Chapter 7 : Electromagnetic Wave

7.1 Plane Waves in Non Conducting Medium.

- Feature of Maxwell eq. for Electromagnetic field is the existance of travelling wave solution which transport of energy from one point to another.

- Most fundumental waves transverse, plane wares. in an infinite medium.

In absence of sources, Maxwell equations in

$$\begin{cases} \vec{\nabla}\cdot\vec{B} = 0 & \vec{\nabla}\times\vec{E} + \dfrac{\partial \vec{B}}{\partial t} = 0 \\ \vec{\nabla}\cdot\vec{D} = 0 & \vec{\nabla}\times\vec{H} - \dfrac{\partial \vec{D}}{\partial t} = 0 \end{cases}$$

Assuming solutions with harmonic time dependence $e^{-i\omega t}$

We can build an arbitary solution by Fourier superposition for amplitude $\vec{E}(\omega, \vec{x})$, etc, read

$$\vec{\nabla}\cdot\vec{B} = 0 \qquad \vec{\nabla}\times\vec{E} - i\omega\vec{B} = 0$$

$$\vec{\nabla}\cdot\vec{D} = 0 \qquad \vec{\nabla}\times\vec{H} + i\omega\vec{D} = 0$$

For uniform isotropic linear media we have
$$\left.\begin{aligned}\vec{D} &= \epsilon \vec{E}\\ \vec{B} &= \mu \vec{H}\end{aligned}\right\}$$

ϵ, μ may in general be complex function of ω.

ابتدا فرض می‌کنیم که ω حقیقی باشد، این بدین معنا است که اتلاف وجود ندارد.

$$1 \quad \boxed{\vec{\nabla} \times \vec{E} - i\omega \vec{B} = 0}$$

$$\vec{\nabla} \times \vec{H} + i\omega \vec{D} = 0 \rightarrow \vec{\nabla} \times \frac{\vec{B}}{\mu} + i\omega\epsilon \vec{E} = 0$$

$$2 \quad \boxed{\vec{\nabla} \times \vec{B} + i\epsilon\mu\omega \vec{E} = 0}$$

The Divergence equations are not independent.

از گرفتن دیورژانس روابط بالا نیز بدست می‌آید.

$$\vec{\nabla} \times \vec{E} = i\omega \vec{B}$$

$$\frac{1}{i\omega} \vec{\nabla} \times (\vec{\nabla} \times \vec{E}) + i\varepsilon\mu\omega \vec{E} = 0$$

$$\left(\frac{1}{i\omega}\left[\vec{\nabla}(\underbrace{\vec{\nabla} \cdot \vec{E}}_{=0}) - \nabla^2 \vec{E}\right] + i\epsilon\mu\omega \vec{E} = 0\right) \times i\omega$$

$$(\nabla^2 + \mu\epsilon\omega^2)\vec{E} = 0$$

$$\vec{E} = \frac{-1}{i\varepsilon\mu\omega} \vec{\nabla} \times \vec{B}$$

$$\frac{\vec{\nabla} \times (\vec{\nabla} \times \vec{B})}{-i\varepsilon\mu\omega} - i\omega \vec{B} = 0$$

$$\frac{\vec{\nabla}(\underbrace{\vec{\nabla} \cdot \vec{B}}_{=0}) - \nabla^2 \vec{B}}{-i\varepsilon\mu\omega} - i\omega \vec{B} = 0$$

$$\nabla^2 \vec{B} + \varepsilon\mu\omega^2 \vec{B} = 0$$

$$\left(\nabla^2 + \mu\epsilon\omega^2\right)\begin{Bmatrix}\vec{E}\\ \vec{B}\end{Bmatrix} = 0 \qquad (7.3)$$

as a possible solution a plane wave travelling in x- direction, $e^{iKx - i\omega t}$, from (7.3) we find the requirment that the wave number K and the frequency ω is related by $\boxed{K = \sqrt{\mu\epsilon}\,\omega}$

The phase velocity of the wave is

$$v = \frac{\omega}{K} = \frac{1}{\sqrt{\mu\epsilon}} = \frac{c}{n} \quad \text{where} \quad n \equiv \sqrt{\frac{\mu}{\mu_0} \cdot \frac{\epsilon}{\epsilon_0}}$$

"index of refraction"

usually a function of frequency

The primordial Solution in one- dimension is

$$u(x,t) = a e^{iKx - i\omega t} + b e^{-iKx - i\omega t}$$

Using $K = \omega/v$: $u_K(x,t) = a e^{iK(x - vt)} + b e^{-iK(x + vt)}$

if the medium is nondispersive ($\mu\epsilon$ independent of frequency)

the Fourier superposition theorem.

$$(2.44)\quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} A(K) e^{iKx} dK$$

$$(2.45)\quad A(K) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-iKx} f(x) dx$$

Construct a general solution of the form

$$u(x,t) = f(x-vt) + g(x+vt)$$

$\to$ wave traveling $-x$ direction.

wave propagation in positive x-direction.

Dispersive medium - Chapter 7.8 / 7.9 / 7.11

ω: EM-wave.

$\vec{K} = K\hat{n}$

Helmholtz equation $(\nabla^2 + \mu\epsilon\omega^2)\begin{Bmatrix}\vec{E}\\ \vec{B}\end{Bmatrix} = 0$

+ Maxwell equation.

$$\vec{E}(\vec{x},t) = \vec{\mathcal{E}} e^{iK\hat{n}\cdot\vec{x} - i\omega t}$$

$$\vec{B}(\vec{x},t) = \vec{\mathcal{B}} e^{iK\hat{n}\cdot\vec{x} - i\omega t}$$

. $\vec{\mathcal{E}}, \vec{\mathcal{B}}, \hat{n}$ are constant vectors.

Each component of $\vec{E}$ and $\vec{B}$ satisfies $(\nabla^2 + \mu\epsilon\omega^2)\begin{Bmatrix}E\\ B\end{Bmatrix} = 0$

$$\to \quad K^2 \hat{n}\cdot\hat{n} = \mu\epsilon\omega^2$$

1 : $K = \sqrt{\mu\epsilon}\,\omega \quad , \quad \hat{n}\cdot\hat{n} = 1$

The divergence equations demand $\hat{n}\cdot\vec{\mathcal{E}} = 0$, $\hat{n}\cdot\vec{\mathcal{B}} = 0$

$\vec{E}$ and $\vec{B}$ are both perpendicular to the direction of propagation $\hat{n}$. Transverse Wave.

The Curl equation provides a

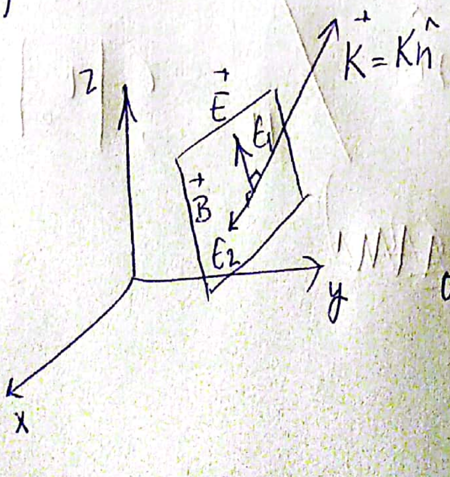
$$\vec{\mathcal{B}} = \sqrt{\mu\epsilon}\;\hat{n}\times\vec{\mathcal{E}} \qquad \sqrt{\mu\epsilon} = \frac{n}{c}$$

$$\vec{\mathcal{H}} = \hat{n}\times\frac{\vec{\mathcal{E}}}{Z} \quad \text{where} \quad Z = \sqrt{\mu/\epsilon} \text{ is an impedance}$$

In Vacuum $\quad Z = Z_0 = \sqrt{\mu_0/\epsilon_0} \approx 376.7$ ohms.
impedance of free space

If $\hat{n}$ is real, implies that $\mathcal{E}$ and $\mathcal{B}$ have the same phase.

$(\hat{\epsilon}_1, \hat{\epsilon}_2, \hat{n})$ mutual orthogonal set



$$\begin{cases} \vec{\mathcal{E}} = \hat{\epsilon}_1 E_0 \\ \vec{\mathcal{B}} = \hat{\epsilon}_2 \sqrt{\epsilon\mu}\, E_0 \end{cases}$$

or

$$\begin{cases} \vec{\mathcal{E}} = \hat{\epsilon}_2 E_0 \\ \vec{\mathcal{B}} = -\epsilon_1 \sqrt{\mu\epsilon}\, E_0 \end{cases}$$

7\) . it represent a time averaged flux of energy given by the real part of the complex Poynting Vector:

$$\vec{S} = \frac{1}{2}\vec{E}\times\vec{H}^*$$

The energy flow (energy per unit area per unit time) is

$$\begin{cases} \vec{E}(\vec{x},t) = \vec{\mathcal{E}}_0 e^{ik\hat{n}\cdot\vec{x}-i\omega t} \\ \vec{B}(\vec{x},t) = \vec{\mathcal{B}}e^{ik\hat{n}\cdot\vec{x}-\omega t} \end{cases} \quad \begin{cases} \vec{\mathcal{B}} = \sqrt{\mu\varepsilon}\times\vec{\mathcal{E}} \\ \mathcal{H} = \hat{n}\times\vec{\mathcal{E}}/Z \\ \text{where} \quad Z = \sqrt{\mu/\epsilon} \end{cases} \Rightarrow \vec{S} = \frac{1}{2}\sqrt{\frac{\epsilon}{\mu}}|E_0|^2\hat{n}$$

The time-averaged energy density u is correspondingly

$$\begin{cases} u = \frac{1}{4}\left(\epsilon\vec{E}\cdot\vec{E}^* + \frac{1}{\mu}\vec{B}\cdot\vec{B}^*\right) \quad \sqrt{\mu\epsilon} \quad \sqrt{\mu\epsilon} \\ u = \frac{\epsilon}{2}|E_0|^2 \end{cases}$$

Ration of $\vec{S}/u \equiv \dfrac{\cancel{\frac{1}{2}}\sqrt{\frac{\epsilon}{\mu}}\cancel{|E_0|^2}\hat{n}}{\frac{\epsilon}{\cancel{2}}\cancel{|E_0|^2}} = \dfrac{1}{\sqrt{\epsilon\mu}} = v$

as expected from phase velocity $v = \dfrac{\omega}{K} = \dfrac{1}{\sqrt{\epsilon\mu}} = \dfrac{c}{n}$

8 Suppose, we consider the most general solution, where $\hat{n}$ is complex:

$$\hat{n} = \hat{n}_R + i\hat{n}_I$$

$$e^{ik\hat{n}\cdot\vec{x} - i\omega t} = e^{-k\hat{n}_I\cdot\vec{x}}\, e^{ik\hat{n}_R\cdot\vec{x} - i\omega t}$$

Inhomogenous plane wave: exponential decay or growth in some direction.

The surface of constant amplitudes or phase is still surface, but they are not parallel.

$\hat{n}\cdot\vec{\mathcal{E}} = 0$ and $\hat{n}\cdot\vec{\mathcal{B}} = 0$ still holds

$$\vec{\mathcal{B}} = \sqrt{\mu\epsilon}\ \hat{n}\times\vec{\mathcal{E}}$$

requirment: $\hat{n}\cdot\hat{n} = 1 \to (\hat{n}_R + i\hat{n}_I)\cdot(\hat{n}_R + i\hat{n}_I)$

$$= |\hat{n}_R|^2 + i\hat{n}_R\cdot\hat{n}_I + i\hat{n}_I\cdot\hat{n}_R - \hat{n}_I^2 = 0$$

$$\begin{cases} \hat{n}_R^2 - \hat{n}_I^2 = 0 \\ \hat{n}_R\cdot\hat{n}_I = 0 \end{cases}$$

n_R and n_I are orthogonal

The coordinate can be oriented in a way that $\hat{n}_R$ is in the x-direction!

$\hat{n}_I$ is in the y-direction!

$$\hat{n} = \hat{e}_1 \cosh\theta + i\hat{e}_2 \sinh\theta$$

real constant

$\hat{e}_1$ and $\hat{e}_2$ real unit vectors in the x and y direction.

not be confused with $\hat{\epsilon}_1$ and $\hat{\epsilon}_2$.

The most general vector $\vec{\mathcal{E}}$ satisfying $\hat{n}\cdot\vec{\mathcal{E}} = 0$ is the

$$\vec{\mathcal{E}} = (i\hat{e}_1 \sinh\theta - \hat{e}_2 \cosh\theta) A + \hat{e}_3 A'$$

where A, A' are complex const.

$$\hat{n}\cdot\vec{\mathcal{E}} = (\hat{e}_1 \cosh\theta + i\hat{e}_2 \sinh\theta)\cdot\vec{\mathcal{E}}$$

$$= i\cosh\theta \sinh\theta A - i\cosh\theta \sinh\theta A = 0 \quad \checkmark$$

In general $\vec{\mathcal{E}}$ has components in direction of $\vec{n}$

10 it is easily verified that for $\theta = 0$

the solutions
$$\begin{cases} \vec{\mathcal{E}} = \hat{\epsilon}_1 E_0 & \vec{\mathcal{B}} = \hat{\epsilon}_2 \sqrt{\mu\epsilon}\, E_0 \\ \text{or } \vec{\mathcal{E}} = \hat{\epsilon}_2 E_0' & \vec{\mathcal{B}} = -\hat{\epsilon}_1 \sqrt{\mu\epsilon}\, E_0' \end{cases}$$
are recovered.

7.2 Linear and Circular Polarization & Stokes parameters.

1 $$\begin{cases} \vec{\mathcal{E}} = \hat{\epsilon}_1 E_0 \qquad \vec{\mathcal{B}} = \hat{\epsilon}_2 \sqrt{\mu\epsilon}\, E_0 \end{cases}$$

electric field vector always in $\hat{\epsilon}_1$ direction.

به این نوع میدان، میدان قطبیده خطی گویند با جهت قطبش ϵ_1

E_0, E_0' constants and complex.

2 $$\begin{cases} \vec{\mathcal{E}} = \hat{\epsilon}_2 E_0' \qquad \vec{\mathcal{B}} = -\hat{\epsilon}_1 \sqrt{\mu\epsilon}\, E_0' \end{cases}$$

linearly polarized with polarization vector $\hat{\epsilon}_2$ and in linearly independent of first one.

Thus the two waves are
$$\begin{cases} \vec{E}_1 = \hat{\epsilon}_1 E_1 e^{i\vec{K}\cdot\vec{x} - i\omega t} \\ \vec{E}_2 = \hat{\epsilon}_2 E_2 e^{i\vec{K}\cdot\vec{x} - i\omega t} \end{cases}$$

$$\vec{B}_j = \sqrt{\mu\epsilon}\, \frac{\vec{K} \times \vec{E}_j}{K} \qquad j = 1, 2$$

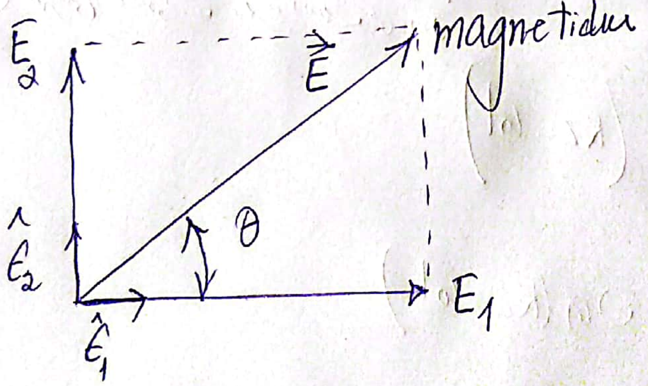
give the most general homogenous plane wave propagation in direction $\vec{K} = K\hat{n}$

$$\vec{E}(\vec{x},t) = (\hat{\epsilon}_1 E_1 + \hat{\epsilon}_2 E_2)\, e^{i\vec{K}\cdot\vec{x} - i\omega t} \qquad (7.19)$$

- Complex quantities to allow the possibility of phase difference.

if E_1 and E_2 have same phase,

(7.19) linear polarized wave with polarization vector making an angle $\theta = \tan^{-1}(E_2/E_1)$ with $\hat{\epsilon}_1$



$$E = \sqrt{E_1^2 + E_2^2}$$

- if E_1 and E_2 have different phase, they represent a wave with "elliptically polarized".

As a specific example, consider circular polarization.

E_1 and E_2 have the same magnitude, but differ in phase by $90°$.

The wave become
$$\begin{cases} E_1 = E_0 = E_0 e^{i\varphi_1} \\ E_2 = E_0 e^{\pm i\pi/2} = E_0 e^{i\varphi_2} \end{cases} \qquad \varphi_1 - \varphi_2 = \pm\frac{\pi}{2}$$

$$\vec{E}(\vec{x},t) = E_0 \left(\hat{\epsilon}_1 \pm i\hat{\epsilon}_2\right) e^{i\vec{K}\cdot\vec{x} - i\omega t}$$

$\uparrow$ real amplitude

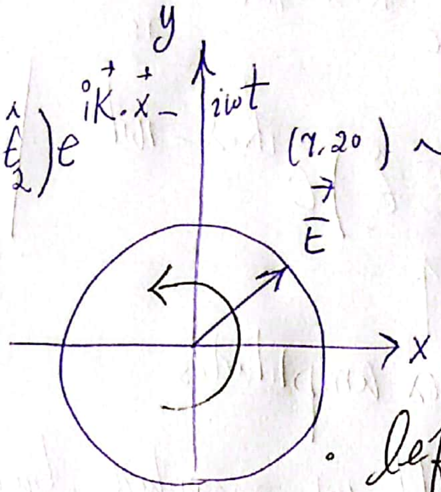
اگر جهت انتشار موج را در راستای z انتخاب کنیم، $\hat{\epsilon}_1$ ، $\hat{\epsilon}_2$ در صفحه x-y قرار بگیرند. بخش حقیقی میدان الکتریکی به صورت زیر به دست می‌آید.

$$\begin{cases} E_x(\vec{x},t) = E_0 \cos(Kz - \omega t) \\ E_y(\vec{x},t) = \mp E_0 \sin(Kz - \omega t) \end{cases}$$

اگر در یک نقطه مشخص از فضا بنشینیم بردار میدان الکتریکی $\vec{E}$ در صفحه x-y حرکت دارد با فرکانس ω می‌پاید.

(7.20)

$$\vec{E}(\vec{x},t) = E_0\left(\hat{\epsilon}_1 \pm i\hat{\epsilon}_2\right)e^{i\vec{k}\cdot\vec{x} - i\omega t}$$



در صورتی که علامت در (7.20) مثبت باشد $\hat{\epsilon}_1 + i\hat{\epsilon}_2$

جهت حرکت میدان $\vec{E}$ پادساعتگرد است

در اپتیک ∘ left circularly polarized

در فیزیک ذرات (ترمینولوژی) ∘ positive helicity

در صورتی که علامت در (7.20) منفی باشد $\hat{\epsilon}_1 - i\hat{\epsilon}_2$

جهت حرکت میدان $\vec{E}$ ساعتگرد است.

در اپتیک right circularly polarized

در فیزیک ذرات ∘ negative helicity

Now we introduce the complex orthogonal unit vector

$$\hat{\epsilon}_{\pm} = \frac{1}{\sqrt{2}}\left(\hat{\epsilon}_1 \pm i\hat{\epsilon}_2\right)$$

With properties:

$$\epsilon_{\pm}^{*}\cdot\epsilon_{\mp} = 0$$

$$\epsilon_{\pm}^{*}\cdot\epsilon_{3} = 0$$

$$\epsilon_{\pm}^{*}\cdot\epsilon_{\pm} = 1$$

The general representation, equivalent to

$$\vec{E}(\vec{x},t) = \left(\hat{\epsilon}_1 E_1 + \hat{\epsilon}_2 E_2\right) e^{i\vec{K}\cdot\vec{x} - i\omega t}$$

is

$$\boxed{\vec{E}(\vec{x},t) = \left(\underset{\uparrow}{E_+} \hat{\epsilon}_+ + \underset{\uparrow}{E_-} \hat{\epsilon}_- \right) e^{i\vec{K}\cdot\vec{x} - i\omega t}} \qquad (7.24)$$

are complex amplitudes

- if E_+ and E_- have different magnitude, but same phase. $\hat{\epsilon}_1$ and $\hat{\epsilon}_2$ represent principal axes of ellipse in the directions of $\hat{\epsilon}_1$ and $\hat{\epsilon}_2$

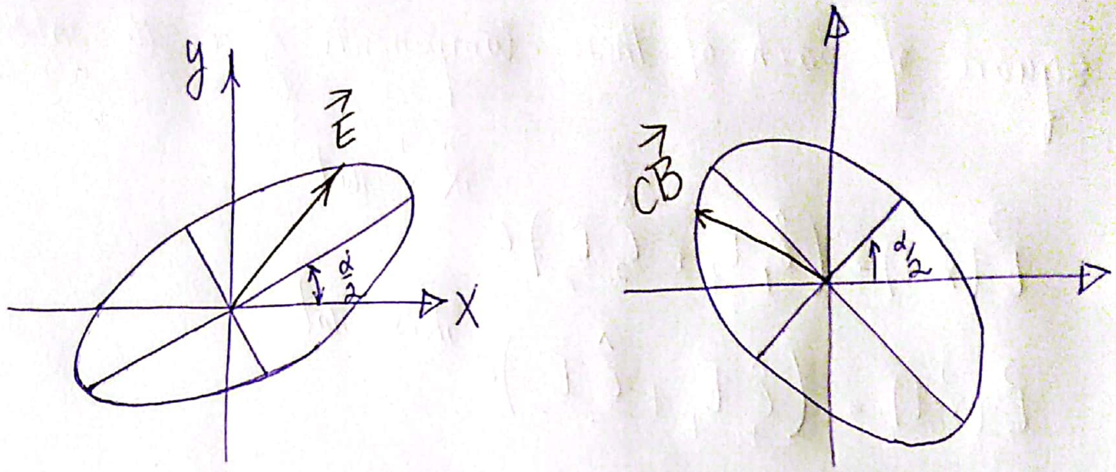
Define $r = \frac{E_-}{E_+} \Rightarrow \frac{\text{semi-major axis}}{\text{semi\_minor axis}} = \frac{1+r}{1-r}$

- if There is a phase difference

$$\frac{E_-}{E_+} = r e^{\underset{\downarrow}{i\alpha}}$$

phase - difference.

ellipse traced out by the vector $\vec{E}$ has its axes rotated by an angle $\frac{\alpha}{2}$.



For $r = \pm 1$ we get back a linearly polarized wave.

اطلاعات مختلف در جهت داشتن E_1, E_2 و یا E_+, E_- در دسترس خواهد بود.

- Four Stokes parameters by G. G. Stokes (1852)

این پارامترها از توان دوم میدان ساخته می‌شوند و با اندازه‌گیری شدت در دسترس خواهند بود.

Assume a propagation of wave in z-direction
the scalar products:

$\hat{\epsilon}_1 \cdot \vec{E}$: amplitude of radiation, linear polariz. in x-direction

$\hat{\epsilon}_2 \cdot \vec{E}$: amplitude of radiation, linear polarz. in y-direction

$\hat{\epsilon}_+^* \cdot \vec{E}$: positive helicity

$\hat{\epsilon}_-^* \cdot \vec{E}$: negative helicity.

square of each of these components give the intensity.

$$(7.19) \quad \vec{E}(\vec{x},t) = (\hat{\epsilon}_1 E_1 + \hat{\epsilon}_2 E_2)\, e^{i\vec{K}\cdot\vec{x} - i\omega t}$$

$$(7.24) \quad \vec{E}(\vec{x},t) = (\hat{\epsilon}_+ E_+ + \hat{\epsilon}_- E_-)\, e^{i\vec{K}\cdot\vec{x} - i\omega t}$$

each scalar coefficient (7.19) and (7.24) as a magnitude times a phase factor:

$$E_1 = a_1 e^{i\delta_1} \qquad E_2 = a_2 e^{i\delta_2}$$

$$E_+ = a_+ e^{i\delta_+} \qquad E_- = a_- e^{i\delta_-}$$

In terms of the linear polarization basis $(\hat{\epsilon}_1, \hat{\epsilon}_2)$ the stokes parameters are

Notation of "Born & Wolf" کتاب اپتیک

(I, Q, U, V)