Online Coloring Co-interval Graphs*

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Abstract

We study the problem of online coloring co-interval graphs. In this problem, a set of intervals on the real line is presented to the algorithm one at a time, and upon receiving each interval I, the algorithm must assign I a color different from the colors of all previously presented intervals not intersecting I. The objective is to use as few colors as possible. It is known that the competitive ratio of the simple First-Fit algorithm on the class of co-interval graphs is at most 2. We show that for the class of unit co-interval graphs, where all intervals have equal length, the 2-bound on the competitive ratio of First-Fit is tight. On the other hand, we show that no deterministic online algorithm for coloring unit co-interval graphs can be better than 3/2-competitive. We then study the effect of randomization on our problem, and show a lower bound of 4/3 on the competitive ratio of any randomized algorithm for the unit co-interval coloring problem. We also prove that for the class of general co-interval graphs no randomized algorithm has competitive ratio better than 3/2.

Keywords: Online Algorithms, Graph Coloring, Co-interval Graphs.

1 Introduction

A variety of optimization problems in scheduling, partitioning and resource allocation can be modeled as graph coloring problems. The graph coloring

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problem involves assigning colors to the vertices of a graph so that adjacent vertices are assigned different colors, with the objective of minimizing the number of colors used. It is well-known that the problem is NP-hard, even for graphs with a fixed chromatic number $k \geq 3$ [13]. Furthermore, it is intractable to approximate the problem to within a factor of n^c for some constant $c > 0$ unless P = NP [18], and to within a factor of $n^{1-\varepsilon}$ for any $\varepsilon > 0$ unless ZPP = NP [7].

In the *online* graph coloring problem, vertices of the graph are presented to the algorithm one at a time. Once a new vertex v is presented, all the edges connecting v to the vertices presented before v are given. The online coloring algorithm must assign a color to the newly presented vertex, different from the colors of all adjacent vertices presented earlier, before the next vertex arrives. The main restriction here is that once a color is assigned to a vertex, the color cannot be changed by the algorithm at a future time. We study online coloring problems in terms of *competitive analysis*, i.e., we compare an online algorithm to an optimal offline algorithm that knows the complete graph in advance. Here, an online coloring algorithm $\mathcal{A}$ is called c-competitive (with *competitive ratio* c) if the number of colors used by $\mathcal{A}$ is at most c times the number of colors used by the optimal offline algorithm.

The online graph coloring problem has been widely studied. Lovász, Saks, and Trotter [17] presented a deterministic online algorithm that achieves a competitive ratio of $O(n/\log^* n)$ on all graphs. Vishwanathan [21] gave a randomized algorithm with a competitive ratio of $O(n/\sqrt{\log n})$ against oblivious adversaries. This competitive ratio was later improved to $O(n/\log n)$ [10]. Halldórsson and Szegedy [11] proved a very close lower bound of $\Omega(n/\log^2 n)$ for both deterministic and randomized algorithms.

Due to the inherent complexity of the online coloring problem on general graphs, many researchers have focused their study on special classes of graphs [4, 5, 12, 15, 19]. For example, Kierstead and Trotter [15] constructed an optimum online algorithm for coloring interval graphs. Their algorithm uses at most $3\omega - 2$ colors, where ω is the maximum clique size of the graph. Since interval graphs are perfect, their chromatic numbers are equal to their clique numbers (see [8] for the definition of graph theory terms used throughout this paper). This means that the algorithm of Kierstead and Trotter is 3-competitive for interval graphs. A matching randomized lower bound of 3 is presented in [16].

Another line of research has been on analyzing the performance of the sim-

	Deterministic		Randomized	
	LB	UB	LB	UB
Coloring co-interval graphs	2 [14]	2 [9]	**3/2**	2 [9]
Coloring unit co-interval graphs	**3/2**	2 [9]	**4/3**	11/6 [23]

Table 1: Summary of the results of this paper and the previous works. Each entry of the table represents a lower/upper bound on the competitive ratio of online algorithms for the corresponding coloring problem.

ple First-Fit algorithm, i.e., the algorithm that simply assigns the smallest available color to each new vertex. For example, it is known that for the class of interval graphs the competitive ratio of First-Fit is at least 4.4 [2] and at most 10 [20].

In this paper, we study the online coloring problem for the class of co-interval graphs. Gyárfás and Lehel [9] have shown that First-Fit uses at most $2\omega - 1$ colors on any co-chordal graph. Since interval graphs are chordal [8], this bound also applies to co-interval graphs. Kierstead and Qin [14] have shown that no online deterministic algorithm can beat this 2-bound on co-interval graphs. We present the first lower bound, up to our best knowledge, for randomized online coloring of co-interval graphs. We show that any randomized algorithm for the problem has an expected competitive ratio of at least 3/2.

For the class of unit co-interval graphs, where all intervals have equal (unit) length, we show that the competitive ratio of First-Fit is still 2. We then show that no deterministic algorithm for this problem can be better than 3/2-competitive. We also prove a lower bound of 4/3 on the competitive ratio of any randomized algorithm for coloring unit co-interval graphs.

The problem of coloring unit co-interval graphs is of special interest because of its connection to some other well-known problems. For example, the problem in the offline setting is equivalent to the problem of finding a largest subset of disjoint intervals among a given set of unit intervals; i.e., finding maximum independent sets in unit interval graphs. This independent set problem can be viewed as a simple scheduling problem, called "activity selection" by Cormen et al. [3]. In the online setting, our problem is equivalent to the problem of online covering a set of points on the line using a minimum number of unit intervals, for which a randomized 11/6-competitive algorithm has been recently announced in [23]. A summary of the lower and upper bounds provided in this paper and the previous works is presented in Table 1.

2 Unit Co-interval Graphs

It is konwn that all comparability graphs are perfect [8], so their chromatic numbers are equal to their clique numbers. Gyárfás and Lehel [9] have shown that for any deterministic online coloring algorithm $\mathcal{A}$ and any positive integer k, there is a tree T such that $\mathcal{A}$ uses at least k colors on T. Since every tree is a comparability graph with clique number two, it follows that the number of colors used by First-Fit on comparability graphs cannot be bounded in general by their clique number.

On the other hand, for the class of co-interval graphs, which is a subclass of comparability graphs, we can obtain better bounds on the competitive ratio of the First-Fit algorithm. The result of [9] shows that First-Fit needs at most $2\omega - 1$ colors on the class of co-chordal graphs, where ω is the clique number of the graph. Since co-interval graphs are special cases of co-chordal graphs, this bound also applies to co-interval graphs.

In the following, we show that the above upper bound on the number of colors used by First-Fit is indeed tight, even on the class of unit co-interval graphs.

Theorem 1 *There exist unit co-interval graphs on which* First-Fit *uses at least* $2\omega - 1$ *colors.*

Proof. For any integer k, we show that there is a unit co-interval graph of size $3k - 2$ with clique number k for which First-Fit uses exactly $2k - 1$ colors. Here is our construction: We first present a sequence of $2k - 2$ intervals of the form $[i, i + 1]$ (for $2 \leq i < 2k$) in order from left to right. First-Fit colors this sequence with exactly $k - 1$ colors: it assigns a unique color to each pair of consecutive intervals $[i, i+1]$ and $[i+1, i+2]$ for $i \in \{2, 4, \ldots, 2k-2\}$. Next, we present k unit intervals of the form $[2i - \frac{1}{2}, 2i + \frac{1}{2}]$, for $1 \leq i \leq k$. Since none of these intervals can be colored with previous colors, First-Fit needs to assign k new colors to these k intervals, summing up to a total number of $2k - 1$ colors. However, it is easy to verify that the co-interval graph represented by this set of intervals has chromatic number exactly k. $\square$

First-Fit is usually referred to as the simplest greedy algorithm for coloring problems. Several other greedy approaches are available for coloring unit co-interval graphs, all leading to a competitive ratio of at most 2. The following is an example of such greedy algorithms.

Algorithm 1 (Grid-Color) *Build a uniform unit grid on the line. Assign each interval to the leftmost grid point intersecting it. Color all intervals assigned to the same grid point with a unique color, different from colors assigned to the other grid points.*

Theorem 2 Grid-Color *is* 2*-competitive on the class of unit co-interval graphs.*

Proof. Let G be a unit co-interval graph, and $\mathcal{C}$ be an optimal coloring for it. Let $\mathcal{I}$ be a subset of intervals receiving the same color in $\mathcal{C}$. We claim that all intervals in $\mathcal{I}$ are assigned to at most two consecutive grid points. Suppose by contradiction that there are two unit intervals $I_1, I_2 \in \mathcal{I}$ that are assigned to two non-consecutive grid points. Then, the enclosing interval of $I_1 \cup I_2$ has length more than 2. Since each of the two intervals has unit length, we conclude that I_1 and I_2 are disjoint. But then, I_1 and I_2 must receive different colors in $\mathcal{C}$, which is a contradiction. It shows that Grid-Color has competitive ratio at most 2. To see why the 2-bound is tight, just consider a sequence of unit intervals $[i - \frac{1}{2}, i + \frac{1}{2}]$ for $1 \le i \le 2k$, on which the Grid-Color algorithm uses $2k$ colors, while the optimal coloring uses only k colors. □

2.1 A Deterministic Lower Bound

As mentioned in the previous section, First-Fit has a competitive ratio of 2 on the class of unit co-interval graphs. An immediate question is whether one can obtain better deterministic algorithms for this problem. The following theorem shows that no such algorithm can be better than 3/2-competitive.

Theorem 3 *There is a lower bound of* 3/2 *on the competitive ratio of any deterministic online algorithm for coloring unit co-interval graphs.*

Proof. Let $I_i = [i, i+1]$, for $i \in \{1, \dots, 4\}$. Consider two input sequences $\sigma_1 = \langle I_2, I_3 \rangle$ and $\sigma_2 = \langle I_2, I_3, I_1, I_4 \rangle$. The adversary chooses one of the two sequences as input. Let $\mathcal{A}$ be any deterministic algorithm for the problem. No matter which sequence is chosen by the adversary, $\mathcal{A}$ receives I_2 and I_3 as the first two intervals. If $\mathcal{A}$ decides to color I_2 and I_3 with two different colors, then $\mathcal{A}$ is 2-competitive on σ_1. If $\mathcal{A}$ decides to color I_2 and I_3 with the same color, then it needs two more colors to color I_1 and I_4. It means that $\mathcal{A}$ needs three colors on σ_2, while the chromatic number of σ_2 is 2. Thus, $\mathcal{A}$ is at least 3/2-competitive on these two input sequences. □

2.2 Randomized Algorithms

Although it is known that no deterministic online algorithm can be better than 2-competitive for coloring co-interval graphs, the recent results in [1] and [23] show that this 2-bound can be beaten for the class of unit co-interval graphs using randomization. The randomized algorithms in [1, 23] indeed solve the following clustering problem:

Problem 1 (1D Unit Clustering) *Given a set of n points on the line, partition the set into clusters (subsets), each of length at most one, so as to minimize the number of clusters used. Here, the* length *of a cluster refers to the length of its smallest enclosing interval.*

In order to apply the results obtained in [1, 23] to our problem, we first need to draw an equivalence between the 1D unit clustering problem and the problem of coloring unit co-interval graphs.

Observation 1 *The one-dimensional unit clustering problem (as defined above) is equivalent to the problem of coloring unit co-interval graphs.*

Proof. It is clear from elementary graph theory that coloring co-interval graphs is equivalent to clique partitioning interval graphs, i.e., partitioning a set of intervals into minimum number of subsets such that intervals in each subset have a common intersection point. In other words, given a set $\mathcal{I}$ of unit intervals, we want to find a minimum cardinality set of points P such that each interval in $\mathcal{I}$ is "pierced" by at least one of the points in P.

Now, we define the following "point-interval" duality: For a given point p, we define p^* to be the unit interval centered at p. For a unit interval I, we define I^* to be its center. It is easy to observe that

$$p \in I \quad \Leftrightarrow \quad I^* \in p^* \ .$$

Given an instance of the unit interval piercing problem, we can map each unit interval to its dual point, and hence, come up with the following equivalent "dual" problem: given a set P of points on the line, find a minimum cardinality set $\mathcal{I}$ of unit intervals such that each point in P is contained in at least one interval of $\mathcal{I}$. This is equivalent to what is defined in Problem 1. □

For the online unit clustering problem in one dimension, a randomized algorithm with competitive ratio 11/6 is presented in [23]. The following is a direct corollary of that result and Observation 1.

Corollary 1 *There is a* $11/6$*-competitive randomized algorithm for online coloring unit co-interval graphs.*

2.3 A Randomized Lower Bound

A randomized lower bound of 4/3 for the online unit clustering problem in one dimension has been proved in [1] using Yao's minimax principle. In the following theorem, we provide a similar result for randomized online coloring unit co-interval graphs. Our proof is direct and does not use Yao's principle.

Theorem 4 *Any randomized algorithm for online coloring unit co-interval graphs is at least* $4/3$*-competitive.*

Proof. Let $\mathcal{A}$ be an arbitrary randomized algorithm for the problem, and let $\rho_{\mathcal{A}}(\sigma)$ be the expected competitive ratio of $\mathcal{A}$ on an input sequence σ. Define $I_i = [i, i+1]$ for $i \in \{1, \dots, 4\}$, and consider two input sequences $\sigma_1 = \langle I_2, I_3 \rangle$ and $\sigma_2 = \langle I_2, I_3, I_1, I_4 \rangle$. No matter which of these two sequences is chosen by the adversary, $\mathcal{A}$ receives I_2 and I_3 as the first two intervals. Let E be the event that $\mathcal{A}$ assigns two different colors to I_2 and I_3. If E occurs, then $\mathcal{A}$ uses 2 colors on each of the input sequences σ_1 and σ_2. If E doesn't occur, then $\mathcal{A}$ uses one color on σ_1 and 3 colors on σ_2. Let $p = \Pr[E]$. Then it is clear that

$$\rho_{\mathcal{A}}(\sigma_1) = 2p + (1-p) = p + 1,$$

and

$$\rho_{\mathcal{A}}(\sigma_2) = \frac{1}{2}(2p + 3(1-p)) = (3-p)/2 \ .$$

If $p > 1/3$ then $\rho_{\mathcal{A}}(\sigma_1) = p + 1 > 4/3$, and hence, $\mathcal{A}$ is not 4/3-competitive on σ_1. Thus we can assume that $p \leq 1/3$. But then $\mathcal{A}$ cannot be better than 4/3-competitive, because $\rho_{\mathcal{A}}(\sigma_2) = (3-p)/2 \geq 4/3$. □

3 General Co-interval Graphs

In this section, we consider the class of general co-interval graphs. Obviously, the 4/3 lower bound proved in the previous section also applies to this class. Here, we obtain a stronger result by proving that no randomized algorithm for coloring (arbitrary rather than unit) co-interval graphs can be better than 3/2-competitive.

Theorem 5 *There is a lower bound of* $3/2$ *on the competitive ratio of any randomized algorithm for online coloring co-interval graphs.*

Proof. By Yao's minimax principle [22], the expected competitive ratio of the optimal deterministic algorithm for an arbitrarily chosen input distribution is a lower bound on the expected competitive ratio of every randomized algorithm. Thus, to show the lower bound, we only need to provide a probability distribution on a set of input sequences such that the expected competitive ratio of any deterministic online algorithm on that distribution is at least 3/2.

Let $k \geq 1$ be a fixed integer. For $1 \leq i \leq k$, we define three types of intervals a_i, b_i, and c_i on the real line as follows:

$$\begin{aligned} a_i &= [3i-3,\, 3i-2], \\ b_i &= [3i+1,\, 3i+2], \\ c_i &= [3i+2,\, +\infty]. \end{aligned}$$

Consider k blocks of intervals $B_1, \ldots, B_k$, where each block is a sequence of two or three intervals defined as follows:

$$B_i = \begin{cases} \langle b_1, c_1 \rangle & i = 1, \\ \langle a_i, b_i, c_i \rangle & 2 \leq i < k, \\ \langle a_k, b_k \rangle & i = k. \end{cases}$$

We construct a set $\mathcal{I}$ of k input sequences σ_1 to σ_k, where each σ_i is obtained by concatenating the first i blocks B_1 to B_i, in order from left to right. It is easy to observe that the co-interval graph represented by each σ_i has chromatic number equal to i.

Now, we define a probability distribution $\mathcal{P}$ over the input set $\mathcal{I}$. Let p_i be the probability that σ_i is chosen as input. We define

$$p_i = \begin{cases} \frac{2^{k-1}}{2^k-1} \cdot \frac{i}{2^i} & 1 \leq i \leq k-1, \\ \frac{k}{2^k-1} & i = k. \end{cases}$$

Note that our probability distribution is properly defined, i.e. $\sum_{i=1}^{k} p_i = 1$.

First-Fit uses exactly $2i-1$ colors to color each σ_i (it opens a new color on every b_j $(1 \le j \le i)$ and a new color on every a_j $(1 < j \le i)$). Thus, the expected competitive ratio of First-Fit on the input distribution $\mathcal{P}$ is

$$\rho_{\mathrm{FF}} = \sum_{i=1}^{k} p_i \left(\frac{2i-1}{i}\right) = \frac{3}{2} - \frac{1}{2(2^k-1)} .$$

Our aim is to show that the expected competitive ratio of any other deterministic online algorithm is at least ρ_{FF} on the input distribution $\mathcal{P}$.

Let $\mathcal{A}$ be an arbitrary deterministic online algorithm. Since all sequences in $\mathcal{I}$ are prefixes of σ_k, $\mathcal{A}$ makes the same decision on any specific interval in all input sequences. Let d_i be the decision made by $\mathcal{A}$ upon receiving the interval c_i $(1 \le i < k)$. We set $d_i = 0$, if $\mathcal{A}$ colors c_i with the same color assigned to b_i, and set $d_i = 1$, otherwise. Note that $\mathcal{A}$ can always color c_i with the same color assigned to its preceding b_i. This is because b_i has no intersection with the intervals of types a and b presented before it, and has intersection with all intervals of type c presented thus far.

We call the sequence of decisions $\langle d_1, \ldots, d_{k-1} \rangle$ the *characteristic sequence* of $\mathcal{A}$. The claim is that knowing the characteristic sequence of an algorithm $\mathcal{A}$, we can determine the minimum number of colors that $\mathcal{A}$ needs on each input sequence, regardless of decisions it makes on intervals of types a and b.

Suppose that a sequence $\delta = \langle d_1, \ldots, d_{k-1} \rangle$ of $k-1$ bits is given, where each $d_i \in \{0, 1\}$. We define a deterministic online algorithm $\mathrm{FF}(\delta)$ as follows. Upon receiving an interval of type c, say c_i, the algorithm checks the corresponding bit d_i in the bit sequence δ and opens a new color for the interval or colors it using the same color assigned to its preceding b_i, depending on whether d_i is 1 or 0, respectively. On the other hand, when $\mathrm{FF}(\delta)$ receives an interval of type a or b, it just behaves like the First-Fit algorithm, i.e., assigns the first available color to the given interval, or opens a new color for the interval provided no previous color is available.

Claim 1 *Among all deterministic online algorithms with characteristic sequence δ, $\mathrm{FF}(\delta)$ uses the minimum number of colors on every input sequence in $\mathcal{I}$.*

Claim 2 *For any arbitrary sequence δ of $k-1$ bits, the expected competitive ratio of $\mathrm{FF}(\delta)$ on the input distribution $\mathcal{P}$ is equal to ρ_{FF}.*

Claim 1 is easy to prove. Here, we provide a proof for Claim 2. Let m be the number of bits which are equal to 1 in the given sequence δ. We prove the claim by induction on m. The base case $m = 0$ is trivial, because in this case all entries in δ are 0, and hence FF(δ) is the same as the FIRST-FIT algorithm.

Now, suppose that the claim is true for all sequences with less than m bits equal to 1, and consider a sequence δ in which m bits are 1. Let j $(1 \le j \le k-1)$ be the position of the last 1-bit in δ. Changing the j-th bit in δ from 1 to 0, we obtain a new sequence which we call $\bar{\delta}$. Since the number of 1-bits in $\bar{\delta}$ is $m-1$, by the induction hypothesis

$$\rho_{\mathrm{FF}(\bar{\delta})} = \rho_{\mathrm{FF}},$$

where $\rho_{\mathrm{FF}(\bar{\delta})}$ is the expected competitive ratio of FF($\bar{\delta}$). Since the first $j-1$ bits of δ and $\bar{\delta}$ are the same, the competitive ratio of FF(δ) is equal to that of FF($\bar{\delta}$) on all input sequences smaller than σ_j in $\mathcal{I}$. For σ_j, FF(δ) uses one color more than FF($\bar{\delta}$), because FF(δ) assigns two different colors to two intervals b_j and c_j, while FF($\bar{\delta}$) colors these two intervals with the same color. On the other hand, FF(δ) uses one color less than FF($\bar{\delta}$) on any input sequence σ_i for all $j < i \le k$. This is because FF($\bar{\delta}$) needs to open two new colors for a_{j+1} and b_{j+1}, while FF(δ) does not open any new color for these two intervals: it colors a_{j+1} with the same color assigned to b_j and colors b_{j+1} with the same color assigned to c_j. Thus,

$$\rho_{\mathrm{FF}(\delta)} \;=\; \rho_{\mathrm{FF}(\bar{\delta})} + \; p_j \left(\frac{1}{j}\right) - \sum_{i=j+1}^{k} p_i \left(\frac{1}{i}\right).$$

But, we know that

$$\begin{aligned}
\sum_{i=j+1}^{k} p_i \left(\frac{1}{i}\right) &= \sum_{i=j+1}^{k-1} \frac{2^{k-1}}{2^k - 1} \cdot \frac{1}{2^i} + \frac{1}{2^k - 1} \\
&= \frac{2^{k-1}}{2^k - 1} \left(\frac{1}{2^j} - \frac{1}{2^{k-1}}\right) + \frac{1}{2^k - 1} \\
&= \frac{2^{k-1}}{2^k - 1} \left(\frac{1}{2^j}\right) \;=\; p_j \left(\frac{1}{j}\right).
\end{aligned}$$

Therefore

$$\rho_{\mathrm{FF}(\delta)} = \rho_{\mathrm{FF}(\bar{\delta})} = \rho_{\mathrm{FF}},$$

and the proof of Claim 2 is complete.

Claims 1 and 2 together show that the expected competitive ratio of any deterministic online algorithm on the input distribution $\mathcal{P}$ is at least ρ_{FF}. If k is chosen arbitrarily large, ρ_{FF} tends to 3/2 and the theorem statement follows. □

4 Conclusions

In this paper, we have provided some lower bounds on the competitive ratio of deterministic and randomized algorithms for online coloring co-interval graphs. Our work raises many open questions concerning the gap between the upper and lower bounds presented in Table 1. Since the submission of this paper, Epstein and van Stee [6] have succeeded to improve the deterministic and randomized lower bounds for coloring unit co-interval graphs to 8/5 and 3/2, respectively. They have also presented a deterministic algorithm for the one-dimensional unit clustering problem (and hence, for coloring unit co-interval graphs) with a competitive ratio of 7/4.

For the class of general co-interval graphs, it is known that no deterministic online coloring algorithm can be better than 2-competitive [14], but we do not see any simple argument that achieves a similar randomized lower bound. An interesting question that remains open is whether one can obtain an online coloring algorithm for general co-interval graphs with a competitive ratio strictly less than 2 using randomization.

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