Theory of Formal Languages and Automata

Lecture 4

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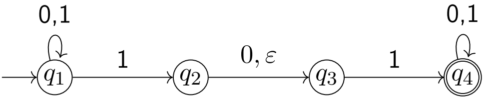
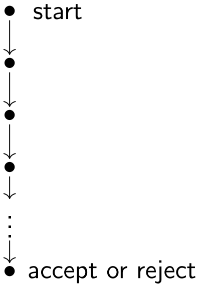
Nondeterminism

- Generalization of determinism: Every deterministic finite automaton (DFA) is a nondeterministic finite automaton (NFA)
  - Several transitions for a symbol
    - Machine splits into multiple copies of itself
    - Similar to parallel computation
  - No transition for every symbol
    - A machine (branch of computation) dies if it can not consume the current symbol on the input tape
  - Transition with ε
    - Make a transition without reading any symbol
  - NFA accepts the input string, if any of copies accept

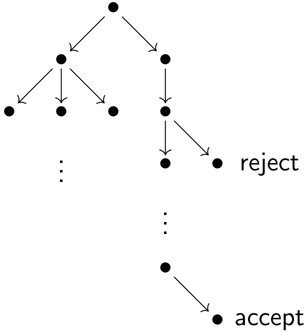


Nondeterminism

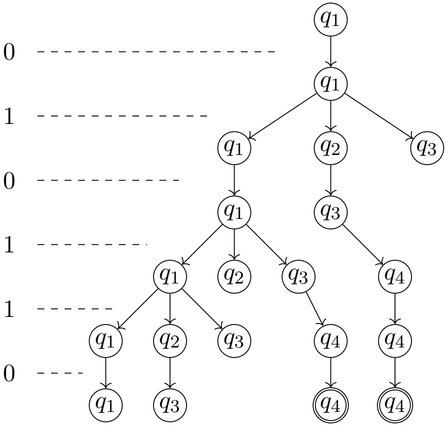
Deterministic computation

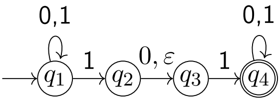


Nondeterministic computation



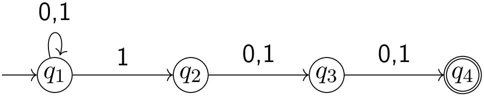
Nondeterminism





Nondeterminism

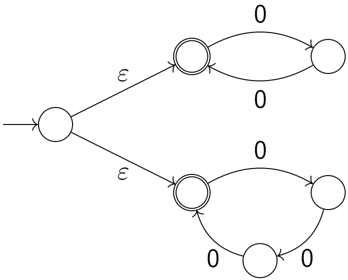
NFAs can guess:



A = { w | w has a 1 in the third position from the end }

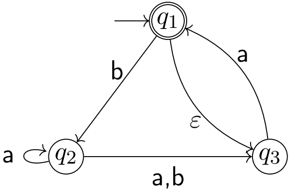
Nondeterminism

NFAs can guess:



$$A = \{0^{2k} | k \in \{0, 1, \dots\}\} \cup \{0^{3k} | k \in \{0, 1, \dots\}\}$$

Nondeterminism



Accepts:

- ε
- a
- baba
- baa

Rejects:

- b
- bb
- babba

Formal definition:

Definition

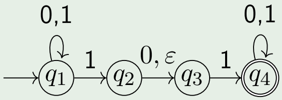
A 5-tuple $(Q, \Sigma, \delta, q_0, F)$, where

1. Q: (finite set of) states,
2. Σ: (finite) alphabet,
3. δ: $Q \times \Sigma_\varepsilon \to \mathcal{P}(Q)$ transition function,
4. $q_0 \in Q$: start state,
5. $F \subseteq Q$: accept states.

- $\mathcal{P}(Q)$: Power set of Q
- $\Sigma_\varepsilon = \Sigma \cup \{\varepsilon\}$

Nondeterministic Finite Automata

Example



1. $Q = \{q_1, q_2, q_3, q_4\}$
2. $\Sigma = \{0, 1\}$
3. δ:

	0	1	ε
q_1	$\{q_1\}$	$\{q_1, q_2\}$	$\emptyset$
q_2	$\{q_3\}$	$\emptyset$	$\{q_3\}$
q_3	$\emptyset$	$\{q_4\}$	$\emptyset$
q_4	$\{q_4\}$	$\{q_4\}$	$\emptyset$

4. q_1: start state,
5. $F = \{q_4\}$

Prove that following languages are regular ($\Sigma = \{0, 1\}$):

- $L_1 = \emptyset$
- $L_2 = \{\varepsilon\}$
- $L_1 = \{w \mid w \text{ starts with } 0 \text{ and ends with } 1\}$

Nondeterministic Finite Automata

Definition (NFA's Computation)

Consider $N = (Q, \Sigma, \delta, q_0, F)$
$w \in L(N)$, if:

1. It is possible to write $w = y_1 y_2 \ldots y_m$ where $y_i \in \Sigma_\varepsilon$,
2. There is a sequence of states $r_1, \ldots, r_m$ with following conditions:
    1. $r_0 = q_0$,
    2. $r_{i+1} \in \delta(r_i, y_{i+1})$, for $i = 0, \ldots, m-1$, and
    3. $r_m \in F$.

Equivalence of NFAs and DFAs

- NFAs and DFAs are equivalent: They recognize the same class of languages

Theorem

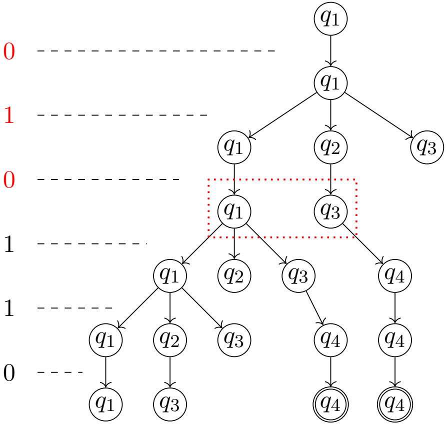
Every NFA has an equivalent DFA.

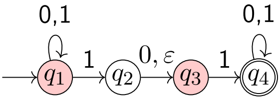
Proof idea:

- Construct a DFA that simulates the NFA,
- Keep track of branches of the computation (active states),
- If the NFA has k states, there are 2^k subsets of active states,
- The DFA needs 2^k states.

Equivalence of NFAs and DFAs

Active states:





Proof (Without ε).

Let $N = (Q, \Sigma, \delta, q_0, F)$ that has not ε arrows and $A = L(N)$.
We construct $M = (Q', \Sigma, \delta', q_0', F')$ that recognizes A.

1. $Q' = \mathcal{P}(Q)$.
2. For $R \in Q'$ and $a \in \Sigma$:

$$\delta'(R, a) = \{q \in Q | q \in \delta(r, a) \text{ for some } r \in R\}. \tag{1}$$

3. $q_0' = \{q_0\}$.
4. $F' = \{R \in Q' | R \text{ contains an accept state of } N\}$

$\square$

Proof (With ε).

$E(R) = \{\ q \mid \text{can reach } q \text{ from } r \in R \text{ via zero or more } \varepsilon \text{ arrows}\ \}$

1. $Q' = \mathcal{P}(Q)$.
2. $R \in Q'$ and $a \in \Sigma$:

$$\delta'(R, a) = \{q \in Q | q \in E(\delta(r, a)) \text{ for some } r \in R\}. \tag{2}$$

3. $q_0' = E(\{q_0\})$.
4. $F' = \{R \in Q' | R \text{ contains an accept state of } N\}$

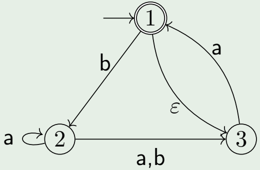
□

Equivalence of NFAs and DFAs

Example

1. $Q' = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\},$
   $\{1,3\}, \{2,3\}, \{1,2,3\}\}.$
2. $q'_0 = E(\{1\}) = \{1,3\}.$
3. $F' = \{\{1\}, \{1,2\}, \{1,3\}, \{1,2,3\}\}.$
4.

	a	b
$\emptyset$	$\emptyset$	$\emptyset$
$\{1\}$	$\emptyset$	$\{2\}$
$\{2\}$	$\{2,3\}$	$\{3\}$
$\{3\}$	$\{1,3\}$	$\emptyset$
$\{1,2\}$	$\{2,3\}$	$\{2,3\}$
$\{1,3\}$	$\{1,3\}$	$\{2\}$
$\{2,3\}$	$\{1,2,3\}$	$\{3\}$
$\{1,2,3\}$	$\{1,2,3\}$	$\{2,3\}$

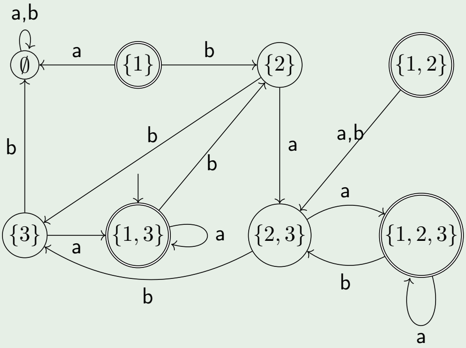


Equivalence of NFAs and DFAs

Example

1. $Q' = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$.
2. $q'_0 = E(\{1\}) = \{1,3\}$.
3. $F' = \{\{1\}, \{1,2\}, \{1,3\}, \{1,2,3\}\}$.
4.

	a	b
$\emptyset$	$\emptyset$	$\emptyset$
$\{1\}$	$\emptyset$	$\{2\}$
$\{2\}$	$\{2,3\}$	$\{3\}$
$\{3\}$	$\{1,3\}$	$\emptyset$
$\{1,2\}$	$\{2,3\}$	$\{2,3\}$
$\{1,3\}$	$\{1,3\}$	$\{2\}$
$\{2,3\}$	$\{1,2,3\}$	$\{3\}$
$\{1,2,3\}$	$\{1,2,3\}$	$\{2,3\}$

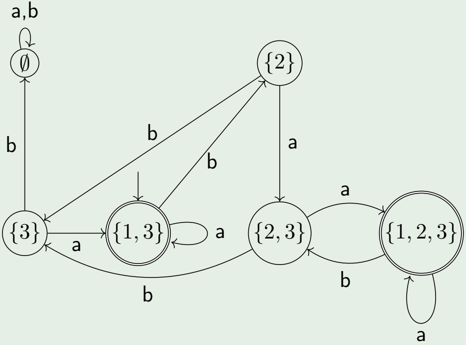


Equivalence of NFAs and DFAs

Example

1. $Q' = \{\emptyset, \{2\}, \{3\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}$.
2. $q'_0 = E(\{1\}) = \{1,3\}$.
3. $F' = \{\{1,3\}, \{1,2,3\}\}$.
4.

	a	b
$\emptyset$	$\emptyset$	$\emptyset$
$\{2\}$	$\{2,3\}$	$\{3\}$
$\{3\}$	$\{1,3\}$	$\emptyset$
$\{1,3\}$	$\{1,3\}$	$\{2\}$
$\{2,3\}$	$\{1,2,3\}$	$\{3\}$
$\{1,2,3\}$	$\{1,2,3\}$	$\{2,3\}$

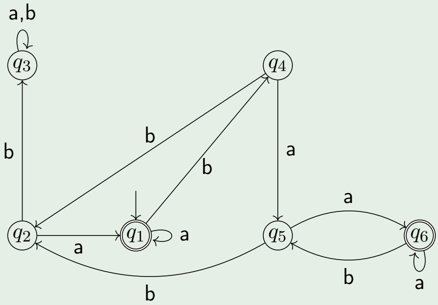


Equivalence of NFAs and DFAs

Example

1. $Q' = \{q_1, q_2, \ldots, q_6\}$.
2. $q'_0 = q_1$.
3. $F' = \{q_1, q_6\}$.
4.

	a	b
q_3	q_3	q_3
q_4	q_5	q_2
q_2	q_1	q_3
q_1	q_1	q_4
q_5	q_6	q_2
q_6	q_6	q_5



Equivalence of NFAs and DFAs

Example

1. $Q' = \{q_1, q_2, \ldots, q_6\}$.
2. $q'_0 = q_1$.
3. $F' = \{q_1, q_6\}$.
4.

	a	b
q_1	q_1	q_4
q_2	q_1	q_3
q_3	q_3	q_3
q_4	q_5	q_2
q_5	q_6	q_2
q_6	q_6	q_5