Deterministic Design of Toeplitz Matrices with Small Coherence Based on Weyl Sums

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***Abstract*—The design of deterministic measurement matrices has been the focus of research in compressed sensing from the early stages. In particular, structured measurement matrices are of great interest as they could be efficiently stored. Our focus in this paper is on Toeplitz structure where naturally arises in linear shift-invariant systems (convolution operator). We design complex-valued Toeplitz matrices with unit modulus elements that have small coherence. The complex phase of the matrix elements are determined by certain polynomials. We provide upper-bounds for the coherence of the resulting matrix using tools from analytic number theory, namely, the Weyl sum theorem. Simulation results confirm that the proposed matrices perform similar to the Gaussian Toeplitz matrices of the same size.**

***Index Terms*—Coherence, Compressive sensing, Toeplitz matrix, Sparse channel estimation, Weyl sum.**

I. Introduction

COMPRESSED sensing (CS) [1]–[3] is a prominent field which aims at sampling sparse signals efficiently using a linear and non-adaptive approach; i.e., projecting sparse signals onto lower-dimensional subspaces. Specifically, if $\mathbf{x}$ denotes a length n vector which is k-sparse, then, in CS the data acquisition process can be described as

$$\mathbf{y}_{m\times 1} = \mathbf{\Phi}_{m\times n}\ \mathbf{x}_{n\times 1} + \mathbf{e}_{m\times 1}, \tag{1}$$

where $\mathbf{e}$ denotes the additive noise vector and $\mathbf{\Phi}$ is called the measurement (sensing) matrix. Obviously, the recovery of $\mathbf{x}$ from $\mathbf{y}$ depends on the size and properties of the matrix $\mathbf{\Phi}$. One of the well-studied sufficient conditions on $\mathbf{\Phi}$ for stable recovery of $\mathbf{x}$ is the so-called *Restricted Isometry Property (RIP)*. We say that $\mathbf{\Phi}$ satisfies RIP of order k with constant δ_k if [4]

$$(1-\delta_k)\|\mathbf{x}_{n\times 1}\|_2^2 \le \|\mathbf{\Phi}_{m\times n}\mathbf{x}_{n\times 1}\|_2^2 \le (1+\delta_k)\|\mathbf{x}_{n\times 1}\|_2^2 \tag{2}$$

holds for all k-sparse vectors $\mathbf{x}_{n\times 1}$. It is shown in [5] that a wide range of random sensing matrices (such as Bernoulli and Gaussian) satisfy the aforementioned property with high probability when $m \ge \mathcal{O}(k\log(\frac{n}{k}))$. However, checking this property for a generic matrix is proved to be computationally NP-hard [6]. This is a great restriction for deterministic designs, as the averaging technique used for random matrices is no longer available. A common alternative to RIP which is computationally feasible, is the coherence measure. The coherence (or mutual coherence) of a matrix $\mathbf{\Phi}$ is defined as

$$\mu(\mathbf{\Phi}) \ = \max_{0\le i<j\le n-1} \frac{|\langle \boldsymbol{\phi}_i, \boldsymbol{\phi}_j\rangle|}{\|\boldsymbol{\phi}_i\|_2\ \|\boldsymbol{\phi}_j\|_2}, \tag{3}$$

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where $\boldsymbol{\phi}_0, \boldsymbol{\phi}_1, \dots, \boldsymbol{\phi}_{n-1}$ represent the columns of $\mathbf{\Phi}$. It is known that if $\mu(\mathbf{\Phi}) < \frac{1}{k-1}$, then, $\mathbf{\Phi}$ satisfies the RIP of order k with $\delta_k < (k-1)\mu(\mathbf{\Phi})$ [7], [8]. Thus, the coherence bound is stronger than the RIP condition. Unfortunately, the guarantees based on the coherence bound are too conservative. The Welch bound for $m\times n$ matrices with $m \le n$ implies that [9]

$$\mu(\mathbf{\Phi}) \ge \sqrt{\frac{n-m}{m(n-1)}}. \tag{4}$$

Therefore, to guarantee the RIP of order k, at least $m = \mathcal{O}(k^2)$ measurements are required. Compared to $m = \mathcal{O}(k\log(\frac{n}{k}))$ for random matrices, this is a considerable increase. With all the shortcomings of the guarantees based on the coherence measure, it is the dominant tool for the design of deterministic matrices (see for instance [8], [10]–[14]). In other words, it is desirable to construct fat matrices with unit-norm columns that have small coherence values. Our approach in this paper is also based on minimizing the coherence value.

In general, the matrix design problem is solved by tuning all the elements of the matrix, possibly independently. In some applications, however, the physics of the problem enforce certain structures on the matrix. An example which will be studied in this paper is the sparse channel estimation, where the Toeplitz structures is involved [15], [16]. While the structure is mainly restrictive for the design, it might bring some advantages. For instance, the existence of the structure simplifies the task of storing the matrix, particularly, in large dimensions. A special advantage of Toeplitz (and circular) structure is the existence of fast matrix-vector multiplication routines, that expedite the recovery algorithms [17].

A. Application

From the theory of communications we know that a digital communication system with band-limited continuous-time pulse shape could be modeled by a discrete-time counterpart [18]. The communication channel in most setups can be fairly approximated as a linear shift-invariant operator (at least for short time spans), e.g., a filter. This operator is initially unknown to the transmitter and receiver sides, and should be estimated before its effect is compensated at the receiver. For this purpose, the transmitter includes certain fixed patterns within the transmitting data to facilitate the estimation procedure at the receiver. Such fixed data are known as pilots.

Considering a fixed-size structure for pilot and data symbols as in Figure 1, we can model the discrete-time communication

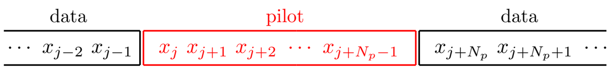

Fig. 1: The transmitted signal.

system as

$$\begin{bmatrix} \vdots \\ y_{i+L-2} \\ y_{i+L-1} \\ \vdots \\ y_{i+N_p-1} \\ y_{i+N_p} \\ \vdots \end{bmatrix} = \begin{bmatrix} & & \vdots & & \\ x_{j+L-2} & x_{j+L-3} & \cdots & x_{j-1} \\ x_{j+L-1} & x_{j+L-2} & \cdots & x_j \\ \vdots & \vdots & \ddots & \vdots \\ x_{j+N_p-1} & x_{j+N_p-2} & \cdots & x_{j+N_p-L} \\ x_{j+N_p} & x_{j+N_p-1} & \cdots & x_{j+N_p-L+1} \\ & & \vdots & & \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \\ \vdots \\ h_{L-2} \\ h_{L-1} \end{bmatrix} + \begin{bmatrix} \vdots \\ z_{i+L-2} \\ z_{i+L-1} \\ \vdots \\ z_{i+N_p-1} \\ z_{i+N_p} \\ \vdots \end{bmatrix} \tag{5}$$

where in the discrete-time model, $h[i]$, $0 \leq i < L$ stand for the finite-length channel taps, $x[i]$ represents the ith transmitting symbol, and $y[i]$ shows the received data corresponding the ith transmitting symbol, which includes an additive noise term $z[i]$ besides the channel effect.

As the receiver is only aware of the pilot values, it is logical to estimate the channel based on $y[i]$s that are solely composed of pilot values:

$$\begin{bmatrix} y_{i+L-1} \\ \vdots \\ y_{i+N_p-1} \end{bmatrix} = \begin{bmatrix} x_{i+L-1} & \cdots & x_i \\ \vdots & \ddots & \vdots \\ x_{i+N_p-1} & \cdots & x_{i+N_p-L} \end{bmatrix} \begin{bmatrix} h_0 \\ \vdots \\ h_{L-1} \end{bmatrix} + \begin{bmatrix} z_{i+L-1} \\ \vdots \\ z_{i+N_p-1} \end{bmatrix}. \tag{6}$$

By using vector and matrix notations, we can rewrite (6) as

$$\mathbf{y} = \mathbf{\Phi}\mathbf{h} + \mathbf{z}, \tag{7}$$

where $\mathbf{\Phi}$ is the Toeplitz matrix formed by pilot values, $\mathbf{h}$ is the length L vector of channel taps, and $\mathbf{z}$ is the vector of additive noise values.

In many real-world applications such as underwater acoustic communications [20], channels tend to exhibit a sparse behavior. Compressed sensing is a tool for such scenarios to increase both accuracy and computation efficiency of the estimates. Alternatively, compressed sensing allows for maintaining the performance of the channel estimation block with fewer pilots, thus, reducing the communication overhead. In summary, we are interested in designing suitable sensing matrices $\mathbf{\Phi}$ with Toeplitz structure that can be converted into pilot sequences.

B. Related Works

The literature of sensing matrix design has a history almost equivalent to that of the compressed sensing; therefore, mentioning all of the designs in this short note is not feasible. Instead, we briefly mention the existing designs related to the Toeplitz structure. The study of random Toeplitz matrices is carried out in [16]. It is shown that if the elements of the first row and column of $\mathbf{\Phi}_{m\times n}$ are set as i.i.d. realizations of a bounded zero-mean distribution with variance $\frac{1}{m}$, then, the matrix satisfies RIP of order k (with high probability) when $m \geq \mathcal{O}(k^2 \log n)$. This bound is improved to $m \geq \mathcal{O}\big(k^{1.5} \log^{1.5} n\big)$ in [21] for random partial circulant matrices (randomly selecting rows from a circulant matrix) drawn from Rademacher distribution. This bound was later improved to $m \geq \mathcal{O}\big(k \log^2 k \, \log^2 n\big)$ in [22]. The bound $m \geq \mathcal{O}\big(k \log^4 n\big)$ is derived in [23] when $\mathbf{\Phi}$ is obtained by random row selection from certain deterministic circulant matrices. A fully deterministic matrix design is presented in [24], where the matrix consists of the elements of the form $\exp(\mathrm{j}2\pi\alpha i^3)$, where α is an algebraic irrational number that is the root of a quadratic polynomial with integer coefficients. For large m, n values, the matrix is guaranteed the RIP order of $\frac{n^{3/8}}{\sqrt{\frac{m}{n}}\log n}$. Unlike the previous random designs, this matrix is Toeplitz (rather than partially Toeplitz). We should highlight that our designed matrices have many similarities with these matrices, except that we avoid using irrational α values (due to practical considerations) which forces us to adopt completely different proof techniques. Other Toeplitz-related matrix designs can be found in [25]–[29].

C. Contributions

In this paper, we shall introduce a fully deterministic matrix design for Toeplitz sensing matrices. The elements of the designed matrices all have unit modulus. We provide an upper-bound for the coherence value of such matrices by invoking some of the known results regarding Weyl sums. The general structure of our matrices is similar to that of [24]; however, besides more degrees of freedom in our approach, we have the advantage of using finitely many elements from the unit complex ball. To be more specific, a critical part of the design in [24] is to use non-repeating complex values on the unit complex ball as matrix elements. In practical scenarios, however, all elements are subject to quantization which violates the non-repeating nature of the design. In contrast, we use only $3n$ different values of the form $\mathrm{e}^{\mathrm{j}\frac{2\pi}{3n}l}$, $0 \leq l < 3n$. Furthermore, our coherence guarantee holds for all prime n, instead of the sufficiently large n in [24]. Finally, when $\frac{m}{n}$ is fixed and n goes to infinity, the scaling order of our guaranteed RIP level is superior than that of [24].

D. Outline

The organization of the paper is as follows: in Section II we will review some of the mathematical tools related to the Weyl sums. Then, we shall propose our design in Section III and derive an upper-bound for its coherence using Weyl sums. We shall investigate the designed matrices via numerical simulations in Section IV. Finally, we conclude the paper in Section V.

II. Mathematical Preliminaries

In this section, we describe a particular family of sums in analytic number theory, widely known as Weyl sums, and express some of the available bounds on such sums. A Weyl sum is of the form [30]

$$\mathrm{S}_h(N) = \sum_{i=1}^{N} \exp\big(\mathrm{j}2\pi h(i)\big), \tag{8}$$

where $h(x)$ is a polynomial of variable x with real coefficients. The following theorems present some of the known upper-bounds on Weyl sums.

Theorem 1 ([31]). *Let $h(x) = \alpha_1 x + \cdots + \alpha_\kappa x^\kappa$ with $\kappa \geq 2$, and*

$$\Big|\alpha_\kappa - \frac{a}{q}\Big| \leq \frac{1}{q^2}, \quad N \leq q \leq N^{\kappa-1}.$$

for relatively prime integers a and q. Then for any $0 < \epsilon < 1$ we have

$$\big|\mathrm{S}_h(N)\big| \leq C(\kappa, \epsilon) N^{1-\frac{1-\epsilon}{2^{\kappa-1}}}, \tag{9}$$

where the constant $C(\kappa, \epsilon)$ does not depend on N.

As we shall see later, we relate the cross correlation of the columns of the sensing matrix to the Weyl sums. Therefore, we prefer to have lowest possible bounds in order to reduce the coherence of the matrix. The upper-bound in (9) is of $\mathcal{O}\big(N^{1-\frac{1-\epsilon}{2^{\kappa-1}}}\big)$ as a function of N, which is increasing in terms of κ. Therefore, we set $\kappa = 2$ in this paper, and focus on a sharper bound for the specific case of $\kappa = 2$:

Theorem 2 ([32]). *Let $h(x) = \alpha_2 x^2 + \alpha_1 x + \alpha_0$ where α_i are real numbers such that*

$$\left|\alpha_2 - \frac{a}{q}\right| \leq \frac{1}{q^2}, \tag{10}$$

for some relatively prime integers a and q. Then

$$\big|\mathrm{S}_h(N)\big| < \sqrt{N + \tfrac{1}{q}\big(2N + q\big)\big(N + q\log q\big)}. \tag{11}$$

III. Main Result

As discussed in Section I, the aim of this paper is to propose a Toeplitz sensing matrix with small coherence measure. In other words, we would like to determine the elements of an $m \times n$ matrix of the form

$$\boldsymbol{\Phi} = \begin{bmatrix} \phi_n & \phi_{n-1} & \cdots & \phi_2 & \phi_1 \\ \phi_{n+1} & \phi_n & \cdots & \phi_3 & \phi_2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_{m+n-1} & \phi_{m+n-2} & \cdots & \phi_{m+1} & \phi_m \end{bmatrix}. \tag{12}$$

In this paper, we set ϕ_is as

$$\phi_i = \frac{1}{\sqrt{m}} \exp\big(\mathrm{j}2\pi f(i)\big), \tag{13}$$

where the function f shall be introduced later. This choice is beneficial in the mentioned communication application (pilot values), since the peak-to-average-power ratio is reduced to its minimum value of 1. In Theorem 3 we bound the coherence of a specific family of sensing matrices formed with (13).

Theorem 3. *Let n be a prime integer and define $\boldsymbol{\Phi}_{m\times n}$ to be the Toeplitz matrix in* (12) *with*

$$\phi_i = \frac{1}{\sqrt{m}} \exp\left(\mathrm{j}2\pi\big(\tfrac{1}{n} i^3/3 + Bi^2/2 + Ci\big)\right). \tag{14}$$

Then, we have that

$$\mu(\boldsymbol{\Phi}) < \frac{1}{m}\sqrt{m + \tfrac{1}{n}\big(2m+n\big)\big(m + n\log n\big)} \tag{15}$$

Proof. First note that $|\phi_i| = \frac{1}{\sqrt{m}}$ for all i. Therefore, $\|\boldsymbol{\phi}_j\|_2 = 1$ for all $1 \leq j \leq n$. Recalling the definition of coherence in (3), we can write that:

$$\begin{aligned} \mu(\boldsymbol{\Phi}) &= \max_{0\leq k_2<k_1\leq n-1} \frac{|\langle \boldsymbol{\phi}_{k_1}, \boldsymbol{\phi}_{k_2}\rangle|}{\|\boldsymbol{\phi}_{k_1}\|_2 \ \|\boldsymbol{\phi}_{k_2}\|_2} \\ &= \max_{0\leq k_2<k_1\leq n-1} \left|\sum_{i=1}^{m} \phi_{i+k_1}\phi^*_{i+k_2}\right| \end{aligned} \tag{16}$$

Using the proposed format of the matrix elements in (14), we can simplify (16) as

$$\mu(\boldsymbol{\Phi}) = \max_{0\leq k_2<k_1\leq n-1} \frac{1}{m}\Bigg|\sum_{i=1}^{m} \exp\Big(\mathrm{j}2\pi\big(f(i+k_1) - f(i+k_2)\big)\Big)\Bigg|, \tag{17}$$

where $f(x) = \frac{1}{n}x^3/3 + Bx^2/2 + Cx$. With this choice of $f(x)$, the difference $f(i+k_1) - f(i+k_2)$ is always a quadratic polynomial of i that can be expressed as $\alpha_2 i^2 + \alpha_1 i + \alpha_0$ where

$$\begin{aligned} &\alpha_2 = \tfrac{1}{n}(k_1 - k_2), \\ &\alpha_1 = \tfrac{1}{n}(k_1^2 - k_2^2) + B(k_1 - k_2), \\ &\alpha_0 = \tfrac{1}{3n}(k_1^3 - k_2^3) + B(k_1^2 - k_2^2)/2 + C(k_1 - k_2). \end{aligned} \tag{18}$$

Hence, (17) can be rewritten as

$$\mu(\boldsymbol{\Phi}) = \max_{0\leq k_2<k_1\leq n-1} \frac{1}{m}\left|\sum_{i=1}^{m} \exp\left(\mathrm{j}2\pi\big(\alpha_2 i^2 + \alpha_1 i + \alpha_0\big)\right)\right|. \tag{19}$$

Since $1 \leq k_1 - k_2 \leq n-1$ and n is a prime integer, $k_1 - k_2$ and n are coprime. Moreover, $|\alpha_2 - \frac{k_1-k_2}{n}| = 0 < \frac{1}{n^2}$. Therefore, the conditions of Theorem 2 are met here, and we can conclude that

$$\mu(\boldsymbol{\Phi}) < \frac{1}{m}\sqrt{m + \tfrac{1}{n}\big(2m+n\big)\big(m + n\log n\big)}, \tag{20}$$

and the proof is complete. □

The coherence bound in (20) can be restated as

$$\mu(\boldsymbol{\Phi}) < \sqrt{2\big(\tfrac{1}{m} + \tfrac{1}{n}\big) + (2+\gamma)\tfrac{\log n}{m}} \approx \sqrt{2+\gamma}\sqrt{\tfrac{\log n}{m}}, \tag{21}$$

if $\gamma = n/m$ and $n, m \gg 1$. Indeed, the parameter γ encodes the aspect ratio of the matrix. For a fixed aspect ratio (fixed γ), the coherence bound in (21) differs from the universal Welch bound (4) by an $\mathcal{O}(\log n)$ factor. Note that the Welch bound is a general bound for all matrices, while here we are focusing on the small subset of Toeplitz matrices. Overall, we find this a fair trade-off.

Corollary 1. *The designed matrix satisfies RIP of order $k < \frac{\sqrt{m/\log n}}{\sqrt{2+\gamma}} + 1$ with $\delta_k < (k-1) \times \sqrt{2+\gamma}\sqrt{\frac{\log n}{m}}$.*

In summary, the introduced Toeplitz matrices in Theorem 3 have relatively small coherence values which makes them suitable for sparse signal recovery. In particular, they are useful in applications such as sparse channel estimation due to their Toeplitz structure. Besides, the matrices are formed of a finite number of equidistant elements on the unit circle in the complex plane (except for the normalizing constant $\frac{1}{\sqrt{m}}$). The equi-modulus property of the elements is desirable from the practical perspective (small PAPR value). Moreover, the minimum distance between the used elements in our design is maximal; this property increases the robustness of the matrix against quantization distortions. Another point is the degrees of freedom provided by the parameters B and C in our matrices; one can set these parameters arbitrarily without affecting the overall coherence bound.

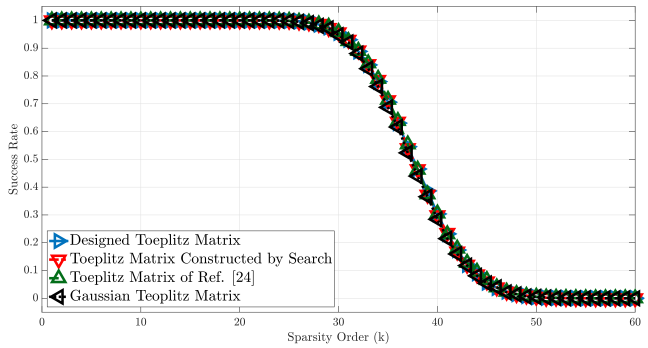

Fig. 2: Perfect recovery probability as a function of sparsity order k. All the matrices are of size 38×113.

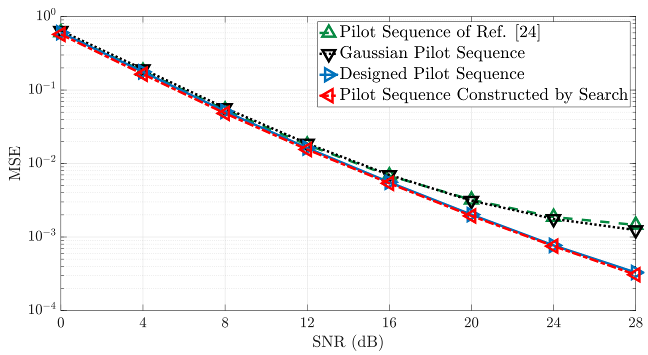

Fig. 3: MSE as a function of SNR. All the sequences are of length 54.

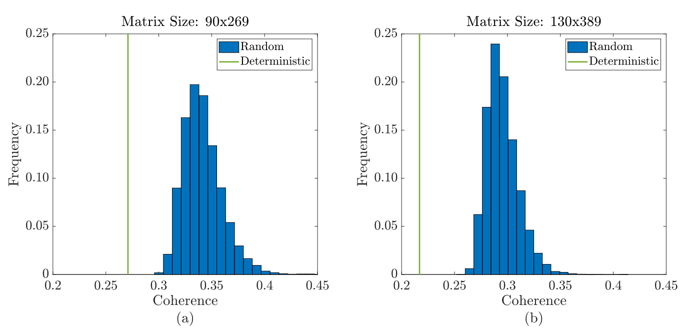

Fig. 4: The comparison between the coherence of the proposed matrices and typical Toeplitz Gaussian matrices of the size a) 90×269 and b) 130×389. The color bars shows the histogram of the coherence of the random matrices for 10000 realizations. The coherence of the deterministic matrices with matching sizes are 0.271 and 0.217 in cases of a) and b), respectively (the vertical lines).

IV. Simulation Results

In this section, we numerically compare the performance of the proposed matrices with some of the existing matrix structures in recovering sparse vectors from compressed measurements. In our comparison, we include complex-valued random Gaussian Toeplitz matrices, the matrices designed in [24], and a Toeplitz matrix found by search. For the latter, we uniformly and independently draw $m + n - 1$ elements from $\{e^{j\frac{2\pi}{n}a}\}_{a=0}^{n-1}$ and form an $m \times n$ Toeplitz matrix (using these values as the elements in the first row and column); we repeat this procedure $100,000$ times and select the matrix with the minimum mutual coherence value. We also generate k-sparse vectors $\mathbf{x}_{n\times 1}$ by first determining the support; indeed, we randomly choose the support with a uniform probability among $\binom{n}{k}$ possibilities. Then, we set the non-zero values by drawing k independent realizations from a standard normal distribution.

In Figure 2, we plot the success rate (percentage of perfect recovery) of the considered matrices in terms of the sparsity level k. A recovery is called successful whenever $\|\mathbf{x} - \hat{\mathbf{x}}\|_2/\|\mathbf{x}\|_2 \leq 10^{-5}$ holds for the reconstructed vector $\hat{\mathbf{x}}$, where $\mathbf{x}$ stands for the original sparse vector. The size of all the matrices in this experiment is 38×113. Note that 113 is a prime and $\gamma = \frac{n}{m} \approx 3$; we have also used $f(x) = \frac{x^3}{339}$ in the design of our matrices. The exact Basis Pursuit (BP) method is used as the recovery method in this experiment. Furthermore, the percentages are found by observing the reconstruction performance over $10,000$ different realizations of the random input vectors $\mathbf{x}$.

Figure 3 depicts the reconstruction performance in the noisy setting where the Orthogonal Matching Pursuit (OMP) method is used for recovery. The matrices are of size 14×41 (or 54 pilot symbols in a communication system) and the sparsity level of the input vector $\mathbf{x}$ is fixed at 2. The curves in this figure depict the MSE of the reconstruction over $1,000,000$ realizations of the input 2-sparse vector.

The curves in both Figures 2 and 3 indicate that the performance of the proposed matrices match the one for random matrices and the structure in [24]. However, there is less degrees of freedom in selecting the elements in the proposed method; besides, the choice of the elements in the proposed method (equidistant points on the unit circle) makes the resulting pilot sequence more suitable for practical implementations.

Finally, we have numerically evaluated the coherence of 10,000 randomly generated Gaussian Toeplitz matrices and plotted their distribution in Figure 4 for matrices with size 90×269 and 130×389. The green vertical line in these plots indicate the coherence of the deterministic matrices constructed with our approach (the same size). These figures reveal that the coherence of the constructed matrices are clearly smaller than typical Gaussian Toeplitz matrices.

V. Conclusion

In this paper, we proposed a deterministic design for Toeplitz matrices with small coherence. The elements of the matrices are chosen from a set of equidistant points on the unit circle in the complex plane. We provided a closed-form upper-bound for the coherence of the designed matrices using tools from analytic number theory. The vanishing nature of this bound enables us to design matrices with arbitrarily small coherence values. Besides, the upper-bound differs from the universal Welch bound by an $\mathcal{O}\big(\log(n)\big)$ factor for fixed aspect ratio of the matrix.

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