Phase velocity and group velocity

This note is part of the supporting online material for the textbook "Solid State Physics: An Introduction" by Philip Hofmann, Wiley-VCH, 2008. You can find more information on www.philiphofmann.net.

The distinction between the phase velocity and the group velocity of a wave is a concept of general significance for many different waves in physics: electromagnetic waves, particle waves, elastic waves and so on. It is explained and illustrated in this document, using little animations in the pdf file. You have to click on the images to start the animations. You need to have Quicktime installed for this to work.

We start by considering a general one-dimensional wave

$$A(x,t) = A_0 e^{i(kx-\omega(k)t)}, \tag{1}$$

where A_0 is the amplitude, k the wave number, $\omega(k)$ the angular frequency, and t the time. The important thing to note is that ω depends on the wave number k or the wavelength $\lambda = 2\pi/k$. This phenomenon is called dispersion and it might be familiar from optics, where the speed of light in a material (or the index of refraction) depends on the wavelength.

Starting from the dispersion $\omega(k)$, we define the phase velocity as

$$v_p = \frac{\omega}{k} \tag{2}$$

and the group velocity as

$$v_g = \frac{\partial \omega}{\partial k} \tag{3}$$

Obviously, these two are the same for a linear dispersion with $\omega(k) = ak$.

We are familiar with different kinds of dispersion relations and these are illustrated in Figure 1. For light in vacuum we have

$$\omega(k) = kc, \tag{4}$$

where c is the (vacuum) speed of light. This dispersion is shown in Figure 1(a). In general matter, $n(\omega)$ is greater than 1 and also depends on ω. This leads to a reduced and energy-dependent speed of light in matter

$$\omega(k) = \frac{kc}{n(\omega)}, \tag{5}$$

with $n(\omega)$ being the index of refraction.

For a free, non-relativistic quantum mechanical particle of mass m, we have

$$E(k) = \hbar\omega(k) = \frac{\hbar^2 k^2}{2m}, \tag{6}$$

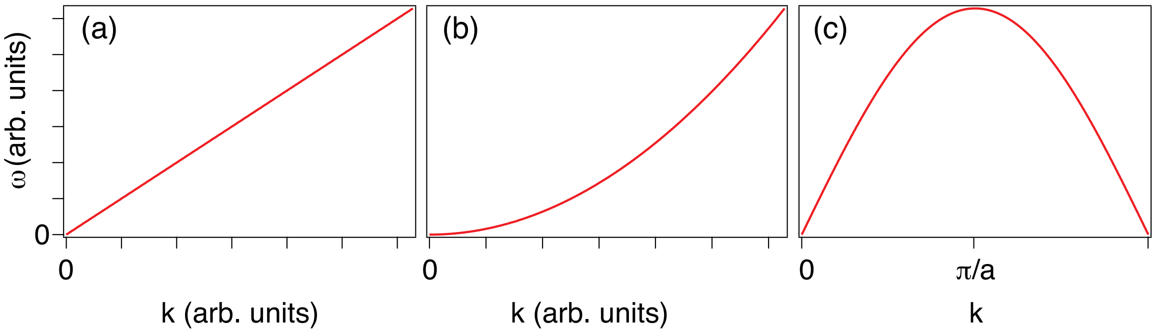

Figure 1: Dispersion relations $\omega(k)$ for different physical situations: (a) light in vacuum (equation 4), (b) a free, non-relativistic quantum mechanical particle (equation 6), (c) the acoustic branch of vibrations in a crystal (equation 8). a is the one-dimensional lattice constant.

so that

$$\omega(k) = \frac{\hbar k^2}{2m}, \tag{7}$$

and for the mechanical wave representing the vibrations in a one-dimensional chain we have

$$\omega(k) = 2\sqrt{\frac{\gamma}{M}}|\sin\frac{ka}{2}|. \tag{8}$$

These cases are illustrated in Figure 1(b) and (c), respectively.

In the following, we shall look at the difference of group velocity and phase velocity for a few situations associated with these different dispersions. In each case, we will study the propagation of waves with different values of k (and thus of ω). Then we form a package of such waves centered around a k-point of interest. Such a package can be interpreted as a "particle" localized in space. For the three case shown in Figure 1, these "particles" could be photons, electrons and phonons, respectively. We will see that our "particle" is not particularly well localized. Indeed, we do not even expect it to be because we will use only a small range Δk to form our wave package and according to Heisenberg's uncertainty principle this implies a large extension Δx in space.

For a linear dispersion, e. g. for light in vacuum, we form a package of partial waves with the different k values indicated by the dots in Figure 2(a). The propagations of the partial wave at the center of the package (blue dot) and of all partial waves are given in Figure 2(b) and (c), respectively. We see that all partial waves propagate with the same phase velocity $v_p = \omega/k = c$. Figure 2(d) shows the propagation of the package formed by these partial waves. The package is strongly peaked around one position and this peak corresponds to our "photon". We see that the "photon" propagates with the same velocity as the partial waves. The velocity of the wave package propagation is the group velocity $v_g = \partial\omega/\partial k (= c)$. In this particular case, it is of course the same as the phase velocity. We also see that that the moving wave package does not change its shape. This is also a consequence of the linear dispersion that makes all partial waves travel with the same velocity.

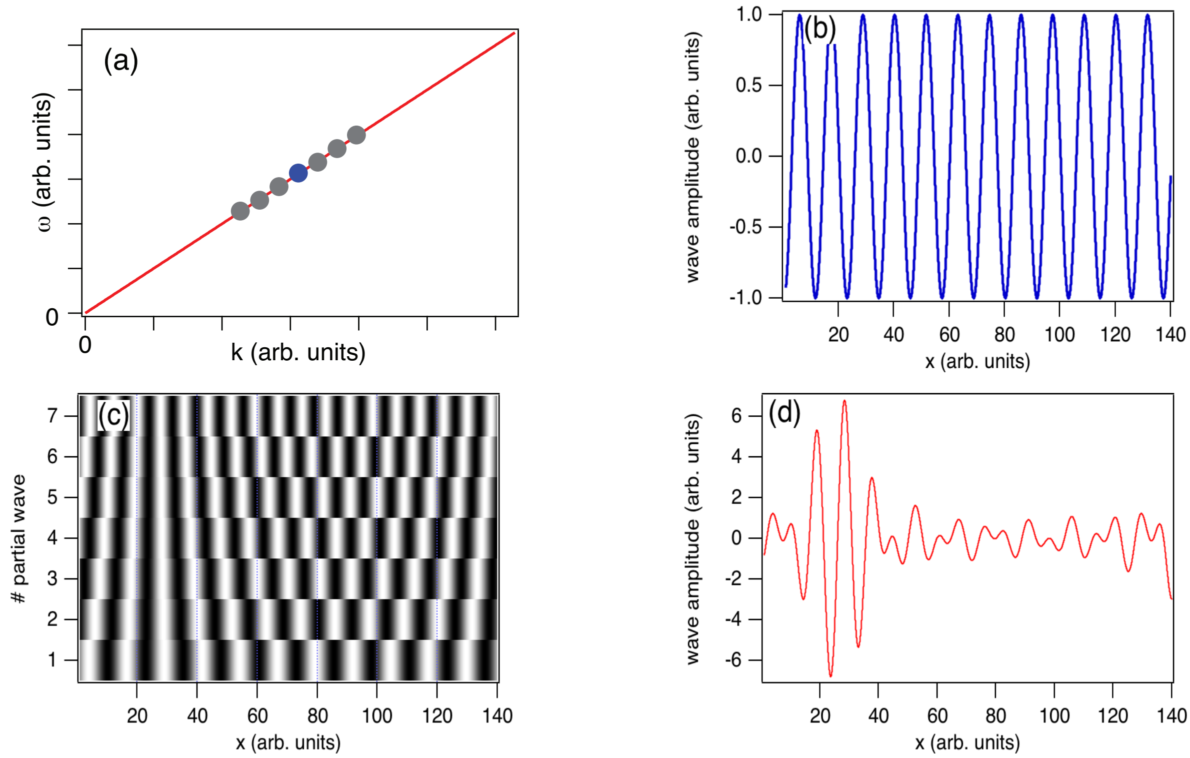

Figure 2: (click on images to activate animations) Propagation of partial waves and wave package for a linear dispersion. (a) Dispersion with markers corresponding to the k values chosen for forming a wave package. (b) Propagation of the partial wave in the center of the package (the one with the blue dot). (c) Propagation of all seven partial waves. (d) Propagation of the package formed from all partial waves.

We now apply the same analysis to the quantum mechanical particle described by equation (6). The result is shown in Figure 3. Now the partial wave do not all move with the same velocity because of the quadratic dispersion. A very important consequence of this is that our initial wave package broadens out with time because the partial waves forming it gradually move out of phase with each other. So even if we start with a fairly localized "particle", it will soon loose this localization. We can calculate the group velocity for this dispersion as

$$v_g = \frac{\partial \omega(k)}{\partial k} = \frac{\hbar k}{m}, \tag{9}$$

and this is perfectly consistent with the movement of a semiclassical particle for which the momentum is $p = \hbar k$ and the group velocity thus $p/m = v_g$.
We now study the dispersion of the lattice vibrations in equation 8 and Figure 1(c). Here, we have three interesting situations. For the first, we can choose a wave package around a small k value where the dispersion is linear. This is the

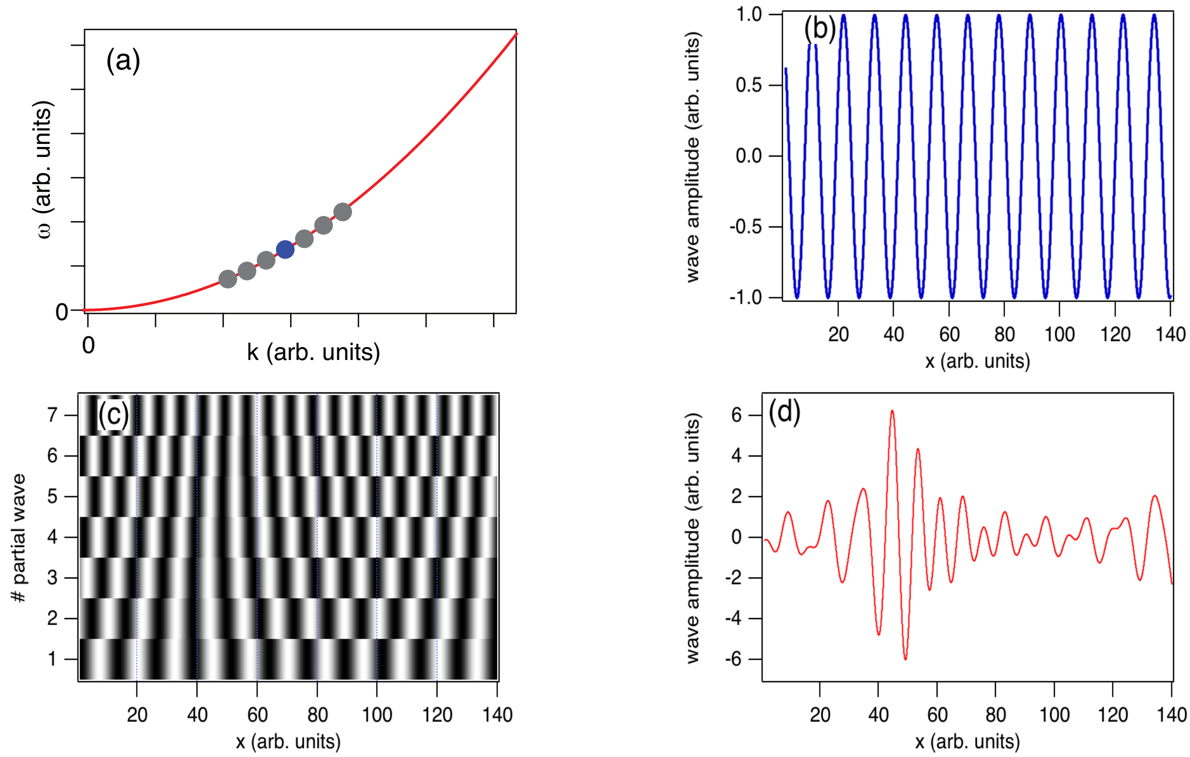

Figure 3: (click on images to activate animations) Propagation of partial waves and wave package for a quadratic dispersion. (a) Dispersion with markers corresponding to the k values chosen for forming a wave package. (b) Propagation of the partial wave in the center of the package (the one with the blue dot). (c) Propagation of all seven partial waves. (d) Propagation of the package formed from all partial waves.

regime corresponding to acoustic waves (sound waves) in the solid. We see that the situation is exactly the same as for the linear dispersion in Figure 2, so we do not have to treat it here again. The only important difference is that the waves do not propagate with the speed of light but with the speed of sound.

The second interesting case is a wave package formed from partial waves around the maximum of the dispersion curve at $k = \pi/a$, that is, at the Brillouin zone boundary. This is illustrated in Figure 4. If we are sufficiently close to the maximum of the dispersion, we would expect the group velocity $v_g = \partial\omega/\partial k$ to be zero, that is, we would expect a standing wave. This is precisely what we find: While all partial waves still have a positive phase velocity, the localized "phonon" does not move at all. When inspecting Figure 4(c), we can see why this is so: The phase velocity of all partial waves is different but it is always so that the faster ones "catch up" with the slower ones in the middle of the image.

A particularly good illustration of how different the group and phase velocity of a particle can be arises for the dispersion of acoustic waves as we go past the maximum, near the next Brillouin zone centre at $2\pi/a$. This situation is shown in Figure 5. Now the group velocity is obviously negative (but linear) and our wave packet does indeed move in the negative direction, even though all partial waves

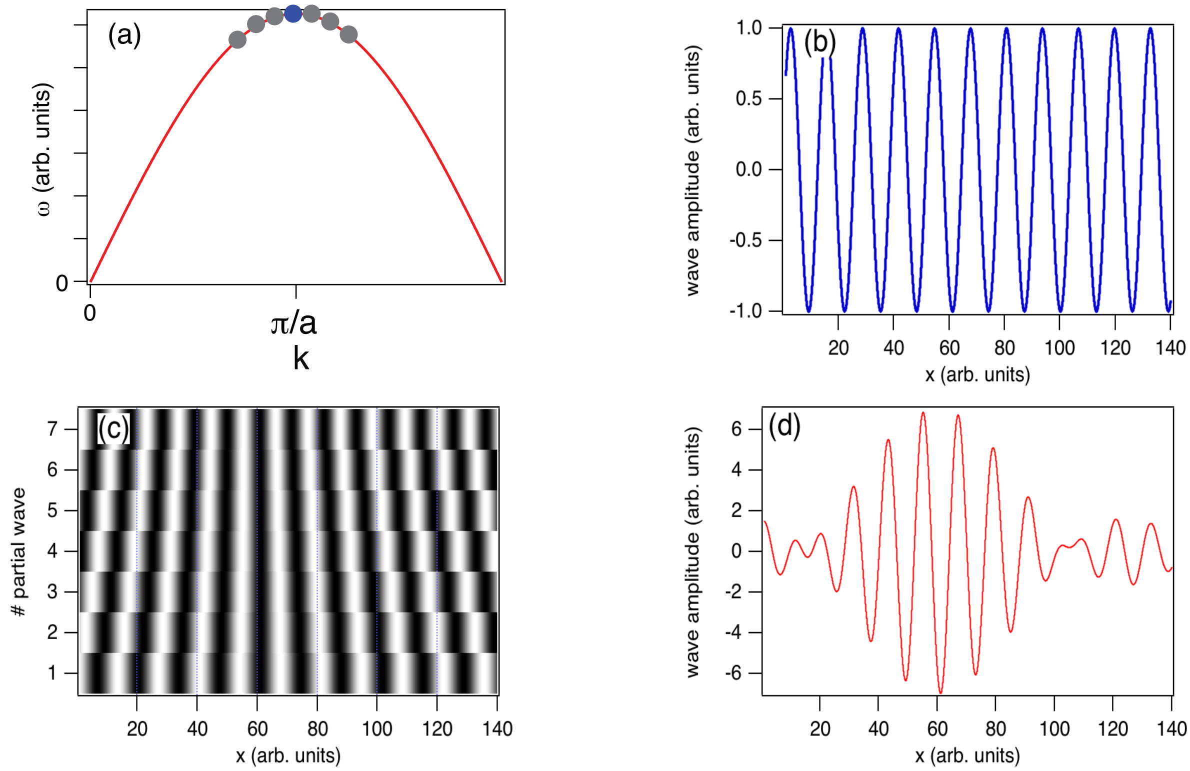

Figure 4: (click on images to activate animations) Propagation of partial waves and wave package for an acoustic phonon dispersion. The wave package is formed from partial waves around the Brillouin zone boundary at $k = \pi/a$. (a) Dispersion with markers corresponding to the k values chosen for forming a wave package. (b) Propagation of the partial wave in the center of the package (the one with the blue dot). (c) Propagation of all seven partial waves. (d) Propagation of the package formed from all partial waves.

move in the positive direction. Note that we could adopt a different point of view and state that we would get the same vibration for a wave packet with a wave vector moved by one reciprocal lattice vector, as illustrated in Figure 6. Then the negative group velocity would not be so surprizing.

Finally, we return to the case of a quantum mechanical particle (equation 6 and Figure 3) and we ask if there is any possibility to prevent the wave function from spreading out. For this, we need a linear dispersion. One way this can be done becomes apparent when we consider the relativistic dispersion relation for a semi-classical particle

$$\omega(k) = \frac{1}{\hbar}\sqrt{(mc^2)^2 + (\hbar kc)^2}.$$

From this, we can achieve a linear dispersion in two ways. We can either choose a particle with rest mass zero or we can choose such a high k that the rest mass contribution to the energy becomes negligible against the kinetic energy. Both ways do not appear very practical in a solid. It is not easy to see how the mass of the electrons should vanish and it is certainly not possible to go to a kinetic energy that is high compared to the rest mass energy (511 keV for a free electron). However,

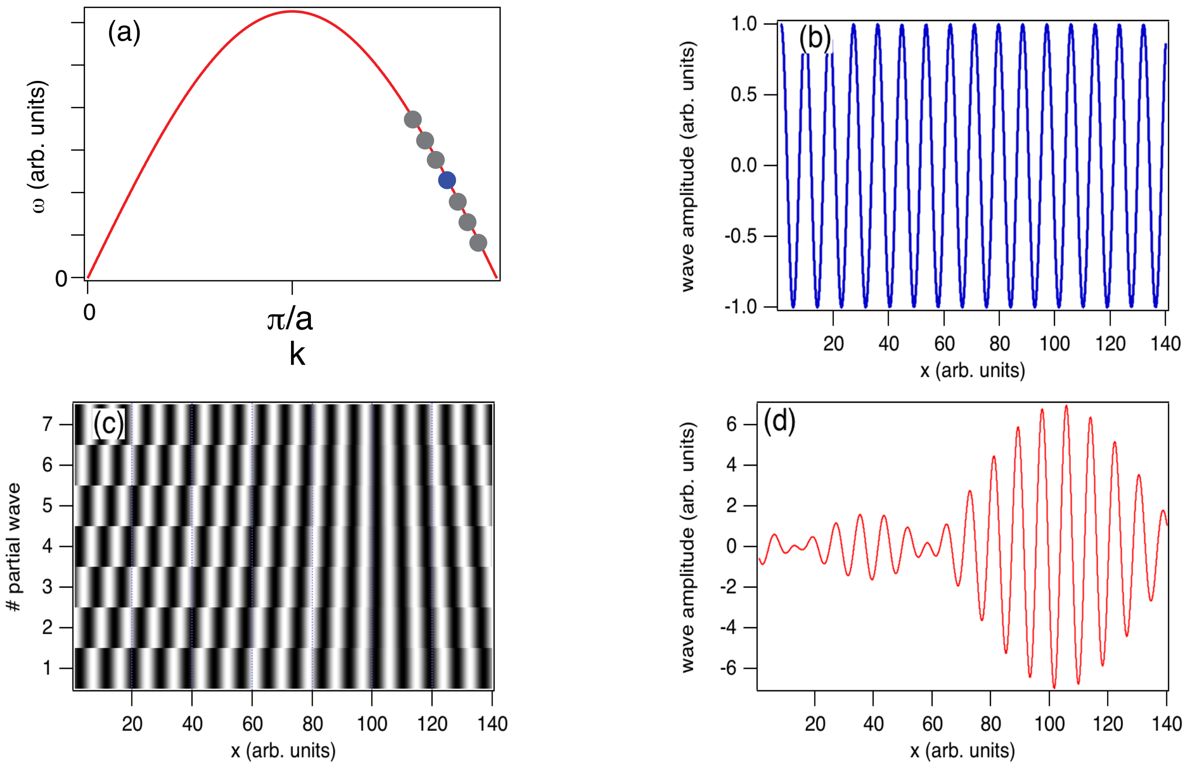

Figure 5: (click on images to activate animations) Propagation of partial waves and wave package for an acoustic phonon dispersion. The wave package is formed from partial waves close to the next Brillouin zone centre at $k = 2\pi/a$. (a) Dispersion with markers corresponding to the k values chosen for forming a wave package. (b) Propagation of the partial wave in the center of the package (the one with the blue dot). (c) Propagation of all seven partial waves. (d) Propagation of the package formed from all partial waves.

the band structure of the solid allows for a way to a situation that corresponds to a vanishing rest mass and a dispersion similar to that of light (4). This is illustrated in Figure 6, that shows a simple band in a one-dimensional solid. As the band disperses only over a limited energy range, the curvature has to change from positive to negative and thus it becomes nearly zero for a small range of k values. There the dispersion is, in fact, locally linear and light-like. The group velocity is, of course, not the speed of light but it is given by the degree of dispersion and the lattice constant. Such a situation is especially important when the linear dispersion is situated at the Fermi energy. The most important example for this appears in the electronic band structure of graphene, a single layer of carbon atoms.

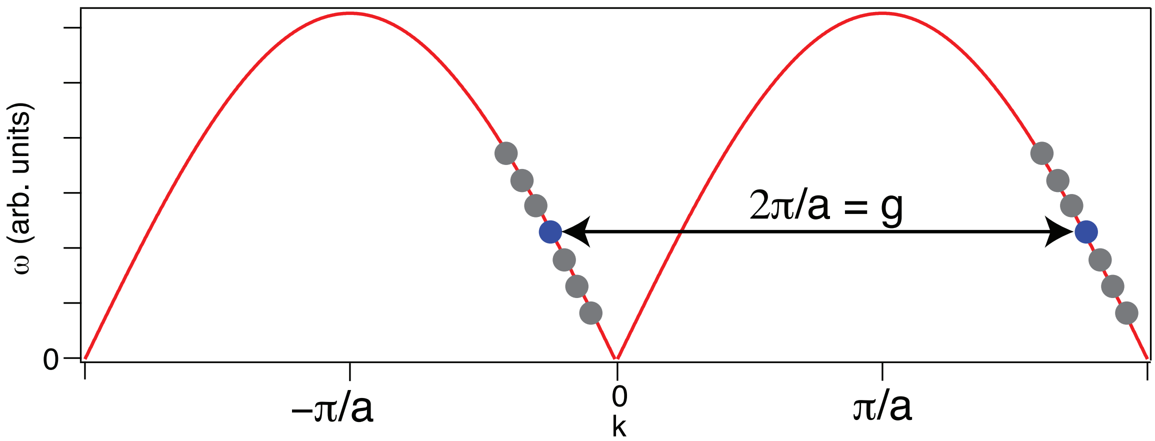

Figure 6: The lattice vibrations corresponding to the situation in Figure 5 could also be represented by a wave package that has been translated by one reciprocal lattice vector. In this case, the wave package still has a negative group velocity but so have all the partial waves.

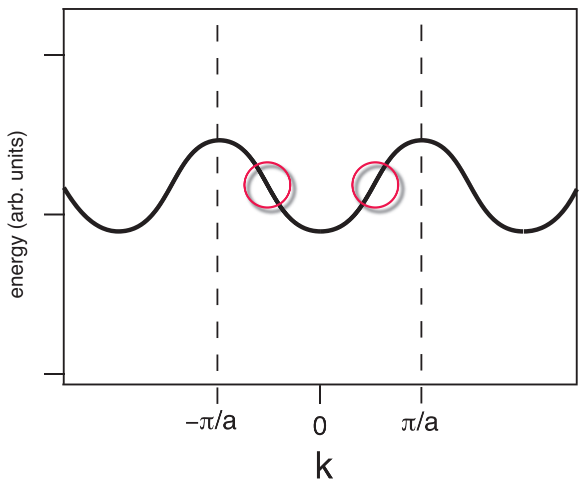

Figure 7: Simple band structure of a solid. Since the curvature of the band has to change sign across the Brillouin zone, the limit of a linear dispersion can be realized in a small range of k values, as marked by the circles.