PROBLEMS

Reference Textbook: E.T. Whittaker, "A History of the Theories of Aether & Electricity: The Modern Theories (1900-1926)".

— Use Planck's law $I(\nu,T) = \frac{2h\nu^3}{c^2}\left(e^{\frac{h\nu}{kT}} - 1\right)^{-1}$ to show that P/A, which is the power emitted per unit area by the emitting source is given by $\int_0^\infty I(\nu,T)\,d\nu \int \cos\theta\, d\Omega$. Prove that $P/A = \sigma T^4$, which is the Stefan-Boltzmann law. Calculate $\sigma = 2\pi^5 k^4/(15c^2h^3) \approx 5.67\times 10^{-8}$ W.m^{-2}.K^{-4}, using the fact that $\int_0^\infty u^3 du/(e^u - 1) = \pi^4/15$.

— Use the Bohr-Sommerfeld quantization rule to determine the energy levels of a simple harmonic oscillator.

— Use de Broglie's relations to prove that wave-particle duality implies: group velocity of waves = particle velocity.

— Use $[\hat{q},\hat{p}] = i\hbar$ to show that in the momentum space $\hat{q} = i\hbar\frac{\partial}{\partial p}$.

— A particle of mass m moves uniformly around a circle of radius r. Show that the action over the period of motion is $A = \pi\, J$, where J is the orbital angular momentum.

— For a stationary circular electron orbit in the Bohr atom, compute the ratio of the circumference of the orbit to the de Broglie wavelength of the electron that follows the orbit.

— What is the formula for the zero-point energy of a harmonic oscillator of frequency ω_0?

— Starting from the Schrödinger equation, derive the continuity equation

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{j} = 0\,, \tag{1}$$

where

$$\rho = \psi^*\psi\,, \qquad \mathbf{j} = \frac{\hbar}{m}\Im\left(\psi^*\frac{d\psi}{dx}\right)\,. \tag{2}$$

What is the physical interpretation of these results?

— Use the Bohr-Sommerfeld quantization rule to extend the Bohr model of the atom to elliptical orbits.

— Starting from $H\psi = i\hbar\frac{\partial\psi}{\partial t}$ and $H = c\,\boldsymbol{\alpha}\cdot\mathbf{p} + mc^2\beta$, find the conditions that $\boldsymbol{\alpha}$ and β have to satisfy in order to get the Dirac equation.