Sharif University of Technology

Electromagnetism 3
Problem Set 0

Mathematical Preliminaries and Maxwell Equations

Department of Physics, Sharif University of Technology.

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1 Vector Identities

Prove,

(a) $\boldsymbol{\nabla}\cdot(f\mathbf{g}) = f\boldsymbol{\nabla}\cdot\mathbf{g} + \mathbf{g}\cdot\boldsymbol{\nabla}f$

(b) $\boldsymbol{\nabla}\times(f\mathbf{g}) = f\boldsymbol{\nabla}\times\mathbf{g} + \mathbf{g}\times\boldsymbol{\nabla}f$

(c) $\boldsymbol{\nabla}\times(\mathbf{g}\times\mathbf{r}) = 2\mathbf{g} + r\frac{\partial\mathbf{g}}{\partial r} - \mathbf{r}(\boldsymbol{\nabla}\cdot\mathbf{g})$

(d) $\boldsymbol{\nabla}\times(\mathbf{g}\cdot\mathbf{r}) = \mathbf{g} + \frac{(\mathbf{r}\cdot\mathbf{g}')\mathbf{r}}{r}$

(e) $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C})(\mathbf{B}\cdot\mathbf{D}) - (\mathbf{A}\cdot\mathbf{D})(\mathbf{B}\cdot\mathbf{C})$

(f) $\boldsymbol{\nabla}\cdot(\mathbf{f}\times\mathbf{g}) = \mathbf{g}\cdot(\boldsymbol{\nabla}\times\mathbf{f}) - \mathbf{f}\cdot(\boldsymbol{\nabla}\times\mathbf{g})$

(g) $(\mathbf{A}\times\mathbf{B})\times(\mathbf{C}\times\mathbf{D}) = (\mathbf{A}\cdot\mathbf{C}\times\mathbf{D})\mathbf{B} - (\mathbf{B}\cdot\mathbf{C}\times\mathbf{D})\mathbf{A}$

(h) $\boldsymbol{\nabla}\cdot(\mathbf{A}\times\mathbf{r}) = 0$

(i) $A_iB_j = \frac{1}{2}\epsilon_{ijk}(\mathbf{A}\times\mathbf{B})_k + \frac{1}{2}(A_iB_j + A_jB_i)$

(Hint: use the Levi-Civita symbol)

2 Identities For $\nabla\times\mathbf{L}$

By putting $\hbar = 1$, The angular momentum operator became $\mathbf{L} = -i\mathbf{r}\times\boldsymbol{\nabla}$. Prove the Identities

(a) $\boldsymbol{\nabla}\times\mathbf{L} = -i\boldsymbol{\nabla}^2 + i\boldsymbol{\nabla}(1+\mathbf{r}\cdot\boldsymbol{\nabla})$

(b) $\boldsymbol{\nabla}\times\mathbf{L} = (\hat{\mathbf{r}}\times\mathbf{L})\left(\frac{1}{r}\frac{\partial}{\partial r}r\right) + \hat{\mathbf{r}}\frac{i}{r}\mathbf{L}^2$

3 The Time Derivative of Flux Integral

Prove,

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbf{S}(t)} d\mathbf{S}\cdot\mathbf{B} = \int_{\mathbf{S}(t)} d\mathbf{S}\cdot\left[\mathbf{v}(\boldsymbol{\nabla}\cdot\mathbf{B}) - \boldsymbol{\nabla}\times(\mathbf{v}\times\mathbf{B}) + \frac{\partial\mathbf{B}}{\partial t}\right]. \tag{1}$$

4 Curl, Div, Grad, Laplacian, and all that in arbitrary coordinate

Consider the Euclidean 3-dimentianal space in Cartesian coordinate,

$$ds^2 = dx^2 + dy^2 + dz^2 \tag{2}$$

Curl, Div, Grad, and Laplacian take the following forms

$$\boldsymbol{\nabla}f = \partial_x f\hat{\mathbf{x}} + \partial_y f\hat{\mathbf{y}} + \partial_z f\hat{\mathbf{z}} \tag{3}$$

$$\boldsymbol{\nabla}\cdot\alpha = \boldsymbol{\nabla}\cdot(A\hat{\mathbf{x}} + B\hat{\mathbf{y}} + C\hat{\mathbf{z}}) = \partial_x A + \partial_y B + \partial_z C \tag{4}$$

$$\boldsymbol{\nabla}\times\omega = \boldsymbol{\nabla}\times(P\hat{\mathbf{x}} + Q\hat{\mathbf{y}} + R\hat{\mathbf{z}}) = (\partial_y R - \partial_z Q)\hat{\mathbf{x}} + (\partial_z P - \partial_x R)\hat{\mathbf{y}} + (\partial_x Q - \partial_y P)\hat{\mathbf{z}} \tag{5}$$

$$\boldsymbol{\nabla}^2 f = \boldsymbol{\nabla}\cdot\boldsymbol{\nabla}f = \partial_x^2 f + \partial_y^2 f + \partial_z^2 f \tag{6}$$

Euclidean metric in arbitrarily orthonormal coordinates takes the following form

$$ds^2 = \lambda^2 du^2 + \mu^2 dv^2 + \nu^2 dw^2 \,. \tag{7}$$

(a) Obtain the form of Curl, Div, Grad, and Laplacian in this coordinate. (Hint: use differential forms.)

(b) Obtain the explicit form of Curl, Div, Grad, and Laplacian in spherical and cylindrical coordinates and compare them with the formula on the Last page of Jackson 1999.

5 Helmholtz Theorem

(a) Show that an arbitrary vector field $\mathbf{C}(\mathbf{r})$ can always be decomposed into the sum of two vector fields; one with zero divergence and one with zero curl. Specifically

$$\mathbf{C} = \mathbf{C}_\perp + \mathbf{C}_\parallel \,, \qquad \text{where,} \qquad \boldsymbol{\nabla}\cdot\mathbf{C}_\perp = 0 \qquad \text{and} \qquad \boldsymbol{\nabla}\times\mathbf{C}_\parallel = 0 \tag{8}$$

We are especially interested in representation, in which

$$\mathbf{C}(\mathbf{r}) = \boldsymbol{\nabla}\times\mathbf{F}(\mathbf{r}) - \boldsymbol{\nabla}\boldsymbol{\Omega}(\mathbf{r}) \tag{9}$$

where $\mathbf{F}(\mathbf{r})$ and $\boldsymbol{\Omega}$ are given uniquely by convergent integrals over all space by

$$\mathbf{F}(\mathbf{r}) = \frac{1}{4\pi}\int d^3\mathbf{r}'\,\frac{\boldsymbol{\nabla}'\times\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,, \qquad \boldsymbol{\Omega}(\mathbf{r}) = \frac{1}{4\pi}\int d^3\mathbf{r}'\,\frac{\boldsymbol{\nabla}'\cdot\mathbf{C}(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,. \tag{10}$$

(b) Prove that for arbitrary scalar function $\phi(\mathbf{r})$,

$$\phi(\mathbf{r}) = -\boldsymbol{\nabla}\cdot\frac{1}{4\pi}\int_V d^3\mathbf{r}'\,\frac{\boldsymbol{\nabla}'\phi(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|} + \boldsymbol{\nabla}\cdot\frac{1}{4\pi}\int_S d\mathbf{S}'\,\frac{\phi(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,. \tag{11}$$

6 $SO(2)$ Symmetry of Maxwell Equations and Magnetic Charge

Maxwell Equations can be generalized to contain magnetic charge and magnetic charge current,

$$\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{\rho_e}{\epsilon_0} \qquad\qquad \boldsymbol{\nabla}\cdot\mathbf{B} = \mu_0\rho_m \tag{12}$$

$$\boldsymbol{\nabla}\times\mathbf{E} = -\mu_0\mathbf{j}_m - \frac{\partial\mathbf{B}}{\partial t} \qquad\qquad \boldsymbol{\nabla}\times\mathbf{B} = -\mu_0\mathbf{j}_e + \frac{1}{c^2}\frac{\partial\mathbf{E}}{\partial t},. \tag{13}$$

Similarly, the Coulomb-Lorentz force can be symmetrized as

$$\mathbf{f} = (\rho_e\mathbf{E} + \mathbf{j}_e\times\mathbf{B}) + (\rho_m\mathbf{B} - \mathbf{j}_m\times\mathbf{E}/c^2) \tag{14}$$

(a) Show that this system of equations are symmetric under $SO(2)$ transformation parametrized by θ, as

$$\mathbf{E}' = \mathbf{E}\cos\theta + c\mathbf{B}\sin\theta \qquad\qquad c\mathbf{B}' = -\mathbf{E}\sin\theta + c\mathbf{B}\cos\theta \tag{15}$$

$$c\rho_e' = c\rho_e\cos\theta + \rho_m\sin\theta \qquad\qquad \rho_m' = -c\rho_e\sin\theta + \rho_m\cos\theta \tag{16}$$

$$c\mathbf{j}_e' = c\mathbf{j}_e\cos\theta + \mathbf{j}_m\sin\theta \qquad\qquad \mathbf{j}_m' = -c\mathbf{j}_e\sin\theta + \mathbf{j}_m\cos\theta \tag{17}$$

and this symmetry imposes $c^2\rho_e^2 + \rho_m^2 = c^2\rho_e'^2 + \rho_m'^2$ condition on charge densities.

(b) Show in the cases ratio of electric to the magnetic charge (e/cg) of all the particles is the same for all the elementary particles it is possible to choose θ such that $\rho_m' = 0$ and the Maxwell equations take their ordinary forms, find that θ.

(c) By considering the influence of $SO(2)$ symmetry on source free Maxwell equation (Electromagnetic waves), discuss the implication of this symmetry.