Fault-Tolerant Spanners in Networks with Symmetric Directional Antennas*

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Abstract

Let P be a set of points in the plane, each equipped with a directional antenna that covers a sector of angle α and range r. In the symmetric model of communication, two antennas u and v can communicate to each other, if and only if v lies in u's coverage area and vice versa. In this paper, we introduce the concept of *fault-tolerant spanners* for directional antennas, which enables us to construct communication networks that retain their connectivity and spanning ratio even if a subset of antennas are removed from the network. We show how to orient the antennas with angle α and range r to obtain a k-fault-tolerant spanner for any positive integer k. For $\alpha \geq \pi$, we show that the range 13 for the antennas is sufficient to obtain a k-fault-tolerant 3-spanner. For $\pi/2 < \alpha < \pi$, we show that using range $6\delta + 19$ for $\delta = \lceil 4/|\cos\alpha| \rceil$, one can direct antennas so that the induced communication graph is a k-fault-tolerant 7-spanner.

Keywords: Wireless network, Directional antenna, Fault-tolerance, Geometric spanner

1 Introduction

Omni-directional antennas, whose coverage area are often modelled by a disk, have been traditionally employed in wireless networks. However, in many recent applications, omni-directional antennas have been replaced by directional antennas, whose coverage region can be modelled as a sector with an angle α and a radius r (also called transmission range), where the orientation of antennas can vary among the nodes of the network. The point is that by a proper orientation of directional antennas, one can generate a network with lower radio wave overlapping and higher security than the traditional networks with omni-directional antennas [6].

There are two main models of communication in networks with directional antennas. In the *asymmetric* model, each antenna has a directed link to any node that lies in its coverage area. In the *symmetric* model, there exists a link between two antennas u and v, if and only if u lies in the coverage area of v, and v lies in the coverage area of u. The symmetric model

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of communication is more practical, especially in networks where two nodes must handshake to each other before transmitting data [8].

In this paper, we consider the symmetric model for communication in directional antennas, and study two properties of the communication graphs: *k-connectivity* and *spanning ratio*. A network is *k-connected* if it remains connected after removing or destroying any $k-1$ of its nodes. Furthermore, if after some failure of nodes, it still has some desirable properties, we say that the network is *fault-tolerant*. Therefore, the fault-tolerance property is more general than the connectivity. A network is called a *spanner*, if there is a short path between any pairs of nodes, within a guaranteed ratio to the shortest paths between those nodes in an underlying base graph. This ratio is called the *stretch factor*. A *fault-tolerant spanner* has the property that when a small number of nodes fail, the remaining network still contains short paths between any pair of nodes. (See [16] for an overview of the properties of geometric spanner networks.)

Related Work. The problem of orienting directional antennas to obtain a strongly connected network was first studied by Caragiannis *et al.* [7] in the asymmetric model. They showed that when $\alpha < 2\pi/3$, the problem of determining the minimum radius to achieve connectivity is NP-hard, and presented a polynomial time algorithm for $\alpha \geq 8\pi/5$ with optimal radius. The problem was later studied for other values of α, and approximation algorithms were provided for minimizing the transmission range of connected networks [3, 9]. However, the communication graphs obtained from these algorithms could have a very large stretch factor, such as $O(n)$, compared to the original unit disk graph (i.e., the omni-directional graph of radius 1). Therefore, subsequent research was shifted towards finding a proper orientation such that the resulting graph becomes a t-hop spanner [6, 13]. In a *t-hop spanner*, the number of hops (i.e., links) in a shortest link path between any pair of nodes is at most t times the number of hops in the shortest link path between those two nodes in the base graph, which happens to be a unit disk graph in this case.

The connectivity of communication graphs in the symmetric model was first studied by Ben-Moshe *et al.* [5] in a limited setting where the orientation of antennas were chosen from a fixed set of directions. Carmi *et al.* [8] later considered the general case, and proved that for $\alpha \geq \pi/3$, it is always possible to orient antennas so that the induced graph is connected. In their presented algorithm, the radius of the antennas were related to the diameter of the nodes. Subsequent work considered the stretch factor of the communication graph. Aschner *et al.* [4] studied the problem for $\alpha = \pi/2$ and obtained a symmetric connected network with radius $14\sqrt{2}$ and a stretch factor of 8, assuming that the unit disk graph of the nodes is connected. Tran *et al.* [19] also studied the case $\alpha = \pi/2$, and proved its NP-hardness when the objective is to determine an orientation of the antennas with a minimum radius such that the induced symmetric communication graph is connected. They also presented an algorithm to improve the radius from $14\sqrt{2}$ to 9 for this case. Recently, Dobrev *et al.* [10] proved that for $\alpha < \pi/3$ and radius one, the problem of determining the existence of an orientation that ensures a connected network is NP-hard. They also showed how to construct spanners for various values of $\alpha \geq \pi/2$. A summary of the current records for the radius and the stretch factor of the communication graphs in the symmetric model is presented in Table 1.

The problem of k-connectivity in wireless networks has been also studied in the literature, mostly for omni-directional networks [14, 15], where the objective is to assign transmission

Table 1: Summary of the previous results for netwroks with symmetric directional antennas. In all these results, the unit disk graph of the nodes (antennas) is assumed to be connected. Here, $\delta = \sqrt{3 - 2\cos\alpha}(1 + 2\sin\frac{\alpha}{2})$.

Angle of Antenna	Stretch Factor	Radius	Ref.
$\pi/2$	–	9	[19]
$\pi/2$	8	$14\sqrt{2}$	[4]
$\pi/2$	7	33	[11]
	5	718	
$\pi/2 \leq \alpha < 2\pi/3$	9	10	[10]
$2\pi/3 \leq \alpha < \pi$	–	5	
$2\pi/3 \leq \alpha < \pi$	6	6	
$\alpha \geq \pi$	–	$\max(2, 2\sin\frac{\alpha}{2} + 1)$	
$\alpha \geq \pi$	3	$\max(2, 2\sin\frac{\alpha}{2} + \delta)$	

range such that the network can sustain fault nodes and remain connected. The stretch factor of the constructed network is also studied in some limited settings. In [12], a setting is studied where antennas are on a unit segment or a unit square, and a sufficient condition is obtained on the angle of directional antennas so that the energy consumption of the k-connected networks is lower when using directed rather than omni-directed antennas. In [18], a tree structure is built on directed antennas, and a fault-tolerance property is maintained by adding additional links to tolerate failure in limited cases, namely, when only a node or a pair of adjacent nodes fail.

Our Results. In this paper, we study the problem of finding fault-tolerant spanners in networks with symmetric directional antennas. The problem is formally defined as follows. Given a set P of n points in the plane, place antennas with angle α and radius r on P, so that the resulting communication graph is a k-fault-tolerant t-spanner. A graph G on the vertex set P is a *k-fault-tolerant t-spanner*, if after removing any subset $S \subseteq P$ of nodes with $|S| < k$, the resulting graph $G \setminus S$ is a t-spanner of the unit disk graph of P. In the rest of the paper, we assume that the unit distance is sufficiently large to ensure that the unit disk graph of P is k-connected. To the best of our knowledge, this is the first time that fault-tolerance is studied in networks with symmetric directional antennas.

We show that for any $\alpha \geq \pi$, we can place antennas with angle α and radius 9, such that the resulting communication graph is k-connected. Moreover, we show that by increasing the radius to 13, we can guarantee that the resulting graph is a k-fault-tolerant 3-spanner. When $\pi/2 < \alpha < \pi$, we consider two cases depending on whether the distribution of antennas is sparse or dense. We prove that for sparse distribution, we can place antennas with angle α and radius $4\delta + 13$, where $\delta = \lceil 4/|\cos\alpha| \rceil$, such that the resulting communication graph is k-connected and then by increasing the radius to $6\delta + 19$, have a k-fault-tolerant 7-spanner. Moreover, for dense distribution, we prove that our algorithm yields a k-fault-tolerant 4-spanner using radius δ. Our results are summarized in Table 2.

Table 2: Summary of our results for netwroks with symmetric directional antennas. In these results, the unit disk graph of the nodes is assumed to be k-connected. Here, $\delta = \lceil 4/|\cos\alpha| \rceil$.

Angle of Antenna	Stretch Factor	Radius	Ref.
$\alpha \geq \pi$	–	9	Theorem 1
$\alpha \geq \pi$	3	13	Theorem 2
$\pi/2 < \alpha < \pi$ (sparse)	–	$4\delta + 13$	Theorem 3
$\pi/2 < \alpha < \pi$ (sparse)	7	$6\delta + 19$	Theorem 4
$\pi/2 < \alpha < \pi$ (dense)	4	δ	Theorem 4

We recall that the k-connectivity of the unit disk graph is assumed in the rest of the paper. In other words, we compared the radius and stretch factor of our k-connected directional network to those of a k-connected omni-directional network. While this assumption is reasonable, it is possible to relax it, and only assume the connectivity of the unit disk graph, which is the minimum requirement assumed in the related (non-fault-tolerant) work. If we replace the k-connectivity assumption with 1-connectivity, the radius and stretch factor of our constructed network is increased by a factor of k, as discussed in Section 5.

2 Preliminaries

Let P be a set of points in the plane, and G be a graph on the vertex set P. For two points $p, q \in P$, we denote by $\delta_G(p, q)$ the shortest hop (link) distance between p and q in G. Throughout this paper, the *length* of a path in a graph refers to the number of edges on that path. For two points p and q in the plane, the Euclidean distance between p and q is denoted by $\|pq\|$.

Let $\mathcal{B}(c, r)$ denote a (closed) disk of radius r centered at c. We define $\mathcal{A}(c, r) \equiv \mathcal{B}(c, r) - \mathcal{B}(c, r-1)$ to be an *annulus* of width 1 enclosed by two concentric circles of radii $r-1$ and r, centered at c. Note that by our definition, $\mathcal{A}(c, r)$ is open from its inner circle, and is closed from the outer circle.

A graph G is k-connected, if removing any set of at most $k-1$ vertices leaves G connected. Given a point set P, we denote by UDG(P) the unit disk graph defined by the set of disks $\mathcal{B}(p, 1)$ for all $p \in P$. We say that P is *k-connected*, if UDG(P) is k-connected. Let $G = \text{UDG}(P)$. A graph H on the vertex set P is a *t-spanner* of G, if for any two vertices u and v in G, we have $\delta_H(u, v) \leq t \cdot \delta_G(u, v)$. We say that the subgraph $H \subseteq G$ is a *k-fault-tolerant t-spanner* of G, if for all sets $S \subseteq P$ with $|S| < k$, the graph $H \setminus S$ is a t-spanner of $G \setminus S$.

Fact 1. *Let G and H be two k-connected graphs, and E be a set of edges between the vertices of G and H. If E contains a matching of size k, then the graph $G \cup H \cup E$ is k-connected.*

Fact 2. *Let G be a k-connected graph, and v be a new vertex adjacent to at least k vertices of G. Then $G + v$ is k-connected.*

Lemma 1. *Let P be a k-connected point set, and r be a positive integer. If $|P| \geq rk$, then for any point $p \in P$, $\mathcal{B}(p, r)$ contains at least rk points of P.*

Proof. Fix a point p, and let q be the furthest point from p in P. If $\|pq\| \leq r$, then $P \subseteq \mathcal{B}(p, r)$, and we are done. Otherwise, consider the annuli $A_i = \mathcal{A}(p, i)$ for $1 \leq i \leq r+1$, and let $A_0 = \{p\}$. Each A_i must be non-empty, because otherwise, p is disconnected from q in UDG(P). Now, we claim that each A_i, for $1 \leq i \leq r$, contains at least k points. Otherwise, if $|A_i| < k$ for some $1 \leq i \leq r$, then removing the points of A_i disconnects A_{i-1} from A_{i+1}, contradicting the fact that P is k-connected. □

3 Antennas with $\alpha \geq \pi$

In this section, we present our algorithm for orienting antennas with angle at least π. The main ingredient of our method is a partitioning algorithm which we describe below. The same partitioning will be used later in Section 4.

3.1 Partitioning Algorithm

The partitioning algorithm is illustrated in Algorithm 1. The algorithm receives as input a point set P, a point $p \in P$, and a positive integer r. It recursively builds a graph H that induces a partitioning on P, as described in Lemma 2.

Algorithm 1 PARTITION(P, p, r)

```
add vertex p to graph H
P = P \ B(p, 2r)
while ∃q ∈ P ∩ B(p, 2r + 1) do
    PARTITION(P, q, r)
    add edge (p, q) to graph H
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Lemma 2. *Let P be a k-connected point set, p be an arbitrary point of P, and $|P| \geq kr$ for a positive integer r. Let $H = (V, E)$ be the graph obtained from* PARTITION(P, p, r)*. For each $v \in V$, we define $Q_v = P \cap \mathcal{B}(v, r)$. Moreover, we define F_v to be the set of all points in $P \setminus \cup_{u \in V} Q_u$ closer to v than any other point in V (ties broken arbitrarily). Then the followings hold:*

(a) H is connected, and for each edge $(u, v) \in E$, $2r < \|uv\| \leq 2r + 1$.

(b) P is partitioned into disjoint sets Q_v and F_v.

(c) Q_v has at least kr points, for all $v \in V$.

(d) F_v is contained in $\mathcal{B}(v, 2r)$, for all $v \in V$.

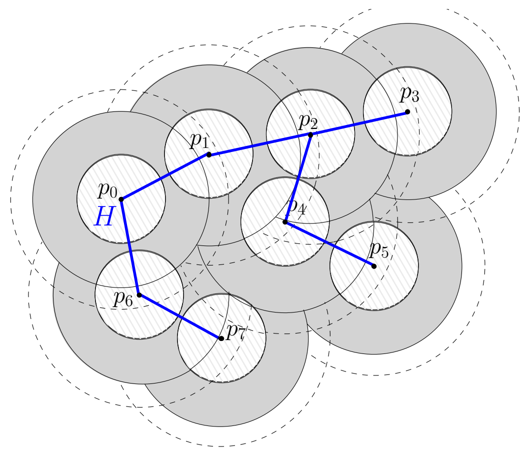

Figure 1: A partitioning obtained by Algorithm 1. The induced graph H is shown by dark edges. Groups and free points are represented by shaded and gray regions, respectively.

Proof. (a) The graph H built by the algorithm is clearly connected, as each new vertex created by calling Partition in line 4 is connected in line 5 to a previous vertex of H. Moreover, lines 2 and 3 of the algorithm enforce that any two adjacent vertices in H have distance between $2r$ and $2r + 1$.

(b) The sets F_v are disjoint by their definition. The sets Q_v are also disjoint, because any two vertices in H have distance more than $2r$ by line 2 of the algorithm.

(c) This is a corollary of Lemma 1.

(d) This is clear from lines 2 and 3 of the algorithm.

□

We call each set Q_v a *group*, and the points in F_v the *free points* associated to the group Q_v (see Figure 1). We call v the *center* of Q_v. Two groups Q_u and Q_v are called *adjacent groups*, if there is an edge (u, v) in the graph H.

Runtime. To find points inside the disks $\mathcal{B}(p, 2r)$ and $\mathcal{B}(p, 2r+1)$, we use a *circular range searching* data structure by Afshani and Chan [2] that preprocesses the input set P of n points in $O(n \log n)$ expected time so that for any query disk, all k points of P inside the disk can be reported in $O(\log n + k)$ time. Whenever we delete a point from set P in line 2 of the algorithm, we do not remove it from the data structure, and only mark it as deleted. Since the radii of the query disks in the algorithm are at most $2r+1$, and their centers are at least $2r$ apart (for a positive integer r), each point of P lies in at most $O(1)$ query disks, and hence, the amortized time for processing each point is $O(\log n/k+1) = O(\log n)$. Therefore, the whole partitioning algorithm takes $O(n \log n)$ expected time.

3.2 Orienting Antennas

Here we show how to place antennas with angle at least π on a point set P, so that the resulting communication graph is k-connected, with a guaranteed stretch factor. In the rest

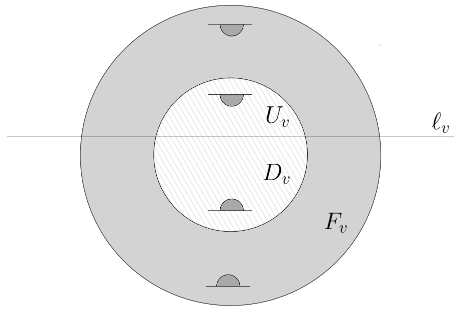

Figure 2: The orientation of antennas with angle π in $Q_v \cup F_v$.

of this section, we describe our method for $\alpha = \pi$. However, the method is clearly valid for any larger angle.

Theorem 1. *Given a k-connected point set P with at least $2k$ points in the plane, we can place antennas with angle π and radius 9 on P, such that the resulting communication network is k-connected.*

Proof. We run Algorithm 1 with $r = 2$ on the point set P to obtain the graph $H = (V, E)$. For each $v \in V$, let Q_v and F_v be the sets defined in Lemma 2. Since $r = 2$, each set Q_v has at least $2k$ points. We partition Q_v by a horizontal line ℓ_v into two equal-size subsets U_v and D_v, each of size at least k, where points in U_v (resp., in D_v) are all above (resp., below) ℓ_v. (Points on ℓ_v can be placed in either U_v or D_v.) Now, we orient antennas in D_v upward, and antennas in U_v downward. Moreover, we orient antennas in F_v upward if they are below ℓ_v, and downward if they are above or on ℓ_v (see Figure 2).

Let G_π be the communication graph obtained by the above orientation, where the radius of each antenna is set to $4r + 1 = 9$. Since each node in D_v has distance at most $2r$ to any node in U_v, Q_v forms a complete bipartite graph, with each part having size at least k, and hence, it is k-connected. Now, we show that the graph on $Q = \cup Q_v$ is k-connected. Note that the distance between the centers of any two adjacent groups Q_u and Q_v is at most $2r+1$, and the farthest points in the groups have distance at most $4r + 1$. By setting the radius of antennas to $4r+1$, either all members of D_u connect to all members of U_v, or all members of U_u connect to all members of D_v. So there is a matching of size k between any two adjacent groups, and hence, Q is k-connected by Fact 1. Since F_v is contained in $\mathcal{B}(v, 2r)$, the farthest points in $Q_v \cup F_v$ are at distance $4r$, and hence, each node in F_v connects to at least k nodes in Q_v. Therefore, the whole communication graph is k-connected by Fact 2. □

Theorem 2. *Given a k-connected point set P with at least $2k$ points in the plane, we can place antennas with angle π and radius 13 on P, such that the resulting communication network is a k-fault-tolerant 3-spanner.*

Proof. We use the same orientation described in the proof of Theorem 1. Now, we show that by setting radius of antennas to $6r + 1 = 13$, the resulting graph G_π is a k-fault-tolerant 3-spanner. Fix a set $S \subseteq P$ with $|S| < k$. We show that for any edge $(p, q) \in \text{UDG}(P) \setminus S$, there is a path between p and q in $G_\pi \setminus S$ of length at most 3. For each $v \in V$, let $T_v = Q_v \cup F_v$, and let ℓ_v be the horizontal line that equipartitions Q_v into sets D_u and U_v. Suppose $p \in T_u$

and $q \in T_v$. Assume w.l.o.g. that ℓ_u is below or equal to ℓ_v. Since $\|pq\| \le 1$, the centers of Q_u and Q_v are at most $4r + 1$ apart. Therefore, by setting the radius to $6r + 1$, we have a matching of size k between D_u and U_v in G_π. Moreover, since $\|pq\| \le 1$, the distance between nodes in Q_u and F_v (resp., Q_v and F_u) is at most $6r + 1$. We distinguish the following four cases based on the order of points and lines on the y-axis. Here, for a point $p = (p_x, p_y)$ and a horizontal line $\ell : y = b$, we say that $p \le \ell$ (resp., $p > \ell$) if $p_y \le b$ (resp., $p_y > b$).

- $p \le \ell_u$ and $q \le \ell_v$. Since $|S| < k$, there is a vertex $w \in U_v \setminus S$ such that p and q are both connected to w. Therefore, $\delta_G(p, q) = 2$ in this case.
- $p \le \ell_u$ and $q > \ell_v$. Since $|S| < k$, there is an edge $(w, x) \in (D_u \setminus S, U_v \setminus S)$. Now, the path $\langle p, x, w, q\rangle$ is a path of length 3 in G.
- $p > \ell_u$ and $q > \ell_v$. Since $|S| < k$, there is a vertex $w \in D_u \setminus S$ such that p and q are both connected to w. Therefore, $\delta_G(p, q) = 2$ in this case.
- $p > \ell_u$ and $q < \ell_v$. This case is analogous to the second case.

□

Runtime. To orient the antennas in Q_v and F_v, we just need to sort the points to determine their position about ℓ_v's, and hence, the orientation can be done in $O(n \log n)$ time. Combined with the partitioning algorithm, the whole runtime is $O(n \log n)$.

4 Antennas with $\pi/2 < \alpha < \pi$

We now consider a more challenging case where the goal is to orient the antennas with angle $\pi/2 < \alpha < \pi$ on a point set P, so that the resulting communication graph is k-connected. Let $\delta = \lceil 4/|\cos\alpha|\rceil$. We distinguish two cases based on the distribution of P on the plane. P is called *α-sparse* if the diameter of P (i.e., the distance of the farthest pair of points in P) is at least δ. Otherwise, P is called *α-dense.*

Algorithm Sketch. We first sketch the whole algorithm, and then go into details of each part. The algorithm is almost similar to the one given in the previous section for $\alpha = \pi$. We run Algorithm 1 with $r = \delta + 3$ on the point set P to obtain the graph $H = (V, E)$. For each $v \in V$, let Q_v and F_v be the sets defined in Lemma 2. We orient antennas in $Q_v \cup F_v$ such that the resulting graph is k-connected. We then make the radius of the antennas large enough, so that for any two adjacent groups Q_u and Q_v, their union (and consequently $Q = \cup Q_v$) becomes k-connected.

Observation 1. *If P is α-dense, then $H = (V, E)$ is a single vertex.*

Lemma 3. *If P is α-sparse, then the diameter of $P \cap \mathcal{B}(p, \delta+1)$ is at least δ, for any $p \in P$.*

Proof. Let (q, q') be the farthest pair of points in P. If both q and q' are contained in $\mathcal{B}(p, \delta + 1)$, we are done. Otherwise, at least one of q and q' (say q) is outside $\mathcal{B}(p, \delta + 1)$. Since UDG($P$) is connected, $\mathcal{A}(p, \delta + 1)$ must contain some point t of P. Since t is inside $\mathcal{B}(p, \delta + 1)$ and $\|tp\| > \delta$, the diameter of $P \cap \mathcal{B}(p, \delta + 1)$ is at least δ. □

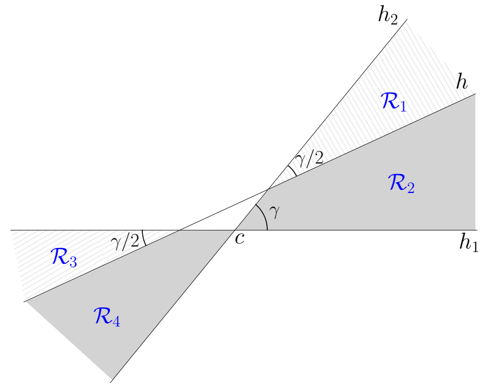

Figure 3: Illustrating the proof of Lemma 4.

We start explaining how to make each Q_v k-connected. We define *γ-cone* to be a cone with angle γ. Let $\sigma(c)$ be an γ-cone with apex c and let $\bar{\sigma}(c)$ be the reflection of $\sigma(c)$ about c. We first state a proposition from [17] and a new lemma, to be used in our construction.

Proposition 1 ([17])**.** *Given a set P of points in the plane, there exist orthogonal lines h_1 and h_2 that equipartition P, i.e., none of the four quadrants obtained by lines h_1 and h_2 has more than $n/4$ of the points of P.*

Lemma 4. *If for some point c, γ-cones $\sigma(c)$ and $\bar{\sigma}(c)$ each contains at least m points, then there exist $(\gamma/2)$-cones $\sigma(c')$ and $\bar{\sigma}(c')$ for some point c', each containing at least $m/2$ points.*

Proof. Let h_1 and h_2 be the lines passing through the sides of $\sigma(c)$. Let h be a line parallel to the bisector of $\sigma(c)$ that partitions points inside $\sigma(c)$ into two equal-size subsets (see Figure 3). Therefore, regions $\mathcal{R}_1$ and $\mathcal{R}_2$ contain $m/2$ points. One of the regions $\mathcal{R}_3$ and $\mathcal{R}_4$ (say $\mathcal{R}_3$) contains at least $m/2$ points. Therefore, if $\mathcal{R}_3$ (resp., $\mathcal{R}_4$) contains at least $m/2$ points, the cone created by h_1 (resp., h_2) and h is the desired cone. □

Now, we present the lemma that our algorithm relies on.

Lemma 5.

(a) If the diameter of Q_v is at least δ, then there is a $(2\alpha-\pi)$-cone $\sigma(c)$ for some point c on the plane such that $\sigma(c)$ and $\bar{\sigma}(c)$ each contains at least $2k$ points of Q_v.

(b) If the diameter of Q_v is less than δ but Q_v contains at least $8k\cdot\pi/(2\alpha-\pi)$ points, then there is a $(2\alpha-\pi)$-cone $\sigma(c)$ for some point c on the plane such that $\sigma(c)$ and $\bar{\sigma}(c)$ each contains at least $2k$ points of Q_v.

Proof. (a) We claim that there are two points p and q in P with distance at least δ such that disks with radius 2 centered at p and q are fully contained in $\mathcal{B}(v,\delta+3)$. If the points defining the diameter of Q_v are inside $\mathcal{B}(v,\delta+1)$, based on the Lemma 3 the claim is simply true. Otherwise, there is a point outside $\mathcal{B}(v,\delta+1)$, and hence there should be a point w in $\mathcal{A}(v,\delta+1)$. By setting $p=v$ and $q=w$, the claim is clearly true. This claim with Lemma 1 implies that each of these two disks contains at least $2k$ points

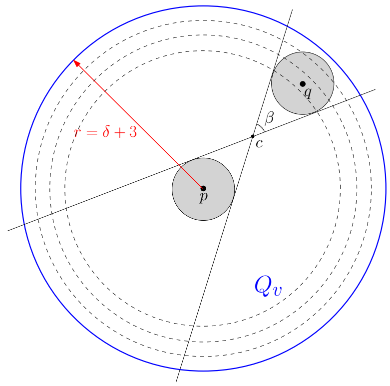

Figure 4: Illustrating the first case in Lemma 5.

of Q_v. Consider the two interior common tangents of these disks with the intersection point c and angle β (see Figure 4). It is simple to see $\sin(\beta/2) \leq 2/(\delta/2)$. This implies $\beta \leq (2\alpha - \pi)$ (note that $\delta = \lceil 4/|\cos\alpha| \rceil$). Therefore, we can locate an $(2\alpha - \pi)$-cone $\sigma(c)$ and its reflection $\bar{\sigma}(c)$ at c such that both cones contain at least $2k$ points of Q_v.

(b) Since Q_v contains at least $8k \cdot \pi/(2\alpha - \pi)$ points, we can apply Proposition 1 and get two orthogonal lines h_1 and h_2 such that each quarter created by h_1 and h_2 contains $2k \cdot \pi/(2\alpha - \pi)$ points. For $3\pi/4 \leq \alpha < \pi$ (or equivalently $\pi/2 \leq 2\alpha - \pi < \pi$), we know $\pi/(2\alpha - \pi) \geq 1$, and therefore each quarter contains at least $2k$ points. Then it suffices to set $\sigma(c)$ to be the $(2\alpha - \pi)$-cone containing one of the four quarters where c is the intersection of h_1 and h_2. For $\pi/2 < \alpha < 3\pi/4$ (or equivalently $0 < 2\alpha - \pi < \pi/2$), we can apply Lemma 4, $(\log_2 (2 \cdot \pi/(2\alpha - \pi))) - 1$ times to get the desired angle and $2k$ points in the desired cone. □

We recall that if P is α-sparse, the diameter of each set Q_v is at least δ. If P is α-dense, we only have one set Q_v and then we only need the extra assumption that P contains at least $8k \cdot \pi/(2\alpha - \pi)$ points, in order to use the lemma in our algorithm.

Orienting $Q_v \cup F_v$. Let $\sigma(c)$ be the $(2\alpha - \pi)$-cone obtained in Lemma 5. Let ℓ_1 and ℓ_2 be the lines passing through the sides of $\sigma(c)$ (and $\bar{\sigma}(c)$ as well), and let ℓ be the bisector of the angle $2\pi - 2\alpha$ whose sides are ℓ_1 and ℓ_2 (see Figure 5 to get more intuition). We define and depict four types of orienting antennas with angle α in Figure 5 naming O_1, O_2, O_3 and O_4. In each type, each side is parallel to one of the lines ℓ_1, ℓ_2, and ℓ.

Backbone Antennas. We select $2k$ point of $Q_v \cap \sigma(c)$ and arbitrarily partition them into two sets D_v and U_v of size k. Similarly, we select $2k$ point of $Q_v \cap \bar{\sigma}(c)$ and arbitrarily partition them into two sets $\bar{D}_v$ and $\bar{U}_v$ of size k. We use types O_1, O_2, O_3, and O_4 for orienting antennas in $D_v, U_v, \bar{D}_v$, and $\bar{U}_v$, respectively. We call each of these four sets a backbone set. Regardless of the antennas radii, this orientation holds the following properties:

- Each antenna in $D_v \cup U_v$ covers each antenna in $\bar{D}_v \cup \bar{U}_v$, and vice versa.
- Each point in the plane is covered by all antennas in one of the backbone sets.

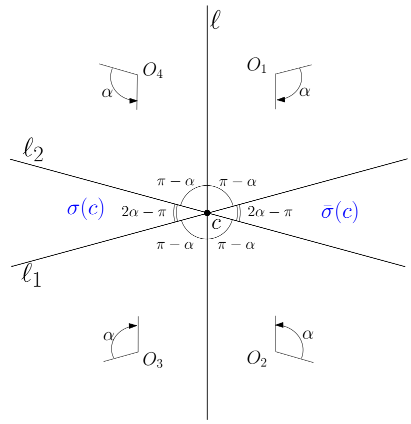

Figure 5: Cones $\sigma(c)$ and $\bar{\sigma}(c)$, and four orientations with angle α.

To orient antenna p in $Q_v \cup F_v$ other than backbone antennas, we detect which backbone set covers p (i.e., p is visible from all antennas in the backbone set). Let O_i be the orientation type used to orient the backbone set. We orient p with type $\bar{O}_i$ where $\bar{O}_i$ is the reflection of O_i about its apex. Figure 6 depicts how to orient antennas depending on their subdivisions induced by ℓ_1, ℓ_2, and ℓ.

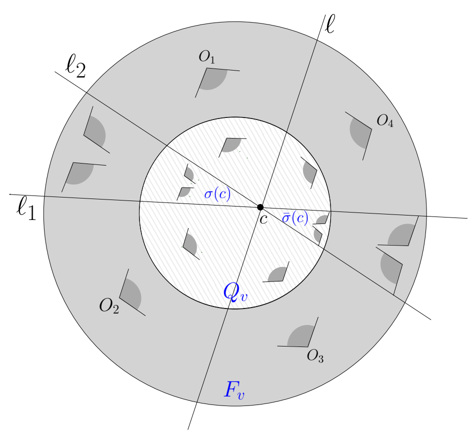

Figure 6: The orientation of antennas with angle $\pi/2 < \alpha < \pi$ in $Q_v \cup F_v$

Radius. If P is α-dense, we set the radius to δ, as the distance of any two antennas is at most δ and hence, the induced graph is k-connected. For the α-sparse set P, we need that any two visible backbone antennas u' and v' from two adjacent groups Q_u and Q_v cover each other. Since their distance is at most $\|u'u\| + \|uv\| + \|vv'\| \leq 4(\delta + 3) + 1$, we set the radius to $4\delta + 13$.

k-Connectivity. For any $v \in V$, the distance between two points in Q_v is at most $2(\delta+3)$. So, by setting the radius to $4\delta + 13$ the induced graph over $(D_v \cup U_v, \bar{D}_v \cup \bar{U}_v)$ is definitely a complete bipartite graph with each part having size more than k and hence it is k-connected. Moreover, by this radius, any antenna in $Q_v \cup F_v$ other than the backbone antennas has a direct connection with at least k backbone antennas as the distance between this antenna and the backbone antennas is at most $3(\delta+3)$. All these simply imply that the induced graph over $Q_v \cup F_v$ is k-connected by Fact 2.

Here, we need to show that the connection of two adjacent groups Q_v and Q_u remains safe even if $k-1$ antennas are destroyed. We partition the backbone antennas in Q_v (similarly in Q_u) into k sets S_v^i ($i = 1, \dots, k$) of size 4, each containing one antenna from the sets D_v, U_v, $\bar{D}_v$, and $\bar{U}_v$. We know each point in the plane is visible from one member of S_v^i, and moreover, two sets S_v^i and S_u^i can be separated by a line. This together with the following proposition implies that there are two backbone antennas $p \in S_v^i$ and $q \in S_u^i$ which are visible to each other, and hence, with the radius specified for antennas, they are in the coverage area of each other.

Proposition 2 ([4])**.** *Let A and B be two sets containing 4 antennas with angle at least $\pi/2$. Suppose both A and B cover the entire plane regardless of the antennas radius. If there exists a line ℓ that separates A and B, then by setting the radius unbounded, the network induced by $A \cup B$ is connected.*

The above discussion shows that by radius $4\delta+13$, there are at least k distinct links between the backbone antennas of two adjacent groups Q_v and Q_u. Therefore, even if $k-1$ antennas are destroyed, the connection between Q_v and Q_u remains safe and by Fact 1, the induced communication graph is k-connected. So, we have the following theorem:

Theorem 3. *Suppose P is a k-connected point set in the plane, and α is a given angle in the range $(\pi/2, \pi)$. Let $\delta = \lceil 4/|\cos\alpha| \rceil$ and P is α-sparse. Then, we can place antennas with angle α and radius $4\delta + 13$ on P, such that the resulting communication network is k-connected.*

Now, we prove that the induced graph for α-dense points, in addition to being k-connected, is also k-fault-tolerant 4-spanner. Moreover, we show that for α-sparse points, by increasing the radius to $6\delta+19$ the resulting communication graph becomes k-fault-tolerant 7-spanner.

Lemma 6. *Suppose $p, q \in Q_v \cup F_v$ and q is a backbone antenna. The points p and q are connected to each other via at most three links, even if $k-1$ antennas are destroyed.*

Proof. Assume w.l.o.g. that $q \in D_v$. We know p is visible from all members of one backbone set. This backbone set can be either $D_v, U_v, \bar{D}_v$, or $\bar{U}_v$. If this backbone set is either $D_v, \bar{D}_v$ or $\bar{U}_v$, we reach q from p with at most two links. Otherwise, with 3 links we can get q from p. Since each backbone set has k members and any member of $D_v \cup U_v$ is visible from $\bar{D}_v \cup \bar{U}_v$ and vice versa, the proof works even if at most $k-1$ antennas are destroyed. □

Graph $H = (V, E)$ has only one vertex if P is α-dense. Therefore, using Lemma 6 we can simply show that any two points are in connection with each other via at most 4 links even if $k-1$ antennas are destroyed. Note that any antenna is either a backbone antenna or directly connected to a backbone antenna. Next we assume that P is α-sparse.

Suppose $p \in Q_u \cup F_u$, $q \in Q_v \cup F_v$ and $\|pq\| \leq 1$. Since $\|pq\| \leq 1$, the centers of Q_u and Q_v are at most $4\delta + 13$ apart. Therefore, by setting the radius to $6(\delta+3)+1$, we have a matching of size k between the backbone antennas of Q_u and Q_v.

This together with Lemma 6 implies that for any two antennas $p \in Q_v \cup F_v$ and $q \in Q_u \cup F_u$ with $\|pq\| \le 1$, there is a connection via at most 7 links.

Stretch Factor. Let p and q be two arbitrary points in P, and let $x_0 = p, x_1, \ldots, x_t = q$ be the shortest link path between p and q in $\mathrm{UDG}(P) \setminus S$, where S is the fault set with size at most $k-1$. Since $\|x_i x_{i+1}\| \le 1$, either there exists $v \in V$ such that $x_i, x_{i+1} \in Q_v \cup F_v$, or there exist two points $u, v \in V$ such that $x_i \in Q_v \cup F_v$ and $x_{i+1} \in Q_u \cup F_u$, and there exists a matching of size k between the backbone antennas of Q_u and Q_v (as just mentioned above). This shows that in the communication graph obtained by our algorithm, each link (x_i, x_{i+1}) either exist or is replaced by a path of length at most 4 in the α-dense set P, and a path of length at most 7 in the α-sparse set P. Therefore, our resulting graph is a 4-spanner and a 7-spanner for the α-dense set P and the α-sparse set P, respectively.

Putting all this together, we get the main theorem of this section.

Theorem 4. *Suppose P is a k-connected point set in the plane, and α is a given angle in the range $(\pi/2, \pi)$. Let $\delta = \lceil 4/|\cos\alpha| \rceil$. Then, the followings hold:*

- *If P is α-sparse, we can place antennas with angle α and radius $6\delta + 19$ on P, such that the resulting communication network is a k-fault-tolerant 7-spanner.*
- *If P is α-dense and contains at least $8k \cdot \pi/(2\alpha - \pi)$ points, we can place antennas with angle α and radius δ on P, such that the resulting communication network is a k-fault-tolerant 4-spanner.*

Runtime. For orienting the antennas, we must first determine whether the point set P is sparse or dense. This can be done by computing the diameter in $O(n \log n)$. Furthermore, due to the following description, the total running time of orienting the antennas in both cases is $O(n \log n)$.

- In α-sparse case, for any $v \in V$, by computing the diameter of Q_v or selecting the points in $\mathcal{A}(v, \delta + 1)$, we can choose p and q with distance at least δ and then finding $(2\alpha - \pi)$-cone and detecting the location of points to this cone can be done in $O(1)$.
- In α-dense case, we first apply Proposition 1, which can be done in $O(n \log n)$ time [17]. We then apply Lemma 4 $O(\log n)$ times, each spending $O(n)$ time. Therefore, the total running time is $O(n \log n)$.

5 Conclusions

In this paper, we introduced the concept of fault-tolerant spanners in networks with symmetric directional antennas, and presented the first algorithms for placing antennas with angles $\alpha > \pi/2$, so that the resulting communication graph is a k-fault-tolerant t-spanner, for small stretch factors $t \le 7$. We compared the radius and stretch factor of our k-connected directional network to those of a k-connected omni-directional network. If we replace the k-connectivity assumption for the input set with a relaxed 1-connectivity one, the radius of antennas implied by Lemma 1 is multiplied by k, and hence, the radius and stretch factor of our constructed network is increased by a factor of k. For instance, on a point set whose unit disk graph is 1-connected, our algorithm constructs a k-fault-tolerant spanner with radius $13k$ and stretch factor $3k$. A natural open problem is to find fault-tolerant spanners with smaller radius and/or stretch factors. The case $\pi/3 \leq \alpha \leq \pi/2$ is also open for further investigation.

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