

Performance Guarantees for Schatten- p Quasi-Norm Minimization in Recovery of Low-Rank Matrices[☆]

Mohammadreza Malek-Mohammadi^{a,*}, Massoud Babaie-Zadeh^a, Mikael Skoglund^b

^aElectrical Engineering Department, Sharif University of Technology, Tehran 1458889694

^bCommunication Theory Lab, KTH- Royal Institute of Technology, Stockholm, 10044, Sweden

Abstract

We address some theoretical guarantees for Schatten- p quasi-norm minimization ($p \in (0, 1]$) in recovering low-rank matrices from compressed linear measurements. Firstly, using null space properties of the measurement operator, we provide a sufficient condition for exact recovery of low-rank matrices. This condition guarantees unique recovery of matrices of ranks equal or larger than what is guaranteed by nuclear norm minimization. Secondly, this sufficient condition leads to a theorem proving that all restricted isometry property (RIP) based sufficient conditions for ℓ_p quasi-norm minimization generalize to Schatten- p quasi-norm minimization. Based on this theorem, we provide a few RIP-based recovery conditions.

Keywords: Affine Rank Minimization (ARM), Nuclear Norm Minimization (NNM), Restricted Isometry Property (RIP), Schatten- p Quasi-Norm Minimization (p SNM).

1. Introduction

Matrix rank minimization constrained to a set of underdetermined linear equations, known as affine rank minimization (ARM), has numerous applications in signal processing and control theory [1, 2]. An important special case of this optimization problem is *Matrix Completion* (MC) in which one aims to recover a matrix from partially observed entries [2]. Applications of ARM and MC include collaborative filtering [2], machine learning [3], quantum state tomography [4], ultrasonic tomography [5], spectrum sensing [6], direction-of-arrival estimation [7], and RADAR [8], among others.

Rank minimization under affine equality constraints is generally formulated as

$$\min_{\mathbf{X}} \text{rank}(\mathbf{X}) \quad \text{subject to} \quad \mathcal{A}(\mathbf{X}) = \mathbf{b}, \quad (1)$$

where $\mathbf{X} \in \mathbb{R}^{n_1 \times n_2}$, $\mathcal{A} : \mathbb{R}^{n_1 \times n_2} \rightarrow \mathbb{R}^m$ is a given linear operator (measurement operator), and $\mathbf{b} \in \mathbb{R}^m$ is the vector of measurements. In case of incomplete measurements, m is less than

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*Corresponding author. Tel: +98 912 5123401.

Email addresses: m.rezamm@ieee.org (Mohammadreza Malek-Mohammadi), mbzadeh@yahoo.com (Massoud Babaie-Zadeh), skoglund@ee.kth.se (Mikael Skoglund)

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$n_1 n_2$, or, usually, $m \ll n_1 n_2$. Problem (1) is generally NP-hard [1], yet there are many efficient algorithms to solve relaxed or approximated versions of it. Nuclear norm minimization (NNM), proposed in [1], replaces the rank with its tightest convex relaxation which leads to

$$\min_{\mathbf{X}} \|\mathbf{X}\|_* \quad \text{subject to} \quad \mathcal{A}(\mathbf{X}) = \mathbf{b}, \quad (2)$$

where $\|\mathbf{X}\|_* \triangleq \sum_{i=1}^r \sigma_i(\mathbf{X})$ denotes matrix nuclear norm in which $\sigma_i(\mathbf{X})$ is the i th largest singular value of \mathbf{X} and r is the rank of the matrix \mathbf{X} . It has been proven that, under some sufficient conditions, (1) and (2) share the same unique solution; see, e.g., [2, 9].

The nuclear norm of a matrix is equal to the ℓ_1 norm of a vector formed by the singular values of the same matrix. Consequently, inspired by experimental observations and theoretical guarantees showing superiority of ℓ_p quasi-norm minimization to ℓ_1 minimization in Compressive Sampling (CS) [10], another approach in [11, 12] replaces the rank function with the Schatten- p quasi-norm resulting in

$$\min_{\mathbf{X}} \|\mathbf{X}\|_p^p \quad \text{subject to} \quad \mathcal{A}(\mathbf{X}) = \mathbf{b}, \quad (3)$$

where $\|\mathbf{X}\|_p \triangleq \left(\sum_{i=1}^r \sigma_i^p(\mathbf{X}) \right)^{1/p}$ for some $p \in (0, 1)$ denotes the Schatten- p quasi-norm. While the above problem is nonconvex, it is observed that numerically efficient implementations of (3) outperforms NNM [11–13].

In practice, there is often some noise in measurements, so measurement model is updated to $\mathcal{A}(\mathbf{X}) + \mathbf{e} = \mathbf{b}$, where \mathbf{e} is the vector of measurement noise. To robustly recover a minimum-rank solution, equality constraints are relaxed to $\|\mathcal{A}(\mathbf{X}) - \mathbf{b}\|_2 \leq \epsilon$, where $\|\cdot\|_2$ denotes the ℓ_2 norm of a vector and $\epsilon \geq \|\mathbf{e}\|_2$ is some constant [2]. Therefore, (3) is modified to

$$\min_{\mathbf{X}} \|\mathbf{X}\|_p^p \quad \text{subject to} \quad \|\mathcal{A}(\mathbf{X}) - \mathbf{b}\|_2 \leq \epsilon. \quad (4)$$

Though there are several theoretical studies concerning ℓ_p quasi-norm minimization in the CS literature (see, for example, [14–17]), only a few papers deal with performance guarantees of Schatten- p quasi-norm minimization (p SNM). In [18], authors propose a necessary and sufficient condition for exact recovery of low-rank matrices using null space properties of \mathcal{A} . However, the sufficient condition is not sharp and seems to be stronger than that of NNM. In contrast, it is well known that finding the global solution of ℓ_p quasi-norm minimization in CS scenario is superior to ℓ_1 minimization [14–16]. Therefore, when one considers the strong parallels between CS and ARM (see [1] for a comprehensive discussion) and superior experimental performance of p SNM in comparison to NNM, he/she expects weaker recovery conditions. We will show that this intuition is indeed the case by providing a sharp sufficient condition, and proving that, using (3), one can uniquely find matrices with equal or larger ranks than those of recoverable by NNM.

In addition, we further exploit this sufficient condition and extend a result from [18] to prove that all restricted isometry property (RIP) based results for recovery of sparse vectors using ℓ_p quasi-norm minimization generalize to Schatten- p quasi-norm minimization with no change. In particular, extending some results of [15], we will show that if $\delta_{2r} < 0.4531$, then all low-rank or approximately low-rank matrices with at most r dominant singular values can be recovered accurately from noisy measurements via (4). This generalization also proves that, for some sufficiently small $p > 0$, if $\delta_{2r+2} < 1$, then, program (4) recovers all matrices with at most r large singular values from noisy measurements accurately. Furthermore, another RIP-based sufficient condition will be presented which is sharper than a threshold in [15] for small values of p .

The rest of this letter is organized as follows. After introducing some notations, in Section 2, we will present our performance analysis. Section 3 is devoted to the proofs of the main results which is followed by conclusion.

Notations: A vector is called k -sparse if it has k nonzero components. \mathbf{x}^\downarrow denotes a vector obtained by sorting elements of \mathbf{x} in terms of magnitude in descending order, and $\mathbf{x}^{(k)}$ designates a vector consisted of the k largest elements (in magnitude) of \mathbf{x} . Let $\langle \mathbf{x}, \mathbf{y} \rangle \triangleq \mathbf{x}^T \mathbf{y}$ be the inner product of \mathbf{x} and \mathbf{y} and $\|\mathbf{x}\|_2 \triangleq \langle \mathbf{x}, \mathbf{x} \rangle^{\frac{1}{2}}$ stand for the Euclidean-norm. ℓ_p quasi-norm of \mathbf{x} for $p \in (0, 1)$ is defined as $\|\mathbf{x}\|_p \triangleq \left(\sum_i x_i^p \right)^{1/p}$, where x_i is the i th entry of \mathbf{x} . For any matrix $\mathbf{X} \in \mathbb{R}^{n_1 \times n_2}$, define $n \triangleq \min(n_1, n_2)$. It is always assumed that singular values of matrices are sorted in descending order, and $\boldsymbol{\sigma}(\mathbf{X}) = (\sigma_1(\mathbf{X}), \dots, \sigma_n(\mathbf{X}))^T$ is the vector of singular values of \mathbf{X} . $\|\mathbf{X}\|_F \triangleq \sqrt{\sum_{i=1}^n \sigma_i^2(\mathbf{X})}$ denotes the Frobenius norm. Furthermore, let $\mathbf{X} = \mathbf{U} \text{diag}(\boldsymbol{\sigma}(\mathbf{X})) \mathbf{V}^T$ denotes the singular value decomposition (SVD) of \mathbf{X} , where $\mathbf{U} \in \mathbb{R}^{n_1 \times n_1}$ and $\mathbf{V} \in \mathbb{R}^{n_2 \times n_2}$. $\mathbf{X}^{(r)} = \mathbf{U} \text{diag}(\sigma_1(\mathbf{X}), \dots, \sigma_r(\mathbf{X}), 0, \dots, 0) \mathbf{V}^T$ represents a matrix obtained by keeping the r largest singular values in the SVD of \mathbf{X} and setting others to 0. For a linear operator $\mathcal{A} : \mathbb{R}^{n_1 \times n_2} \rightarrow \mathbb{R}^m$, let $\mathcal{N}(\mathcal{A}) \triangleq \{\mathbf{X} \in \mathbb{R}^{n_1 \times n_2} : \mathcal{A}(\mathbf{X}) = \mathbf{0}, \mathbf{X} \neq \mathbf{0}\} = \text{null}(\mathcal{A}) \setminus \{\mathbf{0}\}$. For a set S , $|S|$ denotes its cardinality.

2. Main Results

2.1. A null space condition

In [18], exploiting null space properties of \mathcal{A} , a necessary and sufficient condition for successful reconstruction of minimum-rank solutions via (3) are derived, yet there is a gap between these conditions. In this paper, we close this gap by introducing the following lemma, which is mainly based on a result from [19], and prove that the necessary condition in [18] is also sufficient. Moreover, we will show that, using p SNM, one can uniquely recover all matrices with equal or larger rank than those of uniquely recoverable by NNM.

Lemma 1. *All matrices $\mathbf{X} \in \mathbb{R}^{n_1 \times n_2}$ of rank at most r can be uniquely recovered by (3), provided that, $\forall \mathbf{W} \in \mathcal{N}(\mathcal{A})$,*

$$\sum_{i=1}^r \sigma_i^p(\mathbf{W}) < \sum_{i=r+1}^n \sigma_i^p(\mathbf{W}).$$

It is worth mentioning that the sufficient condition in Lemma 1 is weaker than the corresponding sufficient condition in [18] which, according to our notations, is formulated as

$$\sum_{i=1}^{2r} \sigma_i^p(\mathbf{W}) < \sum_{i=2r+1}^n \sigma_i^p(\mathbf{W}).$$

Since $\sum_{i=1}^r \sigma_i^p(\mathbf{W}) \leq \sum_{i=1}^{2r} \sigma_i^p(\mathbf{W})$ and $\sum_{i=r+1}^n \sigma_i^p(\mathbf{W}) \geq \sum_{i=2r+1}^n \sigma_i^p(\mathbf{W})$, the sufficient condition in Lemma 1 is less restrictive than [18, Theorem 3]. Based on the above sufficient condition, we have the following proposition which is a routine extension of [14, Theorem 5].

Proposition 1. *Let $r_p^*(\mathcal{A})$ and $r_1^*(\mathcal{A})$ denote the maximum ranks such that all matrices \mathbf{X} with $\text{rank}(\mathbf{X}) \leq r_p^*(\mathcal{A})$ and $\text{rank}(\mathbf{X}) \leq r_1^*(\mathcal{A})$ can be uniquely recovered by (3) and (2), respectively. Then $r_p^*(\mathcal{A}) \geq r_1^*(\mathcal{A})$ for any $p \in (0, 1)$.*

2.2. RIP-based conditions

Inspired by the strong parallels between CS and ARM, [18] simplifies generalization of some results on ℓ_1 norm minimization to nuclear norm minimization. Remarkably, it shows that all RIP-based conditions for stable and robust recovery of sparse vectors through ℓ_1 norm minimization directly generalize to nuclear norm minimization. Furthermore, [18] proves a similar equivalence between RIP-based conditions for recovery of sparse vectors via ℓ_p quasi-norm minimization and recovery of low-rank matrices using p SNM. Nevertheless, the established equivalence in [18, Lemma 14] is not as strong as one might expect. In essence, it shows an equivalence between RIP conditions for recovery of $2k$ -sparse vectors and RIP conditions for reconstruction of rank k matrices. However, it is natural to have the equivalence between sparsity and rank of the same order. Utilizing Lemma 1, we make the order of sparsity and rank equal to k in the aforementioned equivalence. To that end, first, formulation of ℓ_p quasi-norm minimization as well as the definitions of RIP for vector and matrix cases are recalled.

In ℓ_p quasi-norm minimization, the program

$$\min_{\mathbf{x}} \|\mathbf{x}\|_p^p \quad \text{subject to} \quad \|\mathbf{A}\mathbf{x} - \mathbf{b}_v\|_2 \leq \epsilon \quad (5)$$

is used to estimate a sparse vector $\mathbf{x} \in \mathbb{R}^{m_v}$ from noisy measurements $\mathbf{b}_v = \mathbf{A}\mathbf{x} + \mathbf{e}_v$ in which $\mathbf{A} \in \mathbb{R}^{n_v \times m_v}$ and $\mathbf{b}_v \in \mathbb{R}^{n_v}$ are known and \mathbf{e}_v is noise vector with $\|\mathbf{e}_v\|_2 \leq \epsilon$.

Definition 1 ([20]). For matrix \mathbf{A} and all integers $k \leq m_v$, the restricted isometry constant (RIC) of order k is the smallest constant $\delta_k(\mathbf{A})$ such that

$$(1 - \delta_k(\mathbf{A}))\|\mathbf{x}\|_2^2 \leq \|\mathbf{A}\mathbf{x}\|_2^2 \leq (1 + \delta_k(\mathbf{A}))\|\mathbf{x}\|_2^2$$

holds for all vectors \mathbf{x} with sparsity at most k .

Definition 2 ([18]). For linear operator \mathcal{A} and all integers $r \leq n$, the RIC of order r is the smallest constant $\delta_r(\mathcal{A})$ such that

$$(1 - \delta_r(\mathcal{A}))\|\mathbf{X}\|_F^2 \leq \|\mathcal{A}(\mathbf{X})\|_2^2 \leq (1 + \delta_r(\mathcal{A}))\|\mathbf{X}\|_F^2$$

holds for all matrices \mathbf{X} with rank at most r .

The following theorem formally shows how the results are extended to p SNM.

Theorem 1. Let $\mathbf{x}_0 \in \mathbb{R}^{m_v}$ be any arbitrary vector, $\mathbf{b}_v = \mathbf{A}\mathbf{x}_0 + \mathbf{e}_v$, and \mathbf{x}^* denote a solution to (5) to recover \mathbf{x}_0 . Likewise, let $\mathbf{X}_0 \in \mathbb{R}^{n_1 \times n_2}$ be any arbitrary matrix, $\mathbf{b} = \mathcal{A}(\mathbf{X}_0) + \mathbf{e}$, and \mathbf{X}^* denote a solution to (4) to recover \mathbf{X}_0 . Assume that RIP condition $f(\delta_{k_1}(\mathbf{A}), \dots, \delta_{k_u}(\mathbf{A})) < \delta_0$, for some function f , is sufficient to have

$$\begin{aligned} \|\mathbf{x}_0 - \mathbf{x}^*\|_p &\leq g_1(\mathbf{x}_0^\downarrow, \epsilon), \\ \|\mathbf{x}_0 - \mathbf{x}^*\|_2 &\leq g_2(\mathbf{x}_0^\downarrow, \epsilon), \end{aligned}$$

for some functions g_1 and g_2 . Then, under the same RIP condition $f(\delta_{k_1}(\mathcal{A}), \dots, \delta_{k_u}(\mathcal{A})) < \delta_0$, we have

$$\begin{aligned} \|\mathbf{X}_0 - \mathbf{X}^*\|_p &\leq g_1(\sigma(\mathbf{X}_0), \epsilon), \\ \|\mathbf{X}_0 - \mathbf{X}^*\|_F &\leq g_2(\sigma(\mathbf{X}_0), \epsilon). \end{aligned}$$

One of the best uniform thresholds on δ_{2k} for finding k -sparse vectors using ℓ_p quasi-norm minimization is given in [15]. This threshold works uniformly for any $p \in (0, 1]$ and covers exact recovery conditions as well as robust and accurate reconstruction of sparse and nearly-sparse vectors from noisy measurements. Theorem 1 simply generalizes the results in [15] to low-rank matrix recovery by means of the following proposition and corollary. To have a more organized presentation, we use the inequality $\gamma_{2t} \geq (1 + \delta_{2t})/(1 - \delta_{2t})$, where γ_{2t} is the asymmetric RIC defined in [15], to state our results in terms of δ_{2t} (the RIC defined herein).

Proposition 2. *Let $\mathbf{X}_0 \in \mathbb{R}^{n_1 \times n_2}$ be any arbitrary matrix and $\mathcal{A}(\mathbf{X}_0) + \mathbf{e} = \mathbf{b}$, where $\mathbf{b} \in \mathbb{R}^m$ is known and \mathbf{e} is noise with $\|\mathbf{e}\|_2 \leq \epsilon$. Suppose that \mathbf{X}^* is a solution to (4) to recover \mathbf{X}_0 for some $p \in (0, 1]$. If*

$$\delta_{2t} < \frac{2(\sqrt{2} - 1)(t/r)^{\frac{1}{p} - \frac{1}{2}}}{2(\sqrt{2} - 1)(t/r)^{\frac{1}{p} - \frac{1}{2}} + 1} \quad (6)$$

holds for some integer $t \geq r$, then

$$\begin{aligned} \|\mathbf{X}_0 - \mathbf{X}^*\|_p &\leq C_1 \|\mathbf{X}_0 - \mathbf{X}_0^{(r)}\|_p + D_1 r^{\frac{1}{p} - \frac{1}{2}} \epsilon, \\ \|\mathbf{X}_0 - \mathbf{X}^*\|_F &\leq C_2 t^{\frac{1}{2} - \frac{1}{p}} \|\mathbf{X}_0 - \mathbf{X}_0^{(r)}\|_p + D_2 \epsilon. \end{aligned}$$

The constants C_1, C_2, D_1, D_2 depend only on $p, \delta_{2t}, t/r$ and are given in [15, Theorem 3.1]. In particular, when $\epsilon = 0$ and $\text{rank}(\mathbf{X}_0) \leq r$, (6) implies that \mathbf{X}_0 is a unique solution to (3).

Two important special cases of the above sufficient condition are summarized in the following corollary.

Corollary 1. *The sufficient condition of Proposition 2 implies the following sufficient conditions too:*

- $\delta_{2r} < 0.4531$ for any $p \in (0, 1]$,
- knowing r and $\delta_{2r+2} < 1$, it is possible to find some p_0 such that inequality (6) holds for all $0 < p < p_0$.

Theorem 1 also generalizes other recent RIP-based conditions in ℓ_p quasi-norm minimization (e.g., the conditions in [21, 22]). In addition to the above conditions, below, we introduce another sufficient condition which guarantees robust and accurate reconstruction of low-rank matrices.

Theorem 2. *Under assumptions of Proposition 2, if*

$$\delta_{2t} < \frac{(t/r)^{\frac{2}{p} - 1} - 1}{(t/r)^{\frac{2}{p} - 1} + 1} \quad (7)$$

holds for some integer $t \geq r$, then

$$\begin{aligned} \|\mathbf{X}_0 - \mathbf{X}^*\|_p &\leq C'_1 \|\mathbf{X}_0 - \mathbf{X}_0^{(r)}\|_p + D'_1 r^{\frac{1}{p} - \frac{1}{2}} \epsilon, \\ \|\mathbf{X}_0 - \mathbf{X}^*\|_F &\leq C'_2 t^{\frac{1}{2} - \frac{1}{p}} \|\mathbf{X}_0 - \mathbf{X}_0^{(r)}\|_p + D'_2 \epsilon, \end{aligned}$$

where the constants C'_1, C'_2, D'_1, D'_2 depend only on $p, \delta_{2t}, t/r$. In particular, when $\epsilon = 0$ and $\text{rank}(\mathbf{X}_0) \leq r$, (7) implies that \mathbf{X}_0 is a unique solution to (3).

Despite the fact that a uniform recovery threshold cannot be obtained from Theorem 2, substituting t with $r + 1$ in (7), we get

$$\delta_{2r+2} < \frac{(1 + 1/r)^{\frac{2}{p}-1} - 1}{(1 + 1/r)^{\frac{2}{p}-1} + 1}. \quad (8)$$

Fixing r and δ_{2r+2} , let p_0 denote the maximum value such that all $p \in (0, p_0)$ satisfy (6) for $t = r + 1$. Respectively, let p'_0 denote the maximum value such that all $p \in (0, p'_0)$ satisfy (8). Neglecting the constant terms, since, with the decrease of p , the power of $(1 + 1/r)$ in (8) grows twice that of in (6), it is expected that (8) guarantees accurate recovery for $p'_0 \approx \sqrt{p_0}$ when thresholds in the right-hand side of (6) and (8) tend to 1. Figure 1 shows δ_{2r+2} thresholds derived from Proposition 2 and Theorem 2 as a function of p for $r = 5$. As it is clear, the threshold given in Theorem 2 becomes sharper than that of given in Proposition 2 after passing $p \approx 0.22$. Furthermore, it reaches to 1 at $p \approx 0.05$, while the one from Proposition 2 approaches to 1 at $p \approx 0.025$. Recall that $\delta_{2r} < 1$ is a sufficient condition for the success of the original rank minimization problem in (1) [1]. Consequently, the above result shows that, for a larger range of p 's, p SNM is almost optimal since $\delta_{2r+2} < 1$ guarantees its success.

3. Proofs of results

3.1. Preliminaries

We begin with a definition and a few lemmas.

Definition 3 ([23]). A function $\Phi(\mathbf{x}) : \mathbb{R}^n \rightarrow \mathbb{R}$ is called symmetric gauge if it is a norm on \mathbb{R}^n and invariant under arbitrary permutations and sign changes of \mathbf{x} elements.

Lemma 2 ([19, Corollary 2.3]). Let Φ be a symmetric gauge function and $f : [0, \infty) \rightarrow [0, \infty)$ be a concave function with $f(0) = 0$. Then for $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n_1 \times n_2}$,

$$\Phi\left(f(\sigma(\mathbf{A})) - f(\sigma(\mathbf{B}))\right) \leq \Phi\left(f(\sigma(\mathbf{A} - \mathbf{B}))\right),$$

where $f(\mathbf{x}) = (f(x_1), \dots, f(x_n))^T$.

Lemma 3. Let $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{n_1 \times n_2}$. For any $p \in (0, 1]$,

$$\sum_{i=1}^n \sigma_i^p(\mathbf{A} - \mathbf{B}) \geq \sum_{i=1}^n |\sigma_i^p(\mathbf{A}) - \sigma_i^p(\mathbf{B})|. \quad (9)$$

Proof: It is obvious that $\Phi(\mathbf{x}) = \sum_{i=1}^n |x_i|$ and $f(x) = x^p$, $p \in (0, 1)$, satisfy conditions of Lemma 2. Thus, (9) is an immediate result for $p \in (0, 1)$. Moreover, (9) holds for $p = 1$ [23]. ■

Lemma 4. Let $\mathbf{W} = \mathbf{U} \text{diag}(\sigma(\mathbf{W})) \mathbf{V}^T$ denote the SVD of \mathbf{W} . If for some \mathbf{X}_0 , $\|\mathbf{X}_0 + \mathbf{W}\|_p \leq \|\mathbf{X}_0\|_p$, then with $\mathbf{X}_1 = -\mathbf{U} \text{diag}(\sigma(\mathbf{X}_0)) \mathbf{V}^T$, we have $\|\mathbf{X}_1 + \mathbf{W}\|_p \leq \|\mathbf{X}_1\|_p$.

Proof: The proof easily follows from [18, Lemma 2] by replacing $\|\cdot\|_*$ with $\|\cdot\|_p$ and applying inequality (9). ■

3.2. Proofs

Proof of Lemma 1: If $\mathcal{A}(\mathbf{X}) = \mathbf{b}$, then all feasible solutions to (3) can be represented as $\mathbf{X} + \mathbf{W}$ for some $\mathbf{W} \in \mathcal{N}(\mathcal{A})$. Consequently, to prove that \mathbf{X} is a unique solution to (3), we need to show that $\|\mathbf{X} + \mathbf{W}\|_p^p > \|\mathbf{X}\|_p^p$ for all $\mathbf{W} \in \mathcal{N}(\mathcal{A})$. Applying Lemma 3, it can be written that

$$\begin{aligned}
\|\mathbf{X} + \mathbf{W}\|_p^p &= \sum_{i=1}^n \sigma_i^p(\mathbf{X} + \mathbf{W}) \\
&\geq \sum_{i=1}^n |\sigma_i^p(\mathbf{X}) - \sigma_i^p(\mathbf{W})| \\
&= \sum_{i=1}^r |\sigma_i^p(\mathbf{X}) - \sigma_i^p(\mathbf{W})| + \sum_{i=r+1}^n \sigma_i^p(\mathbf{W}) \\
&\geq \sum_{i=1}^r \sigma_i^p(\mathbf{X}) - \sum_{i=1}^r \sigma_i^p(\mathbf{W}) + \sum_{i=r+1}^n \sigma_i^p(\mathbf{W}) \\
&> \sum_{i=1}^r \sigma_i^p(\mathbf{X}) = \|\mathbf{X}\|_p^p,
\end{aligned}$$

which confirms that \mathbf{X} is the unique solution. \blacksquare

Proof of Theorem 1: The proof is a direct consequence of integrating Lemma 4 of this paper and Theorem 1 and Lemma 5 of [18]. \blacksquare

Proof of Theorem 2: For the sake of simplicity, we prove this theorem for the vector case and by virtue of Theorem 1 matrix case will follow. Let \mathbf{x}^* denote a solution to (5) and $\mathbf{v} = \mathbf{x}^* - \mathbf{x}_0$, where \mathbf{x}_0 is the arbitrary vector we want to recover. Furthermore, let $S_0 \subset \{1, \dots, n_v\}$ with $|S_0| \leq r$. We partition $S_0^c = \{1, \dots, n_v\} \setminus S_0$ to S_1, S_2, \dots with $|S_i| = t$ probably except for the last set. As a result, $\mathbf{v}_{S_i}, i \geq 0$ denote a vector obtained by keeping entries of \mathbf{v} indexed by S_i and setting all other elements to 0.

Our proof is the same as in [15, Theorem 3.1] except the way in which $\|\mathbf{v}_{S_0}\|_2$ and $\|\mathbf{v}_{S_1}\|_2$ are bounded. Hence, we use the same notation and only focus on the bounding and omit other details. By applying the RIP definition, we get

$$\begin{aligned}
\|\mathbf{v}_{S_0} + \mathbf{v}_{S_1}\|_2^2 &\leq \frac{1}{1 - \delta_{2t}} \|\mathbf{A}(\mathbf{v}_{S_0} + \mathbf{v}_{S_1})\|_2^2 \\
&= \frac{1}{1 - \delta_{2t}} \langle \mathbf{A}(\mathbf{v} - \sum_{i \geq 2} \mathbf{v}_{S_i}), \mathbf{A}(\mathbf{v} - \sum_{i \geq 2} \mathbf{v}_{S_i}) \rangle \\
&= \frac{1}{1 - \delta_{2t}} \left[\|\mathbf{A}\mathbf{v}\|_2^2 + 2 \sum_{i \geq 2} \langle \mathbf{A}\mathbf{v}, -\mathbf{A}\mathbf{v}_{S_i} \rangle \right. \\
&\quad \left. + \sum_{i, j \geq 2} \langle \mathbf{A}\mathbf{v}_{S_i}, \mathbf{A}\mathbf{v}_{S_j} \rangle \right]. \tag{10}
\end{aligned}$$

Now, we find upper bounds for the terms in (10). Considering the second term in (10), it can be written that

$$\langle \mathbf{A}\mathbf{v}, -\mathbf{A}\mathbf{v}_{S_i} \rangle \leq \sqrt{1 + \delta_{2t}} \|\mathbf{A}\mathbf{v}\|_2 \|\mathbf{v}_{S_i}\|_2. \tag{11}$$

Since $\langle \mathbf{v}_{S_i}, \mathbf{v}_{S_j} \rangle = 0$ for $i \neq j$, [20, Lemma 2.1] implies that

$$\langle \mathbf{A}\mathbf{v}_{S_i}, \mathbf{A}\mathbf{v}_{S_j} \rangle \leq \delta_{2t} \|\mathbf{v}_{S_i}\|_2 \|\mathbf{v}_{S_j}\|_2, \quad \forall i \neq j. \tag{12}$$

Also,

$$\langle \mathbf{A}\mathbf{v}_{S_i}, \mathbf{A}\mathbf{v}_{S_i} \rangle \leq (1 + \delta_{2t}) \|\mathbf{v}_{S_i}\|_2^2. \quad (13)$$

Putting (11)-(13) in (10) and letting $\Sigma = \sum_{i \geq 2} \|\mathbf{v}_{S_i}\|_2$, we get

$$\begin{aligned} & \|\mathbf{v}_{S_0}\|_2^2 + \|\mathbf{v}_{S_1}\|_2^2 \\ & \leq \frac{1}{1 - \delta_{2t}} \left[\|\mathbf{A}\mathbf{v}\|_2^2 + 2\sqrt{1 + \delta_{2t}} \|\mathbf{A}\mathbf{v}\|_2 \Sigma \right. \\ & \quad \left. + \delta_{2t} \sum_{\substack{i, j \geq 2 \\ i \neq j}} \|\mathbf{v}_{S_i}\|_2 \|\mathbf{v}_{S_j}\|_2 + (1 + \delta_{2t}) \sum_{i \geq 2} \|\mathbf{v}_{S_i}\|_2^2 \right] \\ & = \frac{1}{1 - \delta_{2t}} \left[\|\mathbf{A}\mathbf{v}\|_2^2 + 2\sqrt{1 + \delta_{2t}} \|\mathbf{A}\mathbf{v}\|_2 \Sigma + \delta_{2t} \Sigma^2 \right. \\ & \quad \left. + \sum_{i \geq 2} \|\mathbf{v}_{S_i}\|_2^2 \right] \\ & \leq \frac{1}{1 - \delta_{2t}} \left[\|\mathbf{A}\mathbf{v}\|_2^2 + 2\sqrt{1 + \delta_{2t}} \|\mathbf{A}\mathbf{v}\|_2 \Sigma \right. \\ & \quad \left. + (\delta_{2t} + 1) \Sigma^2 \right] \end{aligned} \quad (14)$$

where, for the last inequality, we use $\sum_{i \geq 2} \|\mathbf{v}_{S_i}\|_2^2 \leq \left(\sum_{i \geq 2} \|\mathbf{v}_{S_i}\|_2 \right)^2$. Inequality (14) can be reduced to

$$\begin{aligned} \|\mathbf{v}_{S_0}\|_2 & \leq \frac{1}{\sqrt{1 - \delta_{2t}}} \left[\|\mathbf{A}\mathbf{v}\|_2 + \sqrt{1 + \delta_{2t}} \Sigma \right], \\ \|\mathbf{v}_{S_1}\|_2 & \leq \frac{1}{\sqrt{1 - \delta_{2t}}} \left[\|\mathbf{A}\mathbf{v}\|_2 + \sqrt{1 + \delta_{2t}} \Sigma \right]. \end{aligned}$$

The rest of the proof is similar to [15, Theorem 3.1] with new parameters $\lambda = 2/\sqrt{1 - \delta_{2t}}$ and $\mu = \sqrt{1 + \delta_{2t}}/\sqrt{1 - \delta_{2t}}(r/t)^{\frac{1}{p} - \frac{1}{2}}$. Therefore, in this proof, from $\mu < 1$, we get

$$\delta_{2t} < \frac{(t/r)^{\frac{2}{p} - 1} - 1}{(t/r)^{\frac{2}{p} - 1} + 1},$$

and, after some simple algebraic manipulations, we obtain

$$\begin{aligned} \|\mathbf{x}_0 - \mathbf{x}^*\|_p & \leq C'_1 \|\mathbf{x}_0 - \mathbf{x}_0^{(r)}\|_p + D'_1 r^{\frac{1}{p} - \frac{1}{2}} \epsilon, \\ \|\mathbf{x}_0 - \mathbf{x}^*\|_2 & \leq C'_2 t^{\frac{1}{2} - \frac{1}{p}} \|\mathbf{x}_0 - \mathbf{x}_0^{(r)}\|_p + D'_2 \epsilon, \end{aligned}$$

with constants

$$\begin{aligned} C'_1 & = \frac{2^{\frac{2}{p} - 1} (1 + \mu^p)^{\frac{1}{p}}}{(1 - \mu^p)^{\frac{1}{p}}}, \quad D'_1 = \frac{2^{\frac{2}{p} - 1} \lambda}{(1 - \mu^p)^{\frac{1}{p}}}, \\ C'_2 & = \left(1 + 2\sqrt{\frac{1 + \delta_{2t}}{1 - \delta_{2t}}} \right) \frac{2^{\frac{2}{p} - 1}}{(1 - \mu^p)^{\frac{1}{p}}}, \\ D'_2 & = 2\lambda + \left(1 + 2\sqrt{\frac{1 + \delta_{2t}}{1 - \delta_{2t}}} \right) \frac{2^{\frac{1}{p} - 1} \lambda}{(1 - \mu^p)^{\frac{1}{p}}}. \end{aligned}$$

■

4. Conclusion

In the affine rank minimization problem, it is experimentally verified that Schatten- p quasi-norm minimization is superior to nuclear norm minimization. In this paper, we established a theoretical background for this observation and proved that, under a weaker sufficient condition than that of nuclear norm minimization, global minimization of the Schatten- p quasi-norm subject to compressed affine measurements leads to unique recovery of low-rank matrices. To show that this approach is robust to noise and being approximately low-rank, we generalized some-RIP based results in ℓ_p quasi-norm minimization to Schatten- p quasi-norm minimization.

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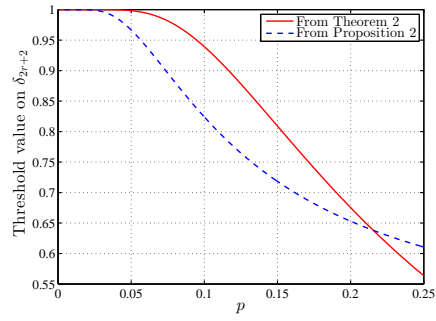


Figure 1: Recovery thresholds from Proposition 2 and Theorem 2 as a function of p . r is fixed to 5, and thresholds are independent of matrix dimensions.