Covering Orthogonal Polygons with Sliding k-transmitters

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Abstract

In this paper we consider a new variant of covering in an orthogonal art gallery problem where each guard is a sliding k-transmitter. Such a guard can travel back and forth along an orthogonal line segment, say s, inside the polygon. A point p is covered by this guard if there exists a point $q \in s$ such that $\overline{pq}$ is a line segment normal to s and has at most k intersections with the polygon's boundary walls. The objective is to minimize the sum of the lengths of the sliding k-transmitters to cover the entire polygon. In other words, the goal is to find the minimum total length of trajectories on which the guards travel to cover the entire polygon. We prove that this problem is NP-complete and present a 2-approximation algorithm for it.

1 Introduction

We study a new version of the art galley problem to cover a simple orthogonal polygon where a new model of covering or visibility, using sliding cameras. Sliding camera guards were introduced by Katz and Morgenstern [8] for guarding orthogonal polygons. A sliding camera can travel back and forth along an axis-aligned segment s inside an orthogonal polygon P. A point p can be viewed by this camera if there exists a point $q \in s$ such that $\overline{pq}$ is a line segment normal to s and is completely inside P. Another variation of coverage that we use for our guards in this paper is "Modem Illumination" where each guard is modeled as an omnidirectional wireless modem with an infinite broadcast range which can penetrate through k (for a fixed integer $k > 0$) walls to reach a client. These modems are also called k-transmitters and were introduced by Fabila-Monroy et al. [6] and Aichholzer et al. [1].

The sliding cameras which we used, can see through at most k walls in the directions perpendicular to their line segment track. As we do here, the walls are most often represented by line segments with no diameters.

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The objective is to find the minimum-length sliding cameras that cover the entire polygon. This is our problem and we denote it as MLSCk . In this paper, we prove that MLSCk is NP-complete and we present a 2-approximation algorithm for it.

Previous Work

The art gallery problem is a classic area of study in computational geometry. Over the years, many variants of this problem have been studied [11, 13], [7, 12]. Most of these variations are proved to be NP-hard [9], including the cases where the target region is a simple orthogonal polygon and the goal is to find the minimum number of vertex guards to guard the entire polygon [11, 12]. Some variations with the limited model of visibility have polynomial time algorithms [10, 14].

In [8], the authors considered the problem of guarding a simple orthogonal polygon with the minimum cardinality sliding cameras (MCSC). They showed that when the cameras are constrained to travel only vertically inside the polygon, MCSC can be solved in polynomial time. They also presented a 2-approximation algorithm for this problem when the trajectories which the cameras travel can be vertical or horizontal and the target region is an x-monotone orthogonal polygon. They left the computation of the complexity of MCSC as an open problem.

In 2013, Durocher and Mehrabi [5] studied the MCSC problem and the minimum-length sliding cameras (MLSC) problem with the goal to minimize the total length of trajectories along which the cameras travel. They proved that MCSC is NP-hard if the orthogonal polygon can have holes. They also proved that MLSC is solvable in polynomial time even for orthogonal polygons with holes. Recently, Durocher et al. [4] gave a (7/2)-approximation algorithm for MCSC.

In 2013, Ballinger et al. [2] considered the guards as k-transmitters. They extended bounds for the number of k-transmitters that are necessary and sufficient to cover a given group of line segments, polygons and polygonal chains.

Notations

Let P be an orthogonal polygon. We refer to the area of P with $\mathcal{A}(P)$ and its edges with $\mathcal{E}(P)$. We extend the endpoints of each edge $e \in \mathcal{E}(P)$ to obtain a line that contains e. Let L be the set of these lines. Obviously, L partitions $\mathcal{A}(P)$ into orthogonal rectangles denoted by $\mathcal{P}(P)$. For each sliding camera c, we denote $\mathcal{V}(c)$ as the set of points in $\mathcal{A}(P)$ which are guarded by c. Similarly, $\mathcal{V}^k(c)$ stands for the same set when we consider the problem using the k-transmitter model. We call a set of cameras C a *candidate set*, if all points in each part $p \in \mathcal{P}(P)$ are covered with the same subset of C. We will prove that in MLSCk there always exists an optimal solution which uses a candidate set.

2 The Hardness of MLSCk Problem

In this section we prove that MLSCk is NP-complete. We present a $poly(n)$ reduction from the problem of tiling an orthogonal polygon by 1×3 rectangles to MLSCk. In the problem of tiling an orthogonal polygon with rectangles, it is assumed that the orthogonal polygon R is drawn on a grid G. The goal is to place non-overlapping 1×3 rectangles to cover all of R. Beauquier et al. showed that this problem is NP-complete [3].

Our proof has two phases. First, we construct a new orthogonal polygon P from R. Next, we prove that for each answer to MLSCk on P, there is a corresponding answer to the tiling problem on R. So, MLSCk is NP-complete.

2.1 The Reduction

In this subsection we construct P from R. The input for this construction is R and a grid G. We denote the vertices of R by V_R. Let n be the number of grid vertices which are inside and on the boundary of R. Let E_G be the set of edges of G that are inside (not on the boundary) R, and EE_G be the duplicated edges of E_G (for each $e \in E_G$, there are $e, e' \in EE_G$). We also denote the vertices of G which are on the boundary of R by $VB_R = \{v_1, ..., v_{n'}\}$ (in clockwise order). At the end, we report the set V_P which contains the set of all vertices and edges of P in clockwise order.

We start traversing VB_R from v_1, until we reach the first vertex v_i such that $v_i \in VB_R$ and $v_i \notin V_R$. Then, we traverse $e_i \in E_G$ which is adjacent to v_i, and reach v_j. If $v_j \notin VB_R$, then we traverse the right most (in clockwise direction) adjacent edge of v_j, if it has not been traversed before. If all adjacent edges of v_j have been traversed or if $v_j \in VB_R$, then we traverse the duplicated edge (traverse e'_i).

We continue this process until we reach v_i again. Then, we traverse the boundary of R (VB_R) until we reach another un-traversed v_x of G, which is not a vertex of R. We repeat the previous steps (traversing the adjacent edge of v_x, if it has not been traversed and performing the other steps) until we arrive back again to v_x. We traverse R and G, until we reach v_1 again.

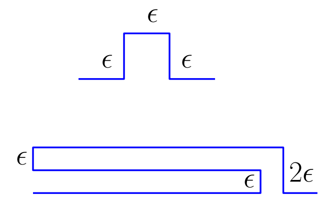

Figure 1: The added part

While traversing any vertex or edge, we add it to the set V_P. So, we construct a new polygon P on G. We add some small parts to P (see figure 1). These new parts are used to avoid having the sliding k-transmitters with the length greater than $1 + \epsilon$ and to avoid having the transmitters which can cover two disconnected part of P. As we at most traverse all vertices and edges of G, the complexity of constructing P is $O(n)$.

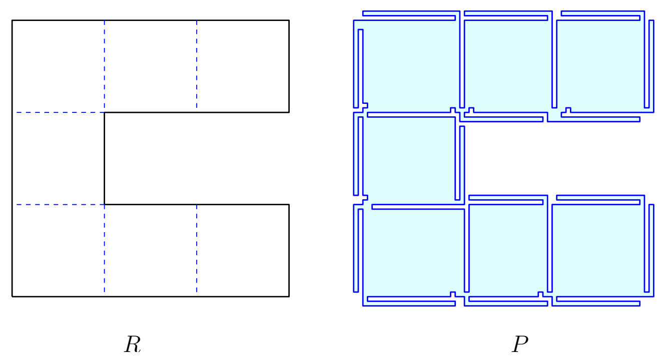

Figure 2: Construction of polygon P from polygon R

2.2 Correctness of the Reduction

In this subsection we prove that for each answer of MLSCk in P there is an answer to the problem of tiling an orthogonal polygon R with 1×3 rectangles and vice versa. So, MLSCk is NP-complete. Let g, the number of the grid cells inside R, be a factor of 3 (otherwise R can not be tiled by 1×3 rectangles). Also, let $k = 2$, which means that the sliding transmitters can see through two walls.

First, assume that we solve MLSCk on P and its answer is denoted as $\{c_1, c_2, ..., c_x\}$. From the construction of P, the length of each transmitter can be $1 \pm \epsilon$ or ϵ. Let m the total length of the transmitters. If $m = g/3 + \epsilon$, then because of the construction of P and the fact that each c_i is a 2-transmitter, the answer to the tiling problem on R is yes. Otherwise, the answer is no.

Second, assume that we solve tiling problem on R. Let $T = \{t_1, t_2, ..., t_m\}$ be the answer. We place the set of sliding k-transmitters $C_1 = \{c_1, c_2, ..., c_m\}$ and C_2, which covers the entire P. From the construction of P, each rectangle $t_i \in T$ of R, is partitioned to three separated squares si_1, si_2, and si_3 in P. We put a sliding k-transmitter $c_i \in C_1$ in the middle of si_2 (see figure 3). Then for covering the added part (which are shown in figure 1), we put some transmitters with length ϵ or 2ϵ in C_2. As c_i can see through at most two walls, so it covers only si_1, si_2, and si_3. Since the rectangles are non-overlapping and they are tiling R, the set $C_1 + C_2$ can cover the entire P and $|C_1 + C_2| = m + \epsilon$, the total length of transmitters, is minimal.

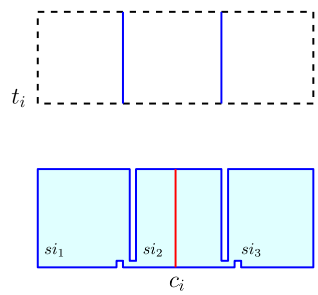

Figure 3: Placing the camera c_i in the middle of rectangle t_i

So, we have the following theorem:

Theorem 1 *The problem of covering a simple orthogonal polygon by the minimum-length sliding k-transmitters is NP-complete.*

3 Approximation Algorithm

In this section we present a 2-approximation algorithm for MLSCk . The algorithm consists of two phases. In the first phase, we relax the problem to the case where each camera has a non-negative density. The goal is to put cameras in the polygon such that, the total density of cameras guarding each point is at least 1 and the total density of all cameras is minimized. Next, we present a polynomial time algorithm for the relaxed MLSCk problem. In the second phase, we add some restrictions to the original problem regarding the answer of the relaxed MLSCk problem on the given polygon. We prove that the objective function of the restricted problem is at most two times the objective function of the original problem. Moreover, we give a polynomial time solution for the restricted problem. At last, we report the solution of the restricted problem as a 2-approximation solution for the original problem.

3.1 Relaxed MLSCk Problem

In this subsection we consider the relaxed version of MLSCk and find an exact solution for this problem. The relaxed MLSCk problem is defined as follows:

Definition 2 *Given an orthogonal polygon P and an integer number k. The relaxed MLSCk problem is to put a set of sliding cameras $C = \{c_1, c_2, \dots, c_{|C|}\}$ in P, each with density of $0 \le d_{c_i} \le 1$, in such a way that for every interior point $p \in \mathcal{A}(P)$, the following constraint is satisfied:*

$$\sum_{c_i \in C, p \in \mathcal{V}^k(c)} d_{c_i} \ge 1.$$

Among all solutions, the one that minimizes $\sum_{c_i \in C} |c_i| d_{c_i}$ is desired.

Let $\mathcal{R}(P)$ be $\sum_{c_i \in C} |c_i| d_{c_i}$ in an optimal solution of the relaxed MLSCk problem on P. If we add the restriction that all d_{c_i}s should be the in set $\{0, 1\}$, the problem would be equivalent to the MLSCk problem. Hence, $\mathcal{R}(P)$ is no more than $\mathcal{M}(P)$ (the optimal solution of the original problem) for all orthogonal polygons P.

Proposition 3 *Any not-necessarily-optimal solution of the relaxed MLSCk problem that the densities of all cameras are 1 is an acceptable but not-necessarily-optimal solution for MLSCk problem.*

Proposition 4 *For any given orthogonal polygon P we have:*

$$\mathcal{R}(P) \le \mathcal{M}(P).$$

Next, we show that the relaxed MLSCk can be solved in polynomial time.

Lemma 5 *There is a polynomial time algorithm that finds an exact solution for the relaxed MLSCk problem.*

Proof. Let $\mathcal{C}(P)$ be a candidate set for the relaxed MLSCk problem on the given orthogonal polygon P. There always exists an optimal solution for the problem using a subset of the cameras in $\mathcal{C}(P)$. The following linear program has $|\mathcal{C}(P)|$ variables d_{c_i} for all $c_i \in \mathcal{C}(P)$.

$$\begin{aligned}
min. \quad & \sum_{c_i \in \mathcal{C}(P)} |c_i| d_{c_i} & & (1)\\
s.t. \quad & \sum_{c_i \in \mathcal{C}(P), p \in \mathcal{V}^k(c_i)} d_{c_i} \ge 1 & \forall p \in \mathcal{A}(P) \quad & (2)\\
& d_{c_i} \ge 0 & \forall c_i \in \mathcal{C}(P) \quad & (3)\\
& d_{c_i} \le 1 & \forall c_i \in \mathcal{C}(P) \quad & (4)
\end{aligned}$$

Constraints of type 2 state that each point in $\mathcal{A}(p)$ should be in the visibility area of cameras that the total sum of their densities is at least 1. Constraints of types 3 and 4 state that density of each camera is between 0 and 1. The objective function is to minimize the total cost of all cameras where cost of each camera c_i is defined as $|c_i| d_{c_i}$. Hence, the above LP finds an optimal solution for the relaxed MLSCk problem. We remark that since $\mathcal{C}(P)$ is a candidate set of cameras for P, every point in each partition of P is in the visibility area of the same set of cameras of $\mathcal{C}(P)$. Hence, we can rewrite the constraints of type 2 in the following way:

$$s.t. \quad \sum_{c_i \in \mathcal{C}(P), p \in \mathcal{V}^k(c_i)} d_{c_i} \geq 1 \qquad \forall p \in \hat{C}(P) \qquad (5)$$

The number of the variables and constraints of the LP is $poly(n)$, therefore we can solve it in time $poly(n)$. □

3.2 Restricted MLSCk problem

In the previous subsection we discussed the relaxed MLSCk problem and showed how can we solve it in polynomial time. Next, we define the restricted MLSCk problem and show that this problem can be solved in polynomial time too.

Definition 6 *Given an orthogonal polygon P and an integer number k and function $f : \mathcal{P}(P) \rightarrow \{H, V\}$, let $\mathcal{V}^{*k}$ be a function that for every horizontal camera c, $\mathcal{V}^{*k}(c)$ is the set of all partitions $p \in \mathcal{V}^k(c)$ such that $f(p) = H$. Similarly $\mathcal{V}^{*k}(c)$ for a vertical camera c is the set of all partitions $p \in \mathcal{V}^k(c)$ such that $f(p) = V$. The restricted MLSCk problem is to put a set of sliding cameras $C = \{c_1, c_2, \ldots, c_{|C|}\}$ in P, each with density $0 \leq d_{c_i} \leq 1$, in such a way that for every interior point $p \in \mathcal{A}(P)$, the following constraint is satisfied:*

$$\sum_{c_i \in C, p \in \mathcal{V}^{*k}(c)} d_{c_i} \geq 1.$$

Among all solutions, the one that minimizes $\sum_{c_i \in C} |c_i| d_{c_i}$ is desired.

Let $\mathcal{R}'(P, f)$ be $\sum_{c_i \in C} |c_i| d_{c_i}$ in an optimal solution of the restricted MLSCk problem on polygon P and function f. We call a solution of the restricted MLSCk problem *Integral* iff all of its guarding cameras have density 1. Next, we show that for every orthogonal polygon P there exists a function $f : \mathcal{P}(P) \rightarrow \{\mathrm{H}, \mathrm{V}\}$ such that $\mathcal{R}'(P, f) \leq 2\mathcal{M}(P)$. Moreover, we show that such a function f can be found in polynomial time.

Lemma 7 *There exists a polynomial time algorithm that for every orthogonal polygon P finds a function $f : \mathcal{P}(P) \rightarrow \{H, V\}$ such that $\mathcal{R}'(P, f) \leq 2\mathcal{M}(P)$.*

Proof. Remark that, we can solve the relaxed MLSCk problem for polygon P in polynomial time. Let $C = \{c_1, c_2, \ldots, c_{|C|}\}$ be the set of the cameras in an optimal solution of the relaxed MLSCk problem and the density of camera c_i be d_{c_i}. Moreover, our algorithm for the relaxed MLSCk problem always selects a candidate set of cameras. We construct function $f : \mathcal{P}(P) \rightarrow \{\mathrm{H}, \mathrm{V}\}$ in the following way:

- For every partition $p \in \mathcal{P}(P)$, that the total densities of horizontal cameras guarding it is not less than 1/2, we set $f(p) = \mathrm{H}$.
- We set $f(p) = \mathrm{V}$, for all other partitions $p \in \mathcal{P}(P)$.

Since the total sum of densities of all cameras guarding each point is at least 1, for each partition $p \in \mathcal{P}(P)$ which $f(p) = \mathrm{V}$, the sum of densities of all vertical cameras guarding it is at least 1/2. Now, we use all cameras $c_i \in C$ with densities $d'_{c_i} = 2d_{c_i}$ as a solution for the restricted MLSCk problem. Therefore, $\sum_{c_i \in C} |c_i| d'_{c_i} = 2\mathcal{R}(P)$ and all the constrains of the restricted MLSCk problem are satisfied. Hence, $\mathcal{R}'(P, f) \leq 2\mathcal{R}(P)$. Therefore, by Proposition (4) we have:

$$\mathcal{R}'(P, f) \leq 2\mathcal{M}(P)$$

□

To obtain a 2-approximation algorithm for the MLSCk problem that runs in polynomial time, we show that every instance of the restricted MLSCk problem has an integral solution which is optimal. Furthermore, we show that such an optimal integral solution can be found in polynomial time.

Lemma 8 *There exists a polynomial time algorithm that finds an optimal integral solution for the restricted MLSCk problem.*

Proof. Since in the restricted MLSCk problem each part of the polygon can be guarded with either vertical or horizontal cameras, we can divide the problem into two separate subproblems. In the first subproblem our aim is to put vertical cameras with minimum total length which guard all the parts of the polygon which can be guarded by vertical cameras. In the other subproblem we want to guard the remaining parts with horizontal cameras such that the total length of cameras is minimized. Since in both subproblems we have only horizontal or only vertical cameras, we can find the integral solutions in polynomial time. Combining the solutions of both subproblems gives us an optimal integral solution for the restricted MLSCk problem. □

Note that, from every integral solution of the restricted MLSCk problem for orthogonal polygon P and arbitrary function f, we can find a solution of the MLSCk problem for polygon P with the same set of cameras. Therefore, Lemmas (7) and (8) show that there exists a polynomial time algorithm that finds a 2-approximation solution for MLSCk problem.

Theorem 9 *There exists a polynomial time algorithm that finds a 2-approximation solution for MLSCk .*

4 Conclusion

In this paper we proved that the problem of covering a simple orthogonal art gallery with the minimum-length sliding k-transmitters, is NP-complete, even for $k = 2$.

Then, we presented a 2-approximation algorithm for this problem. The hardness of guarding an orthogonal polygon with the minimum cardinality sliding cameras and covering a polygon with the minimum cardinality sliding k-transmitters remain open.

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