible states of the world, then markets are arbitrage-free* if and only if *there exists a probability measure $\widetilde{P}$ with respect to which the discounted asset price is a martingale; that is, for all $t \ge 0$ and all $u \ge 0$*

$$E_{\widetilde{P}} \left(e^{-r(t+u)} S_{t+u} | I_t \right) \;=\; e^{-rt} S_t.$$

The probability measures P and $\widetilde{P}$ are related by $d\widetilde{P} = \xi \; dP$, with the function ξ the Radon-Nykodym derivative of $\widetilde{P}$ with respect to P.

See [3] for details. Probability measures P and $\widetilde{P}$ are equivalent in the sense that they have the same domain (set of all possible states of the world). The advantage of $\widetilde{P}$ over P is that discounted asset price is a martingale under $\widetilde{P}$. However, one should note that $\widetilde{P}$ is a "synthetic" probability not representing the real world probabilities.

It is important to note that asset prices in the real world can change in quite complicated ways. Some assets display even fractal-like highly stochastic behavior, requiring results of chaos theory for their analysis; see [4].

3 Risk-Adjusted Probabilities

The important probability measure $\widetilde{P}$ is called the **risk-adjusted synthetic probability measure** of the states of the world. The reason why it is called "risk-adjusted" is the following important property:

Theorem. *The synthetic probability measure $\widetilde{P}$, which converts the discounted asset price into a martingale, switches the drift parameter of the asset price to the risk-free interest rate.*

In other words, with respect to synthetic probabilities, all assets have the same expected value of return equal to the return under risk free interest rates (this is the important "equalization of rates of return" result; see [3]). We demonstrate this important result in the case of asset price being a "geometric Wiener process":

$$S_t = S_0 \; e^{Y_t};$$

with $\{ \; Y_t : \; t \geq 0 \; \}$ a Wiener process with drift μ and variance σ; that is

(1) $Y_0 = 0$.

(2) $\{ \; Y_t : \; t \geq 0 \; \}$ has the *independent increments property*; that is, for every two *disjoint* time intervals (s, t) and (u, v), the random variables $Y_t - Y_s$ and $Y_v - Y_u$ are independent.

(3) $\{ \; Y_t : \; t \geq 0 \; \}$ has the *stationary increments property*; that is, for every time interval (s, t), the random variable $Y_t - Y_s$ has the same probability distribution as the random variable Y_{t-s}.

(4) For every $t \geq 0$,

$$Y_t \sim N\left(\mu t, \sigma^2 t\right);$$

that is, for every $t \geq 0$, Y_t is a normal random variable with mean μt and variance $\sigma^2 t$, so with standard deviation proportional to $\sqrt{t}$, not t.

So here P, the true probability measure of states of the world, is the normal distribution with mean μt and variance $\sigma^2 t$ at time t. Now for every $t \geq 0$ and $u \geq 0$, the quantity S_t is fully determined given I_t, the set of all market information up to time t. Hence it can be taken inside the expectation operator conditioned on I_t:

$$\begin{aligned} E_P\left(S_{t+u}|I_t\right) &= S_t\ E_P\left(\frac{S_{t+u}}{S_t}|I_t\right) \\ &= S_t\ E_P\left(\exp\left(Y_{t+u}-Y_t\right)|I_t\right). \end{aligned}$$

Since $Y_{t+u} - Y_t$ and $Y_{(t+u)-t} = Y_u$ have the same distribution, we get

$$E_P\left(S_{t+u}|I_t\right) = S_t\ E_P\left(\exp\left(Y_u\right)|I_t\right).$$

Next We use the fact that the moment generating function of the normal random variable X with mean μ and variance σ^2 is

$$E\left(e^{\lambda X}\right) = \exp\left(\mu\lambda + \frac{1}{2}\sigma^2\lambda^2\right).$$

so

$$E\left(e^{X}\right) = \exp\left(\mu + \frac{1}{2}\sigma^2\right).$$

Noting that the conditional random variable $Y_u|I_t$ has normal distribution with mean μu and variance $\sigma^2 u$, by the above corollary we have

$$E_P\left(\exp\left(Y_u\right)|I_t\right) = \exp\left[\left(\mu + \frac{1}{2}\sigma^2\right)u\right],$$

so

$$E_P\left(S_{t+u}|I_t\right) = \exp\left[\left(\mu + \frac{1}{2}\sigma^2\right)u\right]\ S_t.$$

Discounting both sides by $e^{-r(t+u)}$, we get

$$E_P\left(e^{-r(t+u)}S_{t+u}|I_t\right) = \exp\left[\left(\mu - r + \frac{1}{2}\sigma^2\right)u\right]e^{-rt}S_t.$$

Now $e^{-rt}S_t$ is a submartingale under the true probability measure P:

$$E_P\left(e^{-r(t+u)}S_{t+u}|I_t\right) \geq e^{-rt}S_t.$$

The above relations give the clue on how to synthesize a probability measure $\widetilde{P}$ under which $e^{-rt}S_t$ is a martingale. Noting that the parameter that makes asset price predictable is the drift not the variance, we assume that at time t, the probability measure $\widetilde{P}$ is a normal distribution like P, with the same variance $\sigma^2 t$ but a different unknown drift (that is, mean) ρt to be determined.

Exactly similar arguments as above lead to

$$E_{\widetilde{P}}\left(e^{-r(t+u)}S_{t+u}|I_t\right) = \exp\left[\left(\rho - r + \frac{1}{2}\sigma^2\right)u\right]e^{-rt}S_t.$$

Since we want the discounted asset price $e^{-rt}S_t$ to be a martingale under $\widetilde{P}$, we should have

$$E_{\widetilde{P}}\left(e^{-r(t+u)}S_{t+u}|I_t\right) = e^{-rt}S_t.$$

The two equalities imply

$$\exp\left[\left(\rho - r + \frac{1}{2}\sigma^2\right)u\right] = 1 \quad \text{for all } u \geq 0 \qquad \Longrightarrow \qquad \rho = r - \frac{1}{2}\sigma^2.$$

making $\widetilde{P}$ fully known. It is of interest to note that the drift parameter μ does not appear in the result; the synthetic probability $\widetilde{P}$ depends only on the variance (also called diffusion parameter) of the true probability P, not its drift. Now $Y_t \sim N(\mu t, \sigma^2 t)$ implies that Y_t is governed by the stochastic differential equation

$$dY_t = \mu\ dt + \sigma\ dW_t;$$

with W_t a standard Wiener Process; that is, $W_t \sim N(0, t)$. The term $\mu\ dt$ models the *deterministic* change in Y_t, and the term $\sigma\ dW_t$ models the *stochastic* change in Y_t.

4 Derivation of Black-Scholes Equation

From this we can obtain the stochastic differential equation governing the dynamics of the asset price. The differential of the asset price is

$$\begin{aligned} dS_t &= \frac{\partial S_t}{\partial Y_t}\ dY_t + \frac{\partial S_t}{\partial t}\ dt \\ &+ \frac{1}{2}\frac{\partial^2 S_t}{\partial Y_t^2}\ dY_t^2 + \frac{1}{2}\frac{\partial^2 S_t}{\partial t^2}dt^2 + \frac{\partial^2 S_t}{\partial Y_t\ \partial t}\ dY_t\ dt + \cdots \end{aligned}$$

In deterministic calculus, one would retain only the first two terms on the righthand side and would ignore all other terms by their being of smaller orders of magnitude. With the objective that we only retain terms of order dt and ignore all $o(dt)$ terms, we cannot ignore the third term on the righthand side. To see this, from the stochastic differential equation for Y_t we have

$$\begin{aligned} dY_t^2 &= (\mu\ dt + \sigma\ dW_t)^2 \\ &= \mu^2\ dt^2 + \sigma^2\ dW_t^2 + 2\mu\sigma\ dt\ dW_t. \end{aligned}$$

By the important property

$$dW_t^2 = dt$$

of the Wiener process $\{\ W_t :\ t \geq 0\ \}$ (see [3]), this leads to

$$dY_t^2 = \sigma^2 dt + o(dt).$$

Plugging in $\sigma^2 dt$ for dY_t^2 in the expression for dS_t, we arrive at the "Itô's formula" for the asset price:

$$dS_t = \frac{\partial S_t}{\partial Y_t}\ dY_t + \frac{\partial S_t}{\partial t}\ dt + \frac{1}{2}\sigma^2\ \frac{\partial^2 S_t}{\partial Y_t^2}\ dt.$$

Now by $S_t = \exp(Y_t)$,

$$\frac{\partial S_t}{\partial Y_t} = \frac{\partial^2 S_t}{\partial Y_t^2} = S_t; \qquad\qquad \frac{\partial S_t}{\partial t} = 0.$$

So

$$dS_t = S_t\,(\mu\ dt + \sigma\ dW_t) + \frac{1}{2}\sigma^2 S_t\ dt.$$

In summary, *under the true probability measure* P, dynamics of the asset price is governed by the stochastic differential equation

$$dS_t = \left[\left(\mu + \frac{1}{2}\sigma^2\right) S_t\right]\ dt + (\sigma S_t)\ dW_t.$$

Unlike the stochastic differential equation for Y_t which is constant-coefficient, the stochastic differential equation for S_t is variable-coefficient, with drift and diffusion coefficients functions of S_t itself.

By the same argument, *under the synthetic probability measure* $\widetilde{P}$, dynamics of the asset price is governed by the stochastic differential equation

$$dS_t = \left[\left(\rho + \frac{1}{2}\sigma^2\right) S_t\right]\ dt + (\sigma S_t)\ d\widetilde{W}_t.$$

Note that the stochastic part of this equation is governed by a standard Wiener process $\widetilde{W}_t$, which is different from W_t. In simulations, values for both of these are drawn from the normal distribution with zero mean and variance t, but they generally take different values. By a martingale analysis, we found $\rho = r - \sigma^2/2$. Substituting this, we obtain the stochastic differential equation

$$dS_t = (rS_t)\ dt + (\sigma S_t)\ d\widetilde{W}_t.$$

As we see, the synthetic probability measure $\widetilde{P}$ has "risk-adjusted" the asset price; the drift parameter is now rS_t, the risk-free interest rate as coefficient instead of μ.

In their breakthrough analysis that led to the well known Black-Scholes partial differential equation (see [1] or [5]), Black and Scholes *assumed* the stochastic differential equation

$$dS_t = (rS_t)\ dt + (\sigma S_t)\ d\widetilde{W}_t.$$

as dynamics of the asset price, as the starting point and derived their partial differential equation by an arbitrage argument.

Generally, the price of an option is a function of asset price S_t and time t, so we can represent its price at time t by $F(S_t, t)$. The current value of this option is then $e^{-rt}F(S_t, t)$, for which, by similar arguments, we have the Itô's formula

$$d\left[e^{-rt}F(S_t,t)\right] = -re^{-rt}F\ dt + e^{-rt}\left(\frac{\partial F}{\partial t}\ dt + \frac{\partial F}{\partial S_t}\ dS_t + \frac{1}{2}\frac{\partial^2 F}{\partial S_t^2}\ dS_t^2\right).$$

Again in a deterministic environment, one would ignore the term $\frac{1}{2}\frac{\partial^2}{\partial S_t^2}\ dS_t^2$. However, here we have

$$dS_t^2 = r^2 S_t^2\ dt^2 + \sigma^2 S_t^2 d\widetilde{W}_t^2 + 2r\sigma S_t^2\ dt\ d\widetilde{W}_t^2 = \sigma^2 S_t^2\ dt + o(dt)$$

again by $d\widetilde{W}_t^2 = dt$. Hence, ignoring all $o(dt)$ terms, we have

$$\begin{aligned} d\left[e^{-rt}F(S_t,t)\right] &= \left[e^{-rt}\left(\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 F}{\partial S_t^2} + rS_t \frac{\partial F}{\partial S_t} - rF\right)\right]\ dt \\ &\quad + \left(e^{-rt}\sigma S_t\right)\ d\widetilde{W}_t. \end{aligned}$$

The discounted option price is a martingale under $\widetilde{P}$, hence we cannot make any forecasts on how it will change. In other words, the discounted option price is a "purely" stochastic process with no deterministic behavior. Translated into the stochastic differential equation , this implies that the drift term in the above equation must be zero:

$$\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 F}{\partial S_t^2} + rS_t \frac{\partial F}{\partial S_t} - rF = 0.$$

This is the fundamental Black-Scholes equation.

5 Conclusion

Financial instruments and derivative securities can be analyzed in a variety of ways and by a variety of techniques, which are theoretically equivalent but practically different. Martingale arguments are generally more complex than arbitrage arguments, but are generally more useful and of wider applicability.

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