Chapter 5

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1- Plate Bending (Review of theory)

Linear elastic small deflections

Assumptions:
1. Plate is thin (h<<L where L=typical length)
2. Normals perpendicular to the mid-surface remain normal to the deflected mid-surface.
3. Small deflections (normal to the plate) so that the mid-surface remains unstretched

Also note that $\tau_{zz} \ll \tau_{xx}$ and $\tau_{yy}$

Deflections:
Displacements of B to B'
$u = -z \sin \alpha = -z \frac{\partial w}{\partial x}$ similarly, $v = -z \frac{\partial w}{\partial y}$

Strains:

$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -z \frac{\partial^2 w}{\partial x^2}$

$\varepsilon_{yy} = \frac{\partial v}{\partial y} = -z \frac{\partial^2 w}{\partial y^2}$

$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z \frac{\partial^2 w}{\partial x \partial y}$
1.1- Stresses (Isotropic Case)

Plane stress in xy plane

\[ \tau_{xx} = \frac{E}{1-\nu^2} \left( \varepsilon_{xx} + \nu \varepsilon_{yy} \right) = -\frac{Ez}{1-\nu^2} \left( \frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2} \right) \]

\[ \tau_{yy} = \frac{E}{1-\nu^2} \left( \nu \varepsilon_{xx} + \varepsilon_{yy} \right) = -\frac{Ez}{1-\nu^2} \left( \nu \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) \]

\[ \tau_{xy} = \frac{E}{2(1+\nu)} \gamma_{xy} = -\frac{Ez}{1+\nu} \frac{\partial^2 w}{\partial x \partial y} \]

Section properties (resultants stresses as bending and twisting moments)

\[ M_{xx} = \int_{-\frac{h}{2}}^{\frac{h}{2}} z \tau_{xx} \, dz = -\frac{Eh^3}{12(1-\nu^2)} \left( w_{xx} + \nu w_{yy} \right) = -D(w_{xx} + \nu w_{yy}) \]

similarly,

\[ M_{yy} = -\frac{Eh^3}{12(1-\nu^2)} \left( (\nu w_{xx} + w_{yy}) = -D(\nu w_{xx} + w_{yy}) \right) \]

\[ M_{xy} = -\int_{-\frac{h}{2}}^{\frac{h}{2}} z \tau_{xy} \, dz = \frac{Eh^3}{12(1+\nu)} w_{xy} = D(1+\nu)w_{xy} = M_{yx} \]

Where D is the bending rigidity and M_x, M_y are bending moments per unit length about x and y axes, respectively. M_{xy} and M_{yx} are the twisting moments per unit length about x and y axes, respectively.

1.2- Shear Forces

\[ Q_x = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{xz} \, dz \quad Q_y = \int_{-\frac{h}{2}}^{\frac{h}{2}} \tau_{yz} \, dz \]

Where Q_x and Q_y are the shear forces per unit length on edges whose normals are x and y axes, respectively.
1.3- Equilibrium Equations

$q(x,y)$ is the transverse load per unit area:

\[ \sum F_z = -Q_x dy - Q_y dx + (Q_x + \frac{\partial Q_x}{\partial x} dx) dy + (Q_y + \frac{\partial Q_y}{\partial y} dy) dz + q(x, y) dx dy = 0 \]

cancelling terms gives:

\[ \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q(x, y) = 0 \]

Moment Equilibrium ignoring the second order effects:

\[ \sum M_x = M_{yy} dx - (M_{yy} + \frac{\partial M_{xx}}{\partial x} dx) dy + Q_x dy dx + M_{xy} dy + (M_{xy} + \frac{\partial M_{xy}}{\partial x} dx) dy = 0 \]

or:

\[ \frac{\partial M_{yy}}{\partial x} + \frac{\partial M_{xy}}{\partial x} + Q_y = 0 \]

\[ \sum M_y = -M_{xx} dy + (M_{xx} + \frac{\partial M_{xx}}{\partial x} dx) dy - Q_x dy dx + M_{xy} dx - (M_{xy} + \frac{\partial M_{xy}}{\partial x} dy) dx = 0 \]

or:

\[ \frac{\partial M_{xx}}{\partial x} - \frac{\partial M_{xy}}{\partial y} = 0 \]

Eliminating $Q_x$ and $Q_y$ and substituting inequations:

\[ \frac{\partial Q_y}{\partial y} = \frac{\partial^2 M_{yy}}{\partial y^2} - \frac{\partial^2 M_{xy}}{\partial x \partial y} \]
\[ \frac{\partial Q_x}{\partial x} = \frac{\partial^2 M_{xx}}{\partial x^2} - \frac{\partial^2 M_{xy}}{\partial y \partial x} \]

Then:

\[ \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q(x, y) = \frac{\partial^2 M_{xx}}{\partial x^2} - 2 \frac{\partial^2 M_{xy}}{\partial y \partial x} + \frac{\partial^2 M_{yy}}{\partial y^2} + q(x, y) = 0 \]

Substitute for $M_{xx}$, $M_{yy}$ and $M_{xy}$:

\[ -D(w_{xxxx} + \nu w_{xxyy}) - 2D(1-\nu)w_{xxyy} - D(\nu w_{xxyy} + w_{yyxy}) + q(x, y) = 0 \]

\[ D(w_{xxxx} + 2w_{xxyy} + w_{yyxy}) = q(x, y) \]

\[ DV^4 w = q(x, y) \quad \text{where} \quad V^4 = \frac{\partial^4}{\partial x^4} + 2 \frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4} \]

is called biharmonic operator
Also encountered in many other engineering problems, e.g. very viscous or creeping incompressible flow, stress analysis using Airy's stress function $\phi$ or $\nabla^4 \phi = 0$ (compatibility equation)

### 1.4- Strain Energy

This is analogous to beam bending.

Energy stored = Elastic work done by moments

$$dU = -\frac{1}{2} (M_{xx} \, dy) w_{xx} \, dx - \frac{1}{2} (M_{yy} \, dx) w_{yy} \, dy + \frac{1}{2} (M_{xy} \, dy) w_{xy} \, dx + \frac{1}{2} (M_{yx} \, dx) w_{yx} \, dy$$

*note:* $M_{xy} = M_{yx}$  $w_{xy} = w_{yx}$

$$U = \frac{1}{2} \int \int_A \left( -M_{xx} w_{xx} - M_{yy} w_{yy} + 2M_{xy} w_{xy} \right) dx \, dy$$

*Substituting for* $M_{xx}$ $M_{yy}$ $M_{xy}$ *from above equations* :

$$U = \frac{1}{2} \int \int_A \left[ D(w_{xx} + \nu w_{yy}) w_{xx} + D(\nu w_{xx} + w_{yy}) + 2D(1 - \nu) w_{xy}^2 \right] dx \, dy$$

$$U = \frac{D}{2} \int \int_A \left[ w_{xx}^2 + w_{yy}^2 + 2\nu w_{xx} w_{yy} + 2(1 - \nu) w_{xy}^2 \right] dx \, dy$$
1.5- Boundary Conditions

a) Simply Supported Edge

1) along x=0 and x=a edges
   \( w=0 \) and \( M_{xx}=0 \)
   But if \( w=0 \) along y on \( x=0 \) then \( w_y=w_{yy}=0 \)
   Therefore \( w_y=w_{yy}=0 \)
   \( M_{xx}=-D(w_{xx}+\nu w_{yy})=0 \) ; \( w_{xx}=0 \)

2) along y=0 and y=b
   \( w=0 \) and \( M_{yy}=0 \) or \( w_{yy}=0 \)
   for \( w=0 \) along \( y=0 \) and \( y=b \), \( w_x=0 \) and \( w_{xx}=0 \)

b) Clamped or built-in edge

1) along x=0 and x=a
   \( w=0 \) and \( \frac{\partial w}{\partial x}=w_x=0 \)

2) along y=0 and y=b
   \( w=0 \) and \( \frac{\partial w}{\partial y}=w_y=0 \)

c) Free edge

There are no restrictions on displacements- no edge forces.
One is tempted to say that along x=a
\( Q_x=0, M_{xx}=0 \) and \( M_{yy}=0 \)
This is wrong because only two independent conditions are allowed.
According to kirchhoff, should use only two, i.e. \( M_{xx}=0 \) and \( T_{xx}=0 \)
(effective shear force) where, \( T_x=Q_x-\frac{\partial M_{xy}}{\partial y}=0 \) is effective shear force
along the edge.

1.6- Potential Energy

Potential energy of the plate bent to transverse load \( q(x,y) \) is given by:
\[
\pi = U - W \\
\pi = \frac{D}{2} \iint_A \left[w_{xx}^2 + w_{yy}^2 + 2w_{xx}w_{yy} + 2(1-\nu)w_{xy}^2\right]dxdy - \iint_A qwdx dy
\]
2- Rectangular Plate Bending Elements

2.1- Non-conforming Rectangular finite element use deflection and two slopes as generalized displacements at each node i.e. use \( w, w_x, w_y \) as nodal degrees of freedom. This element has wide use application and performs very well.

\[
w(x,y) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 y + a_6 x y + a_7 x^3 + a_8 y^2 + a_9 x y^2 + a_{10} y^3 + a_{11} x^3 y + a_{12} x y^3
\]

i.e. complete cubic plus two terms \((x^3 y\) and \(xy^3\)) polynomial of above equation satisfies the homogeneous plate equation \(D\nabla^4 w=0\), this fact is of little significance in the finite element formulation.

**Rigid Body Modes**

\(W=\text{constant} \) (translation) and also need two rotations. Three rigid body modes required are included through \(a_1+a_2 x+a_3 y\) in the polynomial of above equation.

**Constant Strain**

In plate bending, the strains are curvatures and twist i.e. \(w_{xx}, w_{xy}\) and \(w_{xy}\). This is provided by the second degree terms i.e. \(a_4 x^2+a_5 x y+a_6 y^2\) which are also included.

**Continuity**

The polynomial in above equation has been chosen carefully and for a very good reason we included \(x^3 y\) and \(xy^3\) terms instead of \(x^4\) and \(y^4\). For constant \(y\), \(w(x,y)\) is cubic in \(x\) and vice-versa. Now a cubic polynomial in one dimension contains four independent parameters or coefficient which may be specified uniquely by two conditions at each end point (i.e. the end
nodes. This particular feature leads to ensuring displacement continuity between adjacent elements. We will look into it in more detail later.

**Generalized Displacement**
The element formulation begins by first solving for generalized displacements from displacement function. This yields the following matrix equation:

\[
\begin{bmatrix}
  w_1 \\
  w_{x1} \\
  w_{y1} \\
  w_2 \\
  w_{x2} \\
  w_{y2} \\
  w_3 \\
  w_{x3} \\
  w_{y3} \\
  w_4 \\
  w_{x4} \\
  w_{y4}
\end{bmatrix} =
\begin{bmatrix}
  1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  1 & a & 0 & a^2 & 0 & 0 & a^3 & 0 & 0 & 0 & 0 & 0 \\
  0 & 1 & 0 & za & 0 & 0 & 3a^2 & 0 & 0 & 0 & 0 & 0 \\
  0 & 0 & 1 & 0 & a & 0 & 0 & a^2 & 0 & 0 & a^3 & 0 \\
  1 & a & b & a^2 & ab & b^2 & a^3 & a^2b & ab^2 & b^3 & ab^3 & ab^3 \\
  0 & 1 & 0 & za & b & 0 & 3a^2 & zab & b^2 & 0 & 3a^{2b} & b^3 \\
  0 & 0 & 1 & 0 & a & zb & 0 & a^2 & 2ab & 3b^2 & a^3 & 3ab \\
  1 & 0 & b & 0 & 0 & b^2 & 0 & 0 & 0 & b^3 & 0 & 0 \\
  0 & 1 & 0 & 0 & b & 0 & 0 & 0 & b^2 & 0 & 0 & b^3 \\
  0 & 0 & 1 & 0 & 0 & 2b & 0 & 0 & 0 & 3b^2 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
  a_1 \\
  a_2 \\
  a_3 \\
  a_4 \\
  a_5 \\
  a_6 \\
  a_7 \\
  a_8 \\
  a_9 \\
  a_{10} \\
  a_{11} \\
  a_{12}
\end{bmatrix}
\]

or \( \{w\} = [T] \{A\} \)

where \( \{A\} = \langle a_1, a_2, \ldots, a_{12} \rangle \)

The matrix \([T]\) in equations above can be inverted in order to solve for \( \{A\} \) as functions of the generalized displacements. Once \( a_1, a_2 \) etc. are substituted back into displacement function, we obtain deflection in terms of the generalized displacements \( w_1, w_{x1}, w_{y1}, \ldots \) etc. as:
Equation 1

\[
\begin{bmatrix}
    a(1-\xi)^2 \xi(1-\eta) \\
    b(1-\xi)\eta(1-\eta)^2 \\
    a\left[1-\xi\eta-(3-2\xi)\xi^2(1-\eta)-(1-\xi)(3-2\eta)\eta^2\right] \\
    -a(1-\xi)^2\eta(1-\eta) \\
    b(1-\eta)^2\xi\eta \\
    a\left[(3-2\xi)\xi^2(1-\eta)+\xi\eta(1-\eta)(1-2\eta)\right] \\
    -a(1-\xi)\xi^2\eta \\
    -b(1-\eta)\eta^2\eta \\
    a\left[(3-2\xi)\xi^2\eta-\xi\eta(1-\eta)(1-2\eta)\right] \\
    a(1-\xi)^2\eta\xi \\
    -b(1-\xi)(1-\eta)\eta^2 \\
    a\left[(1-\xi)(3-2\eta)\eta^2+\xi(1-\xi)(1-2\eta)\eta\right]
\end{bmatrix}
\]

\[w(\xi, \eta) = \{W\}^T\]

where \{w\} is the column vector of nondimensional generalized displacements:

\[\{w\}^T = [w_{x1} \quad w_{y1} \quad w_{1}/a \quad w_{x2} \quad w_{y2} \quad w_{2}/a \quad w_{x3} \quad \ldots \quad w_{x4} \quad \ldots]\]

\[\xi = \frac{x}{a} \quad \eta = \frac{y}{b}\] are non-dimensional coordinates.

Each row in equation of \(w(\xi, \eta)\) represent a shape function or interpolation function \(N_i\). We may write it as:

\[w(\xi, \eta) = \sum_{i=1}^{12} N_i(\xi, \eta) \delta_i \] where \(\delta_1 = w_{x1} \quad \delta_2 = w_{y1} \quad \delta_3 = \frac{w_1}{a} \quad \text{etc.}\]
Now that we have the displacement distribution within the element defined by displacement equation, we may ask what continuity will be provided by the element. Consider for example, the joining of two elements A and B together as illustrated in the Figure.

\[ w = w_1 H_{01}(x) + w_2 H_{02}(x) + w_{x1} H_{11}(x) + w_{x2} H_{12}(x) \]

For element A, the displacement along edge 3-4 is obtained by putting \( \eta = 1 \) in equation 1.

For element B, the displacement along edge 1-2 (\( \eta = 0 \)) is:

\[ w_B(\xi) = w_1 (1 - 3\xi^2 + 2\xi^3) + aw_{x1}\xi(1 - \xi)^2 + w_3 (3 - 2\xi)\xi^2 - aw_{x3}\xi^2 (1 - \xi) \]

It may be seen that the same function of \( \xi \) occur in these two equations. Therefore, if \( w_1, w_{x1}, w_2, w_{x2} \) of element B are equated to \( w_4, w_{x4}, w_3, w_{x3} \), respectively of element A, \( w \) will have to be continuous between the elements. Similar arguments are easily made for edges parallel to the y-axis.

What about slopes normal to the edges?
There is no continuity of slopes normal to the element edges. This can be shown by taking derivatives of \( w \) with respect to \( \eta \) and substituting \( \eta = 1 \) for element A and \( \eta = 0 \) for element B.

It will be found that terms along those edges are cubic and there is no way we can make normal slopes continuous by equating \( w_{y3} \) and \( w_{y4} \) of element A to \( w_{y2} \) and \( w_{y3} \) of element B, respectively.

**Therefore, the element is called non-conforming**
2.1.1- Stiffness Matrix
Calculate the stiffness matrix for the non-conforming plate bending element by substituting equation 1 into expression for strain energy. After, carrying out integration over the area of the element, we obtain the quadratic form in term of generalized displacements (as expected) for strain energy:

\[ U_e = \frac{1}{2} \{W\}^T [K] \{W\} \]

Here, \([K]\) is the 12 by 12 stiffness matrix for the element and is given in the following page.

Note, this matrix has been derived for \(\{W\}\) as given in equation of \(\delta_3 = w_1/a, \delta_6 = w_2/a, \delta_9 = w_3/a\) and \(\delta_{12} = w_4/a\) i.e. in dimensionless displacements. To allow \(w\)'s to take on dimensionless displacements, the 3\(^{rd}\), 6\(^{th}\), 9\(^{th}\) and 12\(^{th}\) row should be multiplied by \(a\) again. Further, if the degrees of freedom are desired to be arranged as:

\[ \begin{bmatrix} \{\tilde{w}\}^T \\ \end{bmatrix} = \begin{bmatrix} w_1 & w_{x1} & w_{y1} & w_2 & w_{x2} & w_{y2} & w_3 & w_{x3} & w_{y3} & w_4 & w_{x4} & w_{y4} \end{bmatrix} \]

Then the rows and columns should be rearranged accordingly, e.g. 1\(^{st}\) and 2\(^{nd}\) rows should be moved into second and 3\(^{rd}\) rows. And 3\(^{rd}\) row should be placed into 1\(^{st}\) row, etc, etc., etc.
The stiffness matrix for the plate bending element may also be derived following the alternative method we discussed for beam element.

**Figure 1** Stiffness Matrix for 12 parameter Rectangular Element (non-conforming)

\[
\begin{align*}
\frac{2}{3m} + \frac{2(1-\nu)m}{15} & \quad \frac{2m + 2(1-\nu)}{3m} + \frac{2(1-\nu)m}{10} \\
\frac{7}{m} + \frac{10m}{10} & \quad \frac{2m + 2(1-\nu)m}{m} + \frac{15m}{10} + \frac{(7-2\nu)m}{5} \\
\frac{1}{m} & \quad \frac{1}{m} + \frac{(1-\nu)m}{10} \\
\frac{1}{3m} & \quad \frac{1}{3m} + \frac{15m}{10} \\
\frac{1}{m} & \quad \frac{2(1-\nu)m}{10} + \frac{10m}{10} \\
\frac{m}{2} & \quad \frac{m(1-\nu)}{2} + \frac{10m}{10} \\
\frac{3m}{2} & \quad \frac{(1-\nu)m}{10} + \frac{3m}{10} \\
\frac{1}{m} & \quad \frac{2m}{10} + \frac{10m}{10} \\
\frac{3m}{2} & \quad \frac{(1-\nu)m}{10} + \frac{3m}{10} \\
\frac{1}{m} & \quad \frac{3m}{10} + \frac{6m}{10} \\
\frac{2m}{10} & \quad \frac{1}{m} + \frac{(1-\nu)m}{10} \\
\frac{6m}{2} & \quad \frac{3m}{10} + \frac{6m}{10} \\
\frac{1}{m} & \quad \frac{2m}{10} + \frac{10m}{10} \\
\frac{3m}{2} & \quad \frac{(1-\nu)m}{10} + \frac{3m}{10} \\
\frac{1}{m} & \quad \frac{3m}{10} + \frac{6m}{10} \\
\frac{2m}{10} & \quad \frac{1}{m} + \frac{(1-\nu)m}{10} \\
\frac{6m}{2} & \quad \frac{3m}{10} + \frac{6m}{10} \\
\end{align*}
\]

\[\text{CONTINUE}\]

\[
\begin{align*}
\frac{2m + 2(1-\nu)}{3m} + \frac{15m}{10} & \quad \frac{2m + 2(1-\nu)m}{3m} + \frac{2(1-\nu)m}{10} \\
\frac{2m + 2(1-\nu)m}{m} + \frac{15m}{10} & \quad \frac{2m + 2(1-\nu)m}{m} + \frac{(7-2\nu)m}{5} \\
\frac{1}{m} + \frac{(1-\nu)m}{10} & \quad \frac{2m}{10} + \frac{10m}{10} \\
\frac{1}{m} + \frac{(1-\nu)m}{10} & \quad \frac{3m}{10} + \frac{6m}{10} \\
\frac{2m}{10} + \frac{10m}{10} & \quad \frac{3m}{10} + \frac{6m}{10} \\
\frac{1}{m} + \frac{(1-\nu)m}{10} & \quad \frac{3m}{10} + \frac{6m}{10} \\
\frac{2m}{10} + \frac{10m}{10} & \quad \frac{3m}{10} + \frac{6m}{10} \\
\end{align*}
\]
2.1.2- Consistent Load Vector
Assume uniform pressure \( q_0 \). Recall from equation of potential energy \( \pi \), the work done \( W \) is given by:

\[
W = \iint_{A_e} q_0 w dx dy = \{p\}^T \{W\}
\]

where \( A_e \) is the element area, and \( \{w\} \) is given by equation 1, when equation 1 is substituted into above equation and integrating, the load vector for the element in dimensional form:

\[
\{p\}^T = ab q_0 \left[ \frac{a}{24} \quad \frac{b}{24} \quad \frac{1}{4} \quad -\frac{a}{24} \quad \frac{b}{24} \quad \frac{1}{4} \quad -\frac{a}{24} \quad \frac{b}{24} \quad \frac{1}{4} \quad \frac{a}{24} \quad \frac{b}{24} \quad \frac{1}{4} \right]
\]

When nonconforming elements are used to obtain an approximate solution for some loading, generally we use reasonably large number of elements and can obtain reasonable answer by using lumped load i.e. \( q_0 ab/4 \) at each corner node. However, for very refined elements, we must use consistent load vector since much fewer elements are used. In such cases, we may be introducing an undesirable error through lumped loads.

2.1.3- Stresses
Bending and twisting moments
Define:

\[
\{\varepsilon\} = \begin{bmatrix} w_{xx} \\ w_{yy} \\ w_{xy} \end{bmatrix} \quad \text{strain and curvature}
\]

\[
\{\tau\} = \begin{bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{bmatrix} \quad \text{stresses and moments}
\]

\[
\{\tau\} = [D] \{\varepsilon\}
\]

\[
[D] = \begin{bmatrix} -1 & -\nu & 0 \\ -\nu & -1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \quad D = \frac{Eh^3}{12(1-\nu^2)}
\]

From the shape functions in equation 1, we can obtain \( w_{xx}, w_{yy} \) and \( w_{xy} \). Further, these can be evaluated at various points \( (x_i, y_i) \) or \( (\xi_i, \eta_i) \) and hence \( M_{xx}, M_{yy} \) and \( M_{xy} \) can be evaluated at specified points.
We must know \( \{w\} \) before we can compute \( \{\tau\} \).
2.1.4- Boundary Conditions (Kinematic)
Along AB and AD, the plate is simply supported,
AB: \( w=0 \) and \( w_x=0 \)
AD: \( w=0 \) and \( w_y=0 \)
Along cd, the plate is clamped \( w=0 \) and \( w_x=0 \) and \( w_y=0 \)
Nothing specified on free boundary.

2.2- Note on Continuity
Both \( w \) and its normal derivatives or normal slope must be uniquely determined by values along an interface or edge of an element in order to ensure, \( C_1 \) continuity.
Consider edge 3-4 of the rectangular element shown.

Here, \( w_n = w_y \), the normal slope. It is desired that \( w \) and \( w_y \) be uniquely determined by the values of \( w \) and \( w_x \) and \( w_y \) at the nodes lying along edge 3-4.
\[ w = a_1 + a_2 x + a_3 x^2 + ... \]
\[ \frac{\partial w}{\partial y} = b_1 + b_2 x + b_3 x^2 + ... \]

along edge 3-4 with the number of constants \(a_i\) and \(b_i\) in each expression just sufficient to determine the expressions by nodal parameters or dof associated with the line.

- With \(w\) and \(w_x\) as nodal dof at each node i.e. two nodes, we can allow only four \(a_i\) (\(a_1, a_2, a_3\) and \(a_4\)) or at most cubic variation in \(x\) along 3-4.

Similarly only a linear variation can be allowed i.e. two terms (\(b_1\) and \(b_2\)) for \(w_y\). In the same manner, \(w_x\) can be made continuous along the edge parallel to the \(y\) axis (\(w_x = c_1 + c_2 y\) along 2-3)

Therefore, along edge 3-4
- \(w_y\) depends on nodal dof of edge 3-4 and along edge 2-3
- \(w_x\) depends on nodal dof of edge 2-3

Differentiate \(w_y\) along edge 3-4 wrt \(x\) \(\rightarrow W_{xy}\)
Differentiate \(w_x\) along edge 2-3 wrt \(y\) \(\rightarrow W_{yx}\)
The first depends on nodal dof of edge 3-4 and the second depends on nodal dof of edge 2-3.

At common node 3: \( w_{xy} \big|_{3-4} \neq w_{yx} \big|_{2-3} \)
Because of arbitrary nodal dof at nodes 2 and 4 where as for continuous functions \(w_{xy} = w_{yx}\) (\(b_2 \neq c_2\))

Assertion: It is therefore, impossible to use simple polynomials for shape functions ensuring full compatibility when only \(w\) and its slopes are used as dof at nodes.

If any functions satisfying compatibility are found with the three nodal variables, they must be such that at corner nodes they are not continuously differentiable and the cross derivative is not unique.

So far we have applied the argument to a rectangular element, we can extend this for any two arbitrary directions of interfaces or common edges at node 3 (triangular or quadrilaterals).

Unfortunately, this extension requires continuity of cross derivatives in several sets of orthogonal directions, which in fact implies a specification of all second derivatives at a node. This leads to excessive continuity that violates the continuity requirement of potential energy theorem, also the physical requirements. If the plate stiffness varies abruptly from element to element then equality of moments normal to the interface cannot be maintained.
3- Elements for C¹ Problems

Constructing two-dimensional elements that can be used for problems requiring continuity of the field variable $\phi$ as well as its normal derivative $\phi_n$ along element boundaries is far more difficult than constructing elements for C⁰ continuity alone. To preserve C¹ continuity, we must be sure that $\phi$ and $\phi_n$ are uniquely specified along the element boundaries by the degrees of freedom assigned to the nodes along a particular boundary. The difficulties arise from the following principles:

1. The interpolation functions must contain at least some cubic terms because the three nodal values $\phi$, $\phi_x$, and $\phi_y$ must be specified at each corner of the element.

2. For non-rectangular elements, C¹ continuity requires the specification of at least the six nodal values $\phi$, $\phi_x$, $\phi_y$, $\phi_{xx}$, $\phi_{yy}$, and $\phi_{xy}$ at the corner nodes. For a rectangular element with sides parallel to the global axes, we need to specify at the corners nodes only $\phi$, $\phi_x$, $\phi_y$ and $\phi_{xy}$.

It is sometimes very convenient to specify only $\phi$, $\phi_x$, and $\phi_y$ at corners, but when this is done, it is impossible to have continuous second derivatives at the corner nodes. In general, the cross derivative $\phi_{xy}$ will be directionally dependent and hence, nonunique at intersections of the sides of the element. Analysts first began to encounter difficulties in formulating elements for C¹ problems when they attempted to apply FE techniques to plate-bending problems. For such problems, the displacement of the mid plane of the plate for Kirchhoff plate bending theory is the field variable in each element, and interelement continuity of the displacement and its slope is a desirable physical requirement. Also, since the functional for plate bending involves second order derivatives, continuity of slope at element interface is a mathematical requirement because it ensures convergence as element size is reduced. For these reasons, analysts have labored to find elements giving continuity of slope and value.

Rectangular Elements

Whereas triangulars are the simplest element shapes to establish C⁰ continuity in two dimensions, rectangles with sides parallel to the global axes are the simplest element shapes of C¹ continuity in 2 dimensions. The reason is that the element boundaries meet at right angles, and imposing continuity of the cross derivatives $\phi_{xy}$ at the corners guarantees continuity of the derivatives that otherwise might be nonunique. A four-node rectangle with $\phi$, $\phi_x$, $\phi_y$ and $\phi_{xy}$ specified at the corner nodes assigns a 16-dof element.
4- Triangular Elements
For C¹ continuity, by assigning 21 dof to element, we can make a complete quintic polynomial to represent the field variable φ. When φ and all first and second derivatives are specified at the corner nodes. There are only 18 dof, so 3 more are needed to specify the 21-term quintic polynomial. The 3 dof are obtained by specifying the normal derivatives φₙ at the midside nodes. This element guarantees continuity of φ along element boundaries because, along a boundary where s is the linear coordinate, φ varies in s as a quintic function, which is uniquely determined by six nodal values, normal, φ, φₛ, and φₚₚ at each end node. Slope continuity is also assured because the normal slope along each edge varies as a quartic function which is uniquely determined by five nodal variables, namely φₙ and φₙₙ at each end node plus φₙ at the midside node. The presence of midside nodes is undesirable because they require special bookkeeping in the coding process, and they increase the bandwidth of the final matrix. Apparantly, C¹ continuity is not always a necessary condition for convergence in C¹ problems. Experience has indicated that convergence is more dependent on the completeness than on the compatibility property of the element. The following table shows a sample of incompatible elements. Any of these elements can be used in the solution of continuum problems involving functionals containing up to second-order derivatives. The analysts may ask, which element should I use to solve my problem? Unfortunately, no general answer can be given because the answer is problem dependent.
### Table 3.5: Some Incompatible Elements for C¹ Problems

<table>
<thead>
<tr>
<th>Element</th>
<th>Order of Polynomial</th>
<th>Degrees of Freedom per Element</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Δ</td>
<td>4 specified at ( a )</td>
<td>6</td>
<td>Simplest possible plate-bending element.</td>
</tr>
<tr>
<td>( \Delta )</td>
<td>( b ) specified at ( a )</td>
<td>4</td>
<td>Shows correct answers comparable to those for more complex triangular elements.</td>
</tr>
<tr>
<td>( \Delta )</td>
<td>( c ) specified at ( a )</td>
<td>4</td>
<td>Geometric incompatibility is not practical.</td>
</tr>
<tr>
<td>( \Delta )</td>
<td>( d ) specified at ( a )</td>
<td>4</td>
<td>Area coordinates can be used to determine a consistent solution and then a solid ( [G] ).</td>
</tr>
<tr>
<td>( \Delta )</td>
<td>( e ) specified at ( a )</td>
<td>4</td>
<td>Gives poor results.</td>
</tr>
<tr>
<td>( \Delta )</td>
<td>( f ) specified at ( a )</td>
<td>4</td>
<td>Incomplete cubic: ( y ) or ( y ) term omitted.</td>
</tr>
<tr>
<td>( \Delta )</td>
<td>( g ) specified at ( a )</td>
<td>4</td>
<td>Complete cubic.</td>
</tr>
<tr>
<td>( \Delta )</td>
<td>( h ) specified at ( a )</td>
<td>4</td>
<td>Incomplete quartic: ( x ), ( y ), ( f ), and ( g ) terms omitted.</td>
</tr>
<tr>
<td>( \Delta )</td>
<td>( i ) specified at ( a )</td>
<td>4</td>
<td>Complete quartic.</td>
</tr>
<tr>
<td>( \Delta )</td>
<td>( j ) specified at ( a )</td>
<td>4</td>
<td>Incomplete quartic.</td>
</tr>
<tr>
<td>( \Delta )</td>
<td>( k ) specified at ( a )</td>
<td>4</td>
<td>Complete quartic.</td>
</tr>
</tbody>
</table>

---

Some Incompatible Elements for C¹ Problems
5- Nonconforming Triangular Plate Bending Elements

- we need an element of more general shape
- Triangular elements fit curved edges more appropriately than the rectangular elements
- Again consider local coordinates $\xi$ and $\eta$. We shall use transformation matrix to go back to x-y system.
- Consider $w$, $w_\xi$, $w_\eta$ as the dof at each node.
- A cubic has 10 generalized parameters:
  \[ w = a_1 + a_2 \xi + a_3 \eta + a_4 \xi^2 + a_5 \xi \eta + a_6 \eta^2 + a_7 \xi^3 + a_8 \xi^2 \eta + a_9 \xi \eta^2 + a_{10} \eta^3 \]
- for the element we have 9 dof but 10 generalized parameters in above equation. Therefore, must delete one of $a_i$ ($i=1,2,\ldots,10$) or add a dof.

Possibilities:

a) use $w$ at centroid as an extra dof
   - this element does not work sometimes and also exhibit poor convergence
   - Certain orientations may lead to less than a cubic along one of the edges and violates $w$ continuity requirement
b) Throwaway one term- say $a_5=0$
   This violates constant curvature or constant strain energy requirement
   i.e. will not work since $w_\xi \eta=$constant not present

c) combine two terms, i.e. equate $a_8=a_9$
   - we get $a_8(\xi^2 \eta + \xi \eta^2)$ which keeps some symmetry.
   - in general, ruins isotrophy of the polynomial so we expect orientation problems.
Recall:
\[
\begin{align*}
\{\vec{w}\} = [T]_{10 \times 10} \{A\}_{10 \times 1} \\
\{A\}^T = [a_1 \ a_2 \ \ldots \ \ldots \ \ldots \ a_{10}] \\
\{\vec{w}\} = [w_1 \ w_{\xi 1} \ w_{\eta 1} \ \ldots \ \ldots \ w_{\eta 3}]
\end{align*}
\]

[T] matrix becomes singular sometimes. This happens when two of edges are parallel to the global axes (x,y).

d) Use area coordinates (Zienkiewics, 9dof triangular element)
- explain lack of full cubic because of only 9 dof. Let us look at (c) in more detail. [T] matrix

\[
[T] = \begin{bmatrix}
1 & -b & 0 & b^2 & \ldots & \ldots & \ldots & a_1 \\
0 & 1 & 0 & -2b & \ldots & \ldots & \ldots & a_2 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & a_8 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & a_9 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & a_{10} \\
1 & -1 & & & & & & \\
\end{bmatrix}
\]

The last equation is a constraint equation i.e. \(a_8 - a_9 = 0\)

This is a more elegant way of doing it.

\[\text{Det}[T] = c^5(a+b)^5(c+b-a)\]

If \(a = c + b\) or \(c + b - a = 0\) then \(\text{det}[T] = 0\) and we cannot invert \([T]\) to formulate the element.

If this situation is avoided then:
\( \{A\} = [T]^{-1} \left\{ \{\bar{w}\} \right\} \\
\)

This can be written as:
\( \{A\}_{10 \times 1} = [T_2]_{10 \times 9} \{\bar{w}\}_{9 \times 1} \)

\( [T_2] \) contains first 9 columns of \([T]^{-1}\)

then
\( w(\xi, \eta) = \left[ 1 \quad \xi \quad \eta \quad \xi^2 \quad \xi \eta \quad \eta^2 \quad \xi^3 \quad \xi^2 \eta \quad \xi \eta^2 \quad \eta^3 \right] [T_2] \{\bar{w}\} \)

\( w(\xi, \eta) = [p]^T_{1 \times 10} [T_2]_{10 \times 9} \{\bar{w}\}_{9 \times 1} \)

to transform from \( \{\bar{w}\} \) to \( \{w\} = [w_1 \ w_{x_1} \ w_{y_1} \ldots \ldots \ldots \ w_{y_3}] \)

\( \{\bar{w}\} = [R] \{w\} \)

where \( [R] \)

\( [R] = \begin{bmatrix} [R_1] & [0] & [0] \\ [0] & [R_1] & [0] \\ [0] & [0] & [R_1] \end{bmatrix} \)

\( [R_1] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & -\sin \theta & \cos \theta \end{bmatrix} \)

where \( \theta \) is the angle between \((\xi, \eta)\) and \((x, y)\) axes.

---

6- Conforming Rectangular Element (16 dof)

Nodal degrees of freedom at each node are \( w, w_x, w_y \) and \( w_{xy} \). Extra dof \( w_{xy} \) is permissible as it does not involve excessive continuity. Thus, we have 16 dof per element and a polynomial expression involving 16 constants could be used. We retain terms which do not produce a higher order variation of \( w \) or its normal slope than cubic along the sides. There are many alternatives as far as choosing the polynomial is concerned. But some of these alternatives may not produce invertible \([T]\) matrix.
An alternative is to use Hermite polynomials. These are one-dimensional polynomials and possess certain properties. A Hermite polynomial $H_\text{mi}(x)$ is a polynomial of order $2n+1$ which gives, where $x=x_i$:

**Equation 2**

\[
\frac{d^k H}{dx^k} = 1 \quad k = m \quad \text{for} \quad m = 0 \text{ to } n
\]

and

\[
\frac{d^k H}{dx^k} = 0 \quad k \neq m \quad \text{or when} \quad x = x_j
\]

A set of first order Hermite polynomials is thus a set of cubics giving shape functions for a line element $ij$ and at the ends, slopes and values of the function are used as nodal degrees of freedom along 1-2

\[
H_{01}^1(x) = \frac{1}{a^3}(2x^3 - 3ax^2 + a^3)
\]

\[
H_{01}^2(x) = -\frac{1}{a^3}(2x^3 - 3ax^2)
\]

\[
H_{11}^1(x) = \frac{1}{a^2}(x^3 - 2ax^2 + a^2 x)
\]

\[
H_{12}^1(x) = \frac{1}{a^2}(x^3 - ax^2)
\]

These polynomials are plotted in the following figure.

Note these polynomials provide unit values of displacements and slopes at one end and zero at the other as was implies by equation 2. Assume $w(x,y)$ of the following form:

\[
w(x,y) = H_{01}(x)H_{01}(y)w_1 + H_{02}(x)H_{01}(y)w_2 + H_{02}(x)H_{02}(y)w_3 + \\
H_{01}(x)H_{02}(y)w_4 + H_{11}(x)H_{01}(y)w_{x1} + H_{12}(x)H_{01}(y)w_{x2} + \\
H_{12}(x)H_{02}(y)w_{x3} + H_{11}(x)H_{02}(y)w_{x4} + H_{01}(x)H_{11}(y)w_{y1} + \\
H_{02}(x)H_{11}(y)w_{y2} + H_{02}(x)H_{12}(y)w_{y3} + H_{01}(x)H_{12}(y)w_{y4} + \\
H_{11}(x)H_{11}(y)w_{xy1} + H_{12}(x)H_{11}(y)w_{xy2} + H_{12}(x)H_{12}(y)w_{xy3} + H_{11}(x)H_{12}(y)w_{xy4}
\]

The superscript for $H$ has been dropped since all $H_{mi}$ are $2\times1+1=3^{rd}$ degree polynomials ($n=1$). Further for $H_{m}(y)$, just replace $x$ with $y$ and $a$ with $b$.

**Checks**

1. We can show that $w(x,y)$ has three rigid body modes (can be shown by performing an eigenvalue analysis)
2. We can also show that $w(x,y)$ has constant strain modes.
3. continuity: look at edge 1-2 of the element:

\[ w = w_1 H_{01}(x) + w_2 H_{02}(x) + w_{x1} H_{11}(x) + w_{x2} H_{12}(x) \]

\[ w_y: \text{only those terms having } H_{11}(y) \text{ will have non-zero values} \]

\[ w_y = w_{y1} H_{01}(x) + w_{y2} H_{02}(x) + w_{xy1} H_{11}(x) + w_{xy2} H_{12}(x) \]

From above two equations, we note \( w \) and \( w_y \) depends on nodal dof at nodes 1 and 2 for edge 1-2.

Similarly, we can show that we get the same expressions for \( w \) and \( w_y \) along edge 3-4 except \( w_4 \) replaces \( w_1 \), \( w_3 \) replaces \( w_2 \), etc.

Therefore, equating the nodal variables along edge 1-2 of element A in the figure to nodal variables along edge 3-4 of element B will ensure continuity of \( w \) and \( w_y \) as required. In exactly the same manner we can show continuity of \( w \) and \( w_x \) along edges parallel to y axis.

Thus, the plate bending element discussed here is conforming in the sense that displacements and normal slopes are continuous so that the potential energy theorem does apply. We expect monotonic convergence of potential energy as well as strain energy. Potential energy will converge to the exact value from above where as strain energy from below as was shown for the beam problem, i.e. potential energy is bounded above and strain energy is bounded below.
7- Alternative Method for Plate Bending Element
The alternative method for deriving the stiffness matrix and the consistent load vector is presented for the conforming element discussed in the previous section. However, the approach is general enough to apply to any rectangular or triangular elements.
Although, we used Hermitian polynomials in deriving the displacement approximation, one can multiply out these polynomials in eq1 of the previous section and obtain the following expression:

\[
\begin{align*}
    w(x, y) &= a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 + a_7 x^3 + a_8 x^2 y + a_9 xy^2 + a_{10} y^3 + a_{11} x^3 y + a_{12} x^2 y^2 + a_{13} xy^3 + a_{14} x^3 y^2 + a_{15} x^2 y^3 + a_{16} x^3 y^3
\end{align*}
\]

In this equation, the polynomial is complete only up to cubic terms. Using Taylo series approach error in \( w \) is \( f(h^4) \) where \( h \) =typical element dimension
Error in strain \( f(h^2) \) (strain are second derivatives)
Error in strain energy is \( f(h^4) \)
For \( h=L/N \), the strain energy error is \( f(N^{-4}) \) where \( n=number \ of \ elements \) along a side of length \( L \). Generally for convergence rate study, use square elements. Asymptotic convergence rate is \( N^{-4} \). When \( w \) is given in the form of above equation, it is obvious that \( w(x,y) \) contains rigid body modes and constant strains.
We can write the polynomial in the following form:

\[
    w(x, y) = \sum_{i=1}^{16} a_i x^{m_i} y^{n_i}
\]

\[
\{m\}^T = [0 \ 1 \ 0 \ 2 \ 1 \ 0 \ 3 \ 2 \ 1 \ 0 \ 3 \ 2 \ 1 \ 3 \ 2 \ 3]
\]

\[
\{n\}^T = [0 \ 0 \ 1 \ 0 \ 1 \ 2 \ 0 \ 1 \ 2 \ 3 \ 1 \ 2 \ 3 \ 2 \ 3 \ 3]
\]

Let us first obtain the stiffness matrix in terms of \( a_i \)'s and later transform to obtain \( [K] \) in terms of \( w_i \)'s.
\{w_i^T\} = \begin{bmatrix} w_1 & w_{x1} & w_{y1} & \ldots & w_4 & w_{x4} & w_{y4} & w_{xy4} \end{bmatrix}
\{w_{16x1}\} = [T]_{16x16} \{A\}_{16x1}
\{A\}^T = \begin{bmatrix} a_1 & a_2 & \ldots & a_{16} \end{bmatrix}

w_{xx} = \sum_{i=1}^{16} m_i(m_i-1)a_i x^{m_i-2} y^{n_i}

w_{yy} = \sum_{i=1}^{16} n_i(n_i-1)a_i x^{n_i} y^{n_i-2}

w_{xy} = \sum_{i=1}^{16} m_i n_i a_i x^{m_i-1} y^{n_i-1}

U = \frac{D}{2} \int_0^h \int_0^a \left[ w_{xx}^2 + w_{yy}^2 + 2\nu w_{xx} w_{yy} + 2(1-\nu)w_{xy}^2 \right] dx dy

U_e = \frac{D}{2} \int_0^h \int_0^a \left\{ m_i m_j (m_i - 1)(m_j - 1)x^{m_i + m_j - 4} y^{n_i + n_j} + n_i n_j (n_i - 1)(n_j - 1)x^{m_i + m_j - 4} y^{n_i + n_j - 4} \right\} dx dy a_j

Define:

G(m,n) = \int_0^a \int_0^h x^m y^n dx dy = \frac{a^{m+1} b^{n+1}}{(m+1)(n+1)}

Note that \( w_{xx}, w_{yy} \) term has been split into two terms to preserve symmetry i.e. if we change \( i \) with \( j \) \( U_e \) is still the same.

It is obvious that this integration is not valid when \( m=-1 \) or \( n=-1 \) and blows up for \( m \leq -1 \) or \( n \leq -1 \) at lower limit i.e. \( x=0 \) (\( m=0,1,2,\ldots \) and \( n=0,1,2,\ldots \)).

Strain Energy Can be written as:
\[ U_\varepsilon = \frac{1}{2} \{ A \} \{ \overline{K} \} \{ A \} \]

\[ \overline{K}_{ij} = D \left\{ \frac{m_i m_j \left( m_i - 1 \right) \left( m_j - 1 \right) G(m_i + m_j - 4, n_i + n_j) + n_i n_j \left( n_i - 1 \right) \left( n_j - 1 \right) G(m_i + m_j, n_i + n_j - 4) +}{vm_i n_j \left( m_j - 1 \right) + vm_j n_i \left( m_i - 1 \right) + 2(1 - \nu)m_i m_j n_i n_j G(m_i + m_j - 2, n_i + n_j - 2) +} \right\} \]

Next the \([T]\) matrix has to be computed:

\[
[T] = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
1 & a & 0 & a^2 & 0 & a^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 1 & 0 & 2a & 0 & 0 & 3a^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 1 & 0 & a & 0 & 0 & a^2 & 0 & 0 & a^3 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 1 & 0 & 0 & 2a & 0 & 0 & 3a^2 & 0 & 0 & 0 & 0 & 0 & 0
1 & a & b & a^2 & ab & b^2 & a^3 & a^2b & ab^2 & b^3 & a^3b & a^2b^2 & a^3b^2 & a^3b^3 & a^3b^3 & a^3b^3 & a^3b^3 & a^3b^3
0 & 1 & 0 & 2a & b & 0 & 3a^2 & 2ab & b^2 & 0 & 3a^2b & 2ab^2 & b^3 & 3a^2b^2 & 2ab^3 & 3a^2b^3 & 3a^2b^3 & 3a^2b^3 & 3a^2b^3
0 & 0 & 1 & 0 & a & 2b & 0 & a^2 & 2ab & 3b^2 & a^3 & 2a^2b & 3ab^2 & 2a^3b & 3a^2b^2 & 3a^3b^2 & 3a^3b^2 & 3a^3b^2 & 3a^3b^2
0 & 0 & 0 & 0 & 1 & 0 & 0 & 2a & 2b & 0 & 3a^2 & 4ab & 3b^2 & 6a^2b & 6ab^2 & 9a^2b^2 & 9a^2b^2 & 9a^2b^2 & 9a^2b^2
1 & 0 & b & 0 & 0 & b^2 & 0 & 0 & 0 & b^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 1 & 0 & b & 0 & 0 & b^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 1 & 0 & 0 & 2b & 0 & 0 & 0 & 3b^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
0 & 0 & 0 & 0 & 1 & 0 & 0 & 2b & 0 & 0 & 0 & 3b^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Either we can program the matrix above or determine in a more general form as follows:

Define: \( x_i, y_i \quad i = 1,2,3,4 \) as the nodal coordinates

\[
\begin{align*}
(x_1, y_1) &= (\varepsilon, \varepsilon) \quad , \quad (x_2, y_2) = (a, \varepsilon) \\
(x_3, y_3) &= (a, b) \quad , \quad (x_4, y_4) = (\varepsilon, b)
\end{align*}
\]

Where \( \varepsilon \) is a very small number, \( \varepsilon = 10^{-13} \), instead of zero. This helps retaining some more accuracy and some times makes the inversion possible especially for triangular elements which may exhibit some orientation preferences.

Then:
The matrix \([T]\) is then inverted and the stiffness matrix is the global coordinates is calculated:

\[
[T] = \begin{bmatrix}
16 & 16 & 16 & 16 \\
\vdots & \vdots & \vdots & \vdots \\
1 & 16 & 16 & 16 \\
\end{bmatrix}
\]

one has to be cautious when computing \(G(m,n)\) or \([G]\) matrix. This is because some of the terms (lower order) in the polynomial of equation 1 may lead to negative or zero \(m\) and \(n\) i.e. terms like \(m_1+n_1=4\), etc. For example, \(m_1=0, n_1=0\) then \(m_1+m_1-4=4\) and \(n_1+n_1-4=-4\). These are the smallest possible indecies for \(G(m,n)\) or \([G]\). This can be avoided by taking a matrix \([F]\) such that:

\[F(m+5,n+5)=G(m,n)\]

Where \([F]\) has dimensions at least 4 larger than \([G]\) would require.

\[F(m+5,n+5) = G(m,n) = \int_0^b \int_0^a x^m y^n \, dx \, dy = \frac{a^{m+1} b^{n+1}}{(m+1)(n+1)}\]

where \(F_{i,j} = 0\) for \(i=1,2,3,4\) and \(j=1,2,3,4\)
Load Vector
Assume constant load $q_0$/unit area applied to the plate. Therefore, work done is given by:

$$W_c = \int_A q_0 w \, dx \, dy = \int_0^a \int_0^b q_0 \sum_{i=1}^{16} a_i x^{m_i} y^{n_i} \, dx \, dy$$

$$W_c = \{A\}^T \{\vec{f}\} = \{w\}^T \left(\{T\}^{-1}\right)^T \{\vec{f}\}$$

$$W_c = q_0 \sum_{i=1}^{16} a_i G(m_i, n_i)$$

$$\vec{f}_i = q_0 G(m_i, n_i) \quad i = 1, 2, \ldots, 16$$

$$\{f\} = \left(\{T\}^{-1}\right)^T \{\vec{f}_i\}_{16 \times 1}$$ load vector in global coordinates

Stress Matrix for Obtaining Moments (for an element)
Recall:

$$\{\varepsilon\} = \begin{bmatrix} w_{xx} \\ w_{yy} \\ w_{xy} \end{bmatrix} = [S] \{A\} \quad \text{strain and curvature}$$

$$\{\tau\} = \begin{bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{bmatrix} \quad \text{stresses and moments}$$

$$\{\tau\} = [D] \{\varepsilon\}$$

$$[D] = \begin{bmatrix} -1 & -\nu & 0 \\ -\nu & -1 & 0 \\ 0 & 0 & 1-\nu \end{bmatrix} \quad D = \frac{Eh^3}{12(1-\nu^2)}$$

$$w = \sum_{i=1}^{16} a_i x^{m_i} y^{n_i}$$

$$w_{xx} = \sum_{i=1}^{16} m_i (m_i - 1) a_i x^{m_i-2} y^{n_i}$$

$$w_{yy} = \sum_{i=1}^{16} n_i (n_i - 1) a_i x^{m_i} y^{n_i-2}$$

$$w_{xy} = \sum_{i=1}^{16} m_i n_i a_i x^{m_i-1} y^{n_i-1}$$

$$S_{1j} = m_j (m_j - 1) x^{m_j-2} y^{n_j}$$

$$S_{2j} = n_j (n_j - 1) x^{m_j} y^{n_j-2}$$

$$S_{3j} = m_j n_j x^{m_j-1} y^{n_j-1}$$

$$\{A\} = [T]^{-1} \{w\}$$

$$\{\varepsilon\} = [S][T]^{-1} \{w\}$$
where \( w \) is the displacement vector for element under consideration and is extracted from the global displacement vector.

\[
\{\tau\} = \begin{pmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{pmatrix} = [D]_{3 \times 3} [S]_{3 \times 16} [T]^{-1}_{16 \times 16} \{w\}_{16 \times 1}
\]

Note that the matrix \([S]\) is function of \( x \) and \( y \) and has to be evaluated at the points \((x_i, y_i)\) where bending moments and twisting moment are desired to be evaluated.

The \([T]^{-1}\) matrices can be stored away e.g. on a file so that these can be used for determining moments later, i.e. after displacements have been calculated.

As mentioned earlier, the procedure is general and only changes need to be made are integration routine, different data for \( m_i \) and \( n_i \) and changing sizes of various matrices. The logic does not change at all.
8- Triangular Element for Conforming C\(^1\) Continuity
Using quintic polynomial for the displacement field:

**Equation 3**

\[ w(x, y) = a_1 + a_2 x + a_3 y + a_4 x^2 + a_5 xy + a_6 y^2 + a_7 x^3 + a_8 x^2 y + a_9 xy^2 + a_{10} y^3 + a_{11} x^4 + a_{12} x^3 y + a_{13} x^2 y^2 + a_{14} xy^3 + a_{15} y^4 + a_{16} x^5 + a_{17} x^4 y + a_{18} x^3 y^2 + a_{19} x^2 y^3 + a_{20} xy^4 + a_{21} y^5 \]

\[ w(x, y) = \sum_{i=1}^{21} a_i x^{m_i} y^{n_i} \]

\[ \{m_i\}^T = [0 \ 1 \ 0 \ 2 \ 1 \ 0 \ 3 \ 2 \ 1 \ 0 \ 4 \ 3 \ 2 \ 1 \ 0 \ 5 \ 4 \ 3 \ 2 \ 1 \ 0] \]

\[ \{n_i\}^T = [0 \ 0 \ 1 \ 0 \ 1 \ 2 \ 0 \ 1 \ 2 \ 3 \ 0 \ 1 \ 2 \ 3 \ 4 \ 0 \ 1 \ 2 \ 3 \ 4 \ 5] \]

There are 21 generalized parameters \(a_i\)'s, therefore either 21 dof are required or 21 independent equations to relates \(a_i\)'s to the dof.

\[ \hat{e}_{ni} \]

**Option One:**

Six dof at corner nodes (1,2,3), i.e. \(w, w_x, w_y, w_{xx}, w_{xy} \) and \(w_{yy}\) and one dof at the mid side nodes (4,5,6) i.e. \(w_n(\ n = \nabla w \hat{e}_n) \)

**Option two:**

Only six dof at corner nodes (1,2,3), i.e. \(w, w_x, w_y, w_{xx}, w_{xy} \) and \(w_{yy}\) for a total of 18 dof per element. Additional three equations come from constraining the normal slope \(w_n\) to vary cubically.
{w} = [T] {a} \\
{a}^T = [a_1 \ldots a_2 \ldots a_{21}] \\
{w}^T = [w_1 \ w_x1 \ w_y1 \ w_{xx1} \ w_{xy1} \ w_{yy1} \ \ldots \ \ldots \ w_{n4} \ w_{n5} \ w_{n6}]

Edge Geometry
Consider the $i^{th}$ edge defined by nodes $i$ and $j$ as shown. Let $s$ be the running coordinate along the edge and $\hat{e}_{ni}$ be the unit outward normal to the $i^{th}$ edge:

\[ l_i = \sqrt{(x_j - x_i)^2 + (y_j - y_i)^2} \]

\[ j = i + 1 \quad (\text{for } j > 3, \ j \neq 9) \]

\[ \hat{e}_{si} = \cos \beta_i \hat{i} + \sin \beta_i \hat{j} \]

where

\[ \cos \beta_i = \frac{x_j - x_i}{l_i} \quad \sin \beta_i = \frac{y_j - y_i}{l_i} \]

\[ \hat{e}_{ni} = \sin \beta_i \hat{i} - \cos \beta_i \hat{j} \]

also along the $i^{th}$ edge:

\[ x = s \cos \beta_i \quad \text{and} \quad y = s \sin \beta_i \]

Option One

Equation 4

\[ T_{k,j} = x_i^{m_j} y_i^{n_j} \quad k = 1 + 6(i - 1) \quad i = 1, 2, 3 \text{ and } j = 1, 2, 3, \ldots, 21 \]

\[ T_{k+1,j} = m_j x_i^{m_j-1} y_i^{n_j} \]

\[ T_{k+2,j} = n_j x_i^{m_j} y_i^{n_j-1} \]

\[ T_{k+3,j} = m_j (m_j - 1) x_i^{m_j-2} y_i^{n_j} \]

\[ T_{k+4,j} = m_j n_j x_i^{m_j-1} y_i^{n_j-1} \]

\[ T_{k+5,j} = n_j (n_j - 1) x_i^{m_j} y_i^{n_j-2} \]

$i = 1, 2, 3$ takes care of 18 dof at the corner nodes.

At mid – side nodes:

\[ \frac{\partial w}{\partial n} = w_n = \nabla w \hat{e}_n = w_x \sin \beta - w_y \cos \beta \]

\[ T_{i+18,j} = m_j x_i^{m_j-1} y_i^{n_j} \sin \beta_j - n_j x_i^{m_j-1} y_i^{n_j-1} \cos \beta_i \quad j = 1, 2, 3, \ldots, 21 \text{ and } i = 1, 2, 3 (\text{at nodes } 4, 5, 6) \]

\[ \{w\}_{21x1} = [T]_{21x21} \{a\}_{21x1} \quad \text{invert} \ [T] \]

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Option Two
Equation 4 (a-f) for corner nodes still apply. These yields 18 eqns and therefore three more equations are still to be accounted for. Note that for a quintic polynomial, the normal slope along all three edges vary as quartic (4th degree polynomial).
"additional three equations arise from constraining the normal slope to vary cubically along each edge."
Consider only the 5th degree term in equation 3 and denote this partial w(x,y) as \( w_p \) i.e.:

\[
   w_p = a_{16} x^5 + a_{17} x^4 y + a_{18} x^3 y^2 + a_{19} x^2 y^3 + a_{20} x y^4 + a_{21} y^5
\]

also along an edge:
\[ x = s \cos \beta_i \quad \text{and} \quad y = s \sin \beta_i \]

\[
   \frac{\partial w_p}{\partial n} = \nabla w_p \cdot e_{\beta} = \frac{\partial w_p}{\partial x} \sin \beta_i - \frac{\partial w_p}{\partial y} \cos \beta_i
\]

\[
   \frac{\partial w_p}{\partial n} = \left[ a_{16} (s \cos \beta_i^4 \sin \beta_i) + a_{17} (4 \cos \beta_i^3 \sin \beta_i^2 - \cos \beta_i^5) + a_{18} (3 \cos \beta_i^2 \sin \beta_i^3 - 2 \cos \beta_i^4 \sin \beta_i) + a_{19} (2 \cos \beta_i \sin \beta_i^4 - 3 \cos \beta_i^3 \sin \beta_i^2) + a_{20} (5 \cos \beta_i^5 - 4 \cos \beta_i^2 \sin \beta_i^3) + a_{21} (-5 \cos \beta_i \sin \beta_i^4) \right] S^2
\]

Note the bracked term […] is the combined coefficient of \( s^4 \). For \( w_n \) to be cubic along an edge […] must be set equal to zero and hence yields three more equations, from each edge.
Hence,
\[
   T_{i+18,j} = m_j (\cos \beta_i)^{m_j-1} (\sin \beta_i)^{n_j} \sin \beta_i - n_j (\cos \beta_i)^{m_j} (\sin \beta_i)^{n_j-1} \cos \beta_i = 0
\]
along edge 1,2,3 \( j = 16,17,18,19,20,21 \) and \( i = 1,2,3 \)
\[
\begin{bmatrix}
   \{w\}_{18x1} \\
   \{0\}_{3x1}
\end{bmatrix} = [T]_{21x21} \{a\}
\]

invert \([T]\) and ignore the last three columns of \([T]^{-1}\) to obtain
\[
\{a\}_{21x1} = [T]^{-1}_{21x18} \{w\}_{18x1}
\]
8.2- Transformation of Nodal DOF along an Inclined Edge

Before any boundary conditions can be applied along an inclined edge, all first and second derivatives must be transformed to perpendicular and parallel to the edge.

For the first derivatives:
\[ w_{ni} = \lambda_{ni} w_i \quad i = 1,2 \quad \text{or} \quad w_{nj} = \lambda_{nj} w_n \]
\[ w_{sn} = \lambda_{si} w_i \quad i = 1,2 \quad \text{or} \quad w_{sj} = \lambda_{sj} w_s \]

where \( \lambda_{ni} \) are direction cosines of the unit outward normal \( \hat{e}_{si} \) and \( \lambda_{si} \) are the direction cosines of the unit tangential vector \( \hat{e}_{si} \).

For second derivatives:
\[ w_{nn} = \lambda_{ni} \lambda_{nj} w_{ij} \quad w_{ns} = \lambda_{ni} \lambda_{sj} w_{ij} \quad w_{ss} = \lambda_{si} \lambda_{sj} w_{ij} \]
\[ w_{ii} = \lambda_{in} \lambda_{is} w_{ns} \quad w_{ij} = \lambda_{in} \lambda_{js} w_{ns} \quad w_{jj} = \lambda_{jn} \lambda_{js} w_{ns} \]
\[ \lambda_{ni} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \quad \lambda_{si} = \begin{bmatrix} -\sin \theta & \cos \theta \end{bmatrix} \]

to obtain \( \lambda_{si} \), replace \( \theta \) by \( \theta + 90 \) in \( \lambda_{ni} \)
\[ \lambda_{in} = \begin{bmatrix} \cos \theta & -\sin \theta \end{bmatrix} \quad n = 1 \text{ for } n, n = 2 \text{ for } s \]
\[ \lambda_{jn} = \begin{bmatrix} \sin \theta & \cos \theta \end{bmatrix} \quad i \text{ for } x, j \text{ for } y \]
\[
\begin{bmatrix}
  w_{,x} \\
  w_{,y}
\end{bmatrix}
= 

\begin{bmatrix}
  \cos \theta & -\sin \theta \\
  \sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
  w_{,n} \\
  w_{,s}
\end{bmatrix}
= 

[T_1]
\begin{bmatrix}
  w_{,n} \\
  w_{,s}
\end{bmatrix}
\]
\[
\begin{bmatrix}
  w_{,,x} \\
  w_{,,y}
\end{bmatrix}
= 

\begin{bmatrix}
  \cos^2 \theta & -2 \sin \theta \cos \theta & \sin^2 \theta \\
  \sin \theta \cos \theta & \cos^2 \theta - \sin^2 \theta & -\sin \theta \cos \theta \\
  \sin^2 \theta & 2 \sin \theta \cos \theta & \cos^2 \theta
\end{bmatrix}
\begin{bmatrix}
  w_{,nn} \\
  w_{,ns} \\
  w_{,ss}
\end{bmatrix}
= 

[T_2]
\begin{bmatrix}
  w_{,nn} \\
  w_{,ns} \\
  w_{,ss}
\end{bmatrix}
\]

Note: \( \cos \theta = \sin \beta_i \), \( \sin \theta = -\cos \beta_i \)

For option 2, for edge 1–2 (k – l) as an inclined edge, the following transformation will apply:

\[
\begin{bmatrix}
  \{w^{3}\}_{xy}^{18 \times 1} \\
  \{w^{3}_{,ns}\}_{18 \times 1}
\end{bmatrix}
= 

\begin{bmatrix}
  [Q] & [0] & [0] \\
  [0] & [Q] & [0] \\
  [0] & [0] & [I]
\end{bmatrix}
\begin{bmatrix}
  \{w^{3}\}_{NS}^{18 \times 1} \\
  \{w^{3}_{,ns}\}_{18 \times 1}
\end{bmatrix}
= 

[Q][Q]_{6 \times 6}
\begin{bmatrix}
  1 & 0 & 0 \\
  0 & [T_1] & [0] \\
  0 & [0] & [T_2]
\end{bmatrix}_{6 \times 6}
\]

\[
[K]_{18 \times 18} = [Q][Q]^{T}_{18 \times 18}[K][Q]_{18 \times 18}
\]
9- Two-Dimensional Creeping Flow

\[ I = \frac{\nu}{2} \iint_{\Omega} \left( \nabla^2 \Psi \right)^2 \, d\Omega \quad \Psi = \text{streamline function} \]

\[ \partial \vec{A} = \nu \iint_{\Omega} \nabla^2 \Psi \delta \left( \nabla^2 \Psi \right) \, d\Omega = \nu \iint_{\Omega} \nabla^2 \Psi \left( \delta \psi_{xx} + \delta \psi_{yy} \right) \nabla \Omega \]

\[ \partial \vec{A} = \nu \iint_{\Omega} \nabla^2 \Psi \delta \psi_{n} \, ds - \nu \iint_{\Omega} \left[ \left( \nabla^2 \Psi \right)_{,x} \delta \psi_{x} + \left( \nabla^2 \Psi \right)_{,y} \delta \psi_{y} \right] \, d\Omega \]

\[ \partial \vec{A} = \nu \iint_{\Omega} \nabla^2 \Psi \delta \psi_{n} \, ds - \nu \iint_{\Omega} \left[ \left( \nabla^2 \Psi \right)_{,xx} \delta \psi_{xx} + \left( \nabla^2 \Psi \right)_{,yy} \delta \psi_{yy} \right] \, d\Omega \]

field eqn

\[ \nu \left( \nabla^2 \Psi \right)_{,xx} + \left( \nabla^2 \Psi \right)_{,yy} = 0 \quad \text{or} \quad \nabla^4 \Psi = 0 \]

Boundary conditions

either: \( \psi^2 \psi = 0 \) or \( \delta \psi_n = 0 \) on \( S \)

either: \( \nu \left( \nabla^2 \Psi \right)_{,n} = 0 \) or \( \delta \psi = 0 \) on \( S \)

on solid boundary

\[ u_n = \frac{\partial \psi}{\partial s} \quad u_s = -\frac{\partial \psi}{\partial n} \]

Also \( \nabla^2 \psi = \nabla \left( \psi_{nn} + \psi_{ss} \right) \) \( \psi_{ss} = 0 \) for straightedge: \( \nabla^2 \psi = \mu \left( \frac{\partial u_s}{\partial n} + \frac{\partial u_n}{\partial s} \right) = -\tau_{ns} \neq 0 \)

Along a centrel ine \( u_n = 0 \) \( u_s \neq 0 \) \( \psi \psi_n = 0 \) hence \( \nabla^2 \psi = 0 \) \( \nabla^2 \psi = \psi = 0 \) (vorticity is zero)

look at:

\[ \nu \left( \nabla^2 \Psi \right)_{,n} \quad \text{or} \quad \mu \left( \nabla^2 \Psi \right)_{,n} \]

from momentum equations

\[ p_x = \mu \nabla^2 u \quad p_y = \mu \nabla^2 v \quad (u = u_x \quad \text{and} \quad v = u_y) \]

\[ \hat{e}_n = \cos \theta \hat{i} + \sin \theta \hat{j} \]

\[ \hat{e}_s = -\sin \theta \hat{i} + \cos \theta \hat{j} \]

\[ \mu \nabla^2 \psi_n = \mu \nabla^2 \left( \psi_x \cos \theta + \psi_y \sin \theta \right) = \mu \nabla^2 \left( -v \cos \theta + u \sin \theta \right) = \mu \nabla^2 v \cos \theta + \mu \nabla^2 u \sin \theta \]

\[ \mu \nabla^2 \psi_n = -p_x \cos \theta + p_y \sin \theta = -p \hat{e}_x \]

\[ \hat{e}_s = \frac{\partial p}{\partial s} \]

\[ \hat{e}_s = \text{Pressure drop across a wake, s normal to the wake or free surface} \]

these second boundaries integral reveal that if \( \psi \) is not fixed along a boundary then \( \delta \psi = 0 \) alongs:

for \( \delta \psi \neq 0 \) then \( \frac{\partial p}{\partial s} = \mu \nabla^2 \psi_n = 0 \)
9.1- Fully Developed Parallel Flow

\[ U = \frac{\partial \Psi}{\partial y} = 6y(1-y) \]
\[ V = 0 \]
\[ \psi = 3y^2 - 2y^3 \quad \psi(0) = 0 \quad \psi(1) = 1 \]

**Bc's**
\[ \psi = 3y^2 - 2y^3 \quad \text{and} \quad \Psi_x = 0 \quad \text{on section one} \]
\[ \psi = 0 \quad \text{and} \quad \Psi_y = 0 \quad \text{at bottom edge} \]
\[ \psi = 1 \quad \text{and} \quad \Psi_y = 0 \quad \text{on top edge} \]
\[ \psi_x = 0 \quad \text{on section two} \]

![Diagram of Fully Developed Parallel Flow](image)

9.2- Flow Past a Cylinder

Computational domain=20xR away, flow can be assumed uniform

**Bc's**
\[ \psi = u_0 y \quad \text{and} \quad \Psi_x = 0 \quad \text{on section one} \]
\[ \psi = 0 \quad \text{at bottom along x} \]
\[ \psi_y = u_0 \quad \text{and} \quad \Psi = 10u_0 \quad \text{on top edge} \]
\[ \psi_x = 0 \quad \text{on section two} \]

![Diagram of Flow Past a Cylinder](image)