The Epiperimetric Inequality Approach for the Regularity of a Free Boundary Problem

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Hilbert's 19th Problem

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- I.G. Petrowsky (1939): C^1 solutions are analytic.
- E. De Giorgi - J. Nash (1957): With merely assumptions the solutions are $C^{1,\alpha}$.

Main Questions

$$u \leftarrow \min \int_{\Omega} |\nabla u|^2 + 2F(x,u)dx$$

The minimizers solve the semilinear problem

$$\Delta u = f(x,u)$$

- Regularity of the minimizers?
- Regularity of the level sets $\{u = c\}$?

Example: Obstacle Problem

$$u \leftarrow \min_{u \geq 0} \int |\nabla u|^2 + 2u \, dx \xrightarrow{\text{satisfies}} \Delta u = \chi_{\{u>0\}}$$

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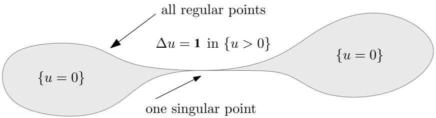

Higher Regularity in a Semilinear Problem ($0 < q < 1$)

$$\min \int |\nabla u|^2 + \frac{2}{1+q}|u|^{1+q}\, dx \xrightarrow{\text{solves}} \Delta u = |u|^{q-1}u$$

Priori Regularity: At least $u \in C^{2,q}$. Moreover, $u \in C^\infty$ in $\{u \neq 0\}$.

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Question: Which regularity for solutions and free boundaries $\partial\{u > 0\}$ and $\partial\{u < 0\}$ do we expect?

Higher Regularity $(0 < q < 1)$

$$\Delta u = |u|^{q-1}u = (u_+)^q - (u_-)^q$$

- M. F. - H. Shahgholian (2017): The optimal regularity of solution on the free boundary $\partial\{u > 0\}$ is $C^{[\kappa],\kappa-[\kappa]}$, where $\kappa = 2/(1-q)$.

Regularity of FB

Blow-up: $u_{r,x_0}(x) := u(x_0 + rx)/r^{\kappa} \longrightarrow u_{x_0}(x) \qquad \text{as } r \to 0.$

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Global Solutions: u_{x_0} is a κ-homogeneous solution in $\mathbb{R}^n$,

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Lipschitz Regularity: On the regular part of FB, we have the normal vector field $x_0 \mapsto \nu_{x_0}$.

Regularity of FB

If $\|u_{r,x_0} - u_{x_0}\|_{L^2(\partial B_1)} \leq Cr^\beta$ for every free boundary point x_0, then

$$|\nu_{x_1} - \nu_{x_2}| \leq C|x_1 - x_2|^{\beta/(\kappa+\beta)},$$

so the free boundary is $C^{1,\beta/(\kappa+\beta)}$.

Epiperimetric Inequality

Define the energy functional

$$M(v) := \int_{B_1} |\nabla v|^2 + \frac{2}{1+q}|v|^{1+q}dx - \kappa \int_{\partial B_1} |v|^2 d\mathcal{H}^{n-1},$$

and the admissible set of blowups

$$\mathbb{H} := \{\alpha_\kappa (x \cdot \nu)_+^\kappa : \nu \in \mathbb{R}^n \text{ is a unit vector}\}.$$

There exist $\epsilon \in (0,1)$ such that if ϕ is a κ-homogeneous function and enough close to $\mathbb{H}$ in topology of $W^{1,2}(B_1)$, then there exists $v \in W^{1,2}(B_1)$ such that $v = \phi$ on ∂B_1 and

$$M(v) - M(\mathbb{H}) \leq (1-\epsilon)(M(\phi) - M(\mathbb{H})).$$

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- M. F. - H. Shahgholian - G.S. Weiss (2021): The regular part of free boundary in problem $\Delta u = |u|^{q-1}u$ is $C^{1,\alpha}$.

Thank you for your attention.