Time-dependent PT

Problem Statement: Given H_0 with eigensystem $\{|\psi_n^{(0)}\rangle, E_n^{(0)}\}$,

we want to $\boxed{\underbrace{\text{study}}_{?}}$ $H = H_0 + \lambda \underbrace{W(t)}$.

The key diff. is that the perturbation is time-dep.

Reminder:

So far, we dealth with time-indep potentials, for which

we could seperate variables and get

$$\frac{d|\psi(t)\rangle}{dt} = -\frac{i}{\hbar} H |\psi(t)\rangle \Rightarrow \frac{\partial \psi(x,t)}{\partial t} = -\frac{i}{\hbar}\left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\psi(x,t)$$

$$\psi(x,t) \to \Phi(t)\,\Psi(x)$$

$$\Rightarrow \left(\frac{\hbar}{i}\right)\frac{1}{\phi(t)}\frac{d\phi(t)}{dt} = \frac{1}{\psi(x)}\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\psi(x) = E$$

const of x & t.

$$\phi(t) = e^{-\frac{i}{\hbar}Et}, \quad \psi(t): -\frac{\hbar^2}{2m}\frac{d^2\psi(x)}{dx^2} + V(x)\psi(x) = E\psi(x)$$

Time-indep-Sch. eq.

But, if $V(x) \to V(x,t)$

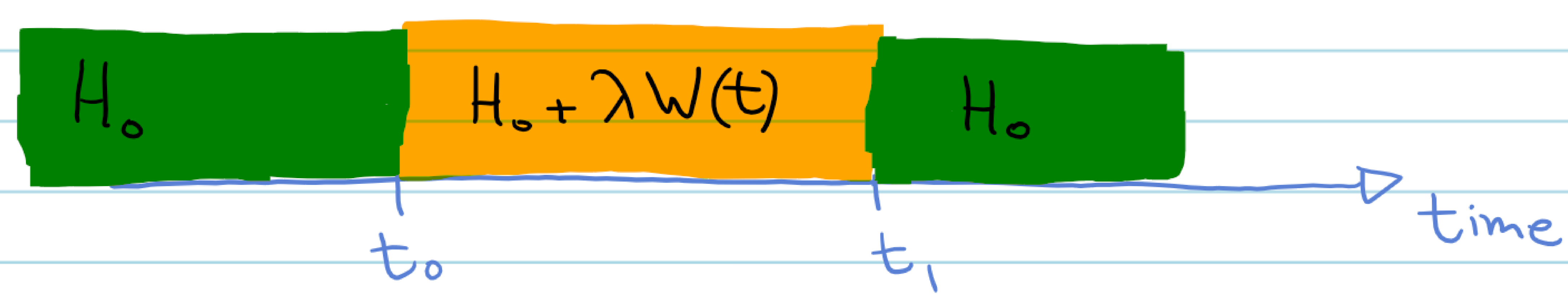
$\triangleright$ This step cannot be done $\to$ Cannot seperate the solution to time & position dep functions.

What do we want?

The energy of an eigenstate is not always constant. → No energy eigen-state

So, we are more interested to check how the perturbation changes some input state.

For the most of this chapter, we consider the following picture, where the time-dep. perturbation is turned on only for a specific period and then turned off again.



Now, starting with a state $|\psi(t_0)\rangle$ before the perturbation, we are interested to understand how the state changes through the perturbation.

For instance, if $|\psi(t_0)\rangle = |\psi_n\rangle$ i.e. an eigenstate of H_0,

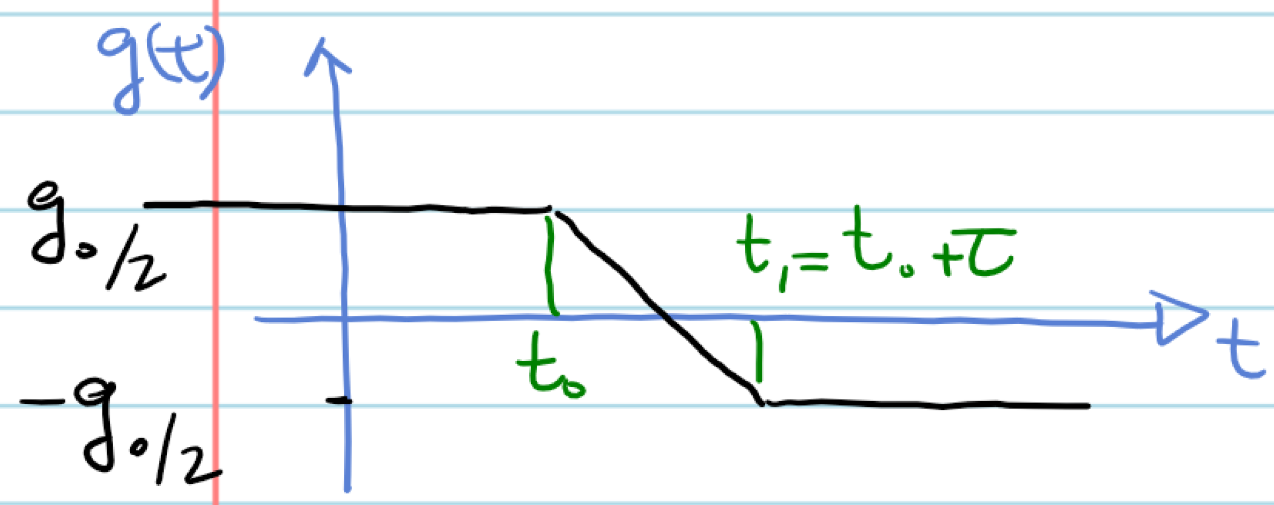
$|\psi(t_1)\rangle$ could be $\sum_m \alpha_m |\psi_m\rangle$ which indicates

that the perturbation could result in transition to other eigenstates of H_0, and the transition probabilities would be given by

$$P_{n\to m} = |\langle \psi_m | \psi(t_1)\rangle|^2 = |\alpha_m|^2$$

$$\boxed{\text{Question: How does the energy change?}}$$

Example: $H = \frac{\omega}{2}\sigma_z + g(t)\,\sigma_x$



* Find the spectrum

* Start with $|gs\rangle$ and see what happens?

Demo

How to find $|\psi(t)\rangle$?

Going fast

Going slow

$$\boxed{\begin{aligned} t_0 &= \frac{-\tau}{2} \\ g(t) &= \frac{g_0 t}{\tau}\end{aligned}} \quad H(t) = \begin{pmatrix} \frac{\omega}{2} & -\frac{g_0 t}{\tau} \\ -\frac{g_0 t}{\tau} & -\frac{\omega}{2}\end{pmatrix} \qquad \lambda = \pm\frac{1}{2}\sqrt{\omega^2 + \left(\frac{g_0 t}{\tau}\right)^2}$$

$$|\lambda_\pm\rangle = R_z(\pm\theta)\,|0\rangle \qquad \theta = \tan^{-1}\left(\frac{g_0 t}{\omega\tau}\right)$$

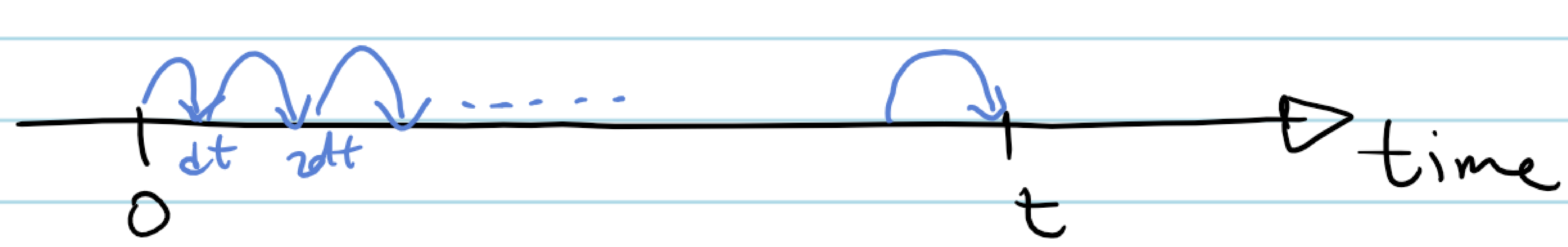
↳ The eigenstates and eigenvalues change with time.

But how do we do the evolution.

For any instance of time, we still have

$$\frac{d|\psi(t)\rangle}{dt} = -\frac{i}{\hbar}H(t)\,|\psi(t)\rangle \longrightarrow |\psi(t+dt)\rangle = e^{-\frac{i}{\hbar}H(t)\,dt}\,|\psi(t)\rangle$$

So $$|\psi(t)\rangle = e^{-\frac{i}{\hbar}H(t-dt)\,dt}\, e^{-\frac{i}{\hbar}H(t-2dt)\,dt} \cdots e^{-\frac{i}{\hbar}H(0)\,dt}\,|\psi(0)\rangle$$



Interaction Picture

This provides a powerful tool to study / understand the dynamics of time-dep. Hamiltonians.

Reminder:	State ψ	Operator A
Schrödingers Picture	$\vert\psi\rangle \xrightarrow{U(t)} \vert\psi(t)\rangle$	A const.
Heisenberg Picture	$\vert\psi\rangle$ const.	$A \rightarrow A(t) = \mathcal{U}^{\dagger}(t)\, A\, \mathcal{U}(t)$

* $\mathcal{U}(t)$ is the evolution operator and for H_0 is given by $e^{-\frac{i}{\hbar}Ht}$.

One of the key challenges with time-dep Hamiltonians (as we saw in the example) is that the evolution is affected time-dep of H on top of the normal time-dep of $\mathcal{U}(t) = e^{-iHt/\hbar}$.

Example: Rotation with moving axis of rotation.

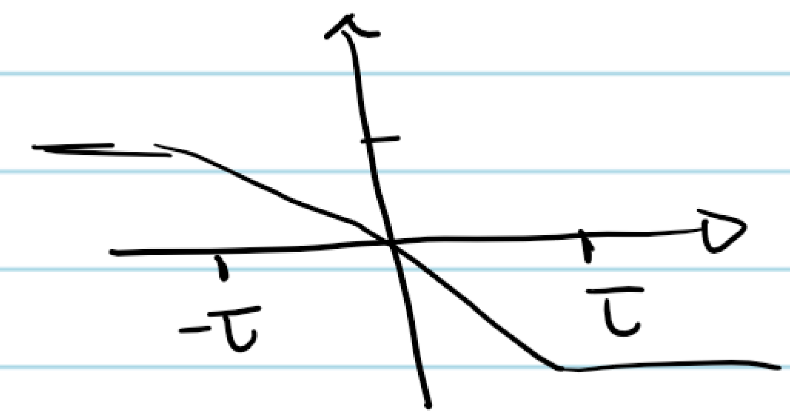
To simplify the problem, it helps to seperate the time-dep of the Hamiltonian and work in a frame $W(t)$ gives the dynamics, i.e.

$$H = H_0 + \lambda W(t)$$

a frame that does not show the dynamic of H_0 (sort of).

Classical example:

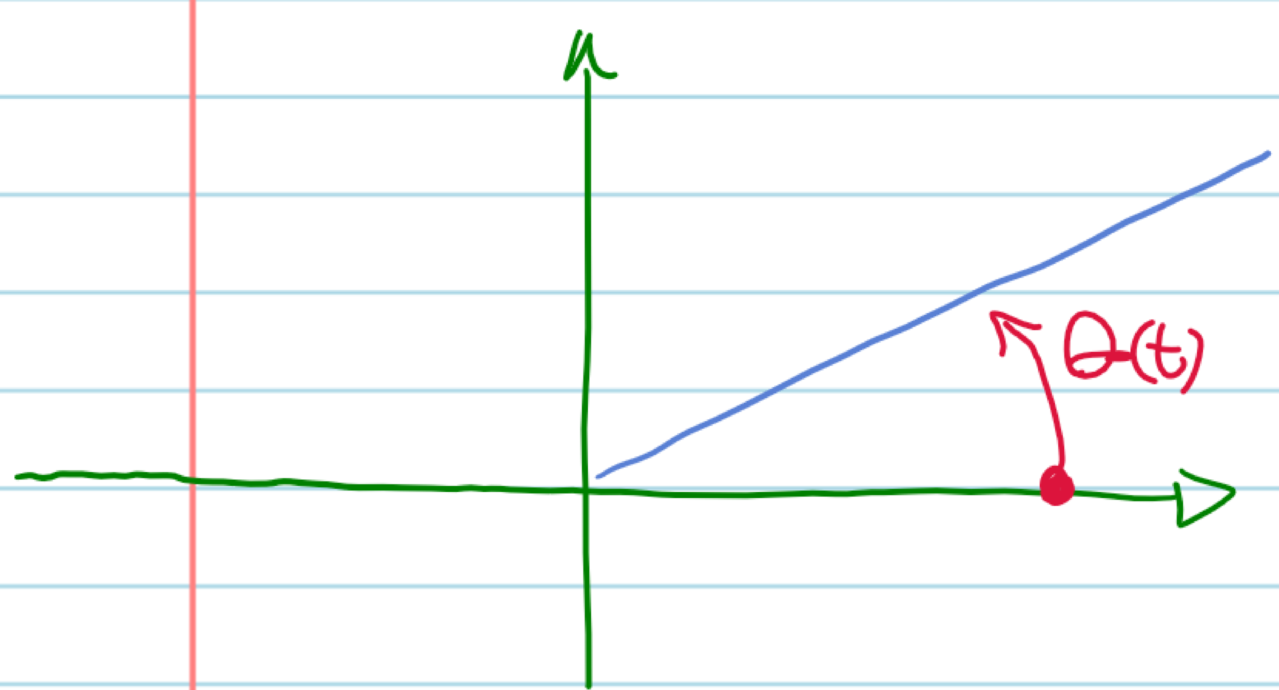
$$H = \frac{\omega}{2}\sigma_z + \frac{\delta t}{2}\sigma_z$$

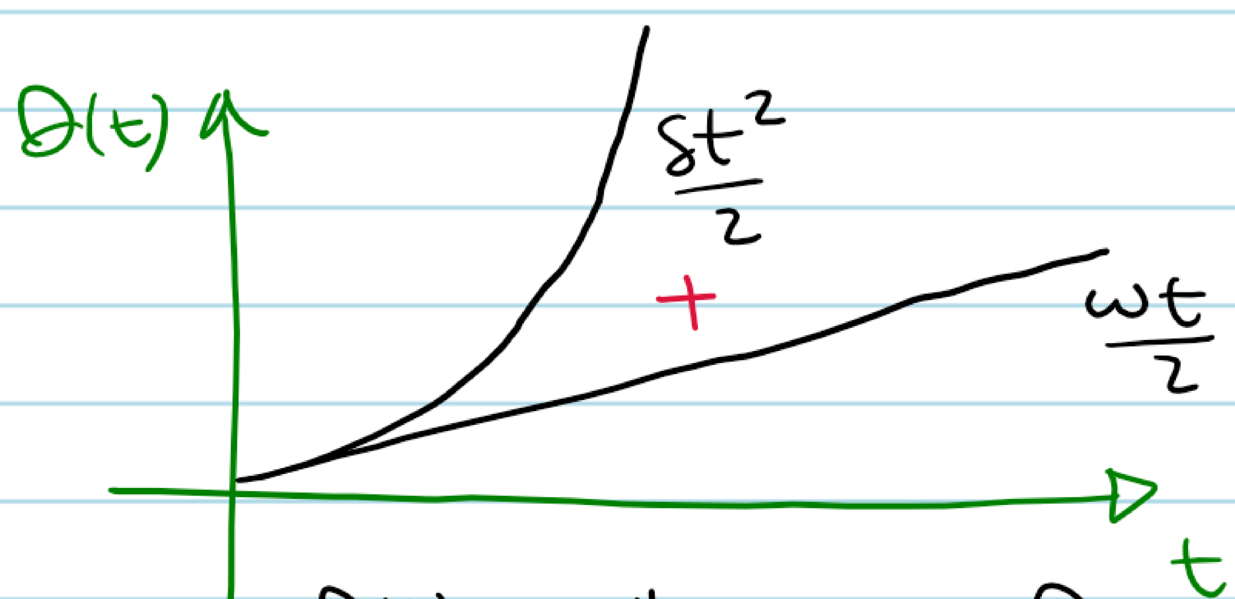


For $|\psi(-\tau)\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$ we get

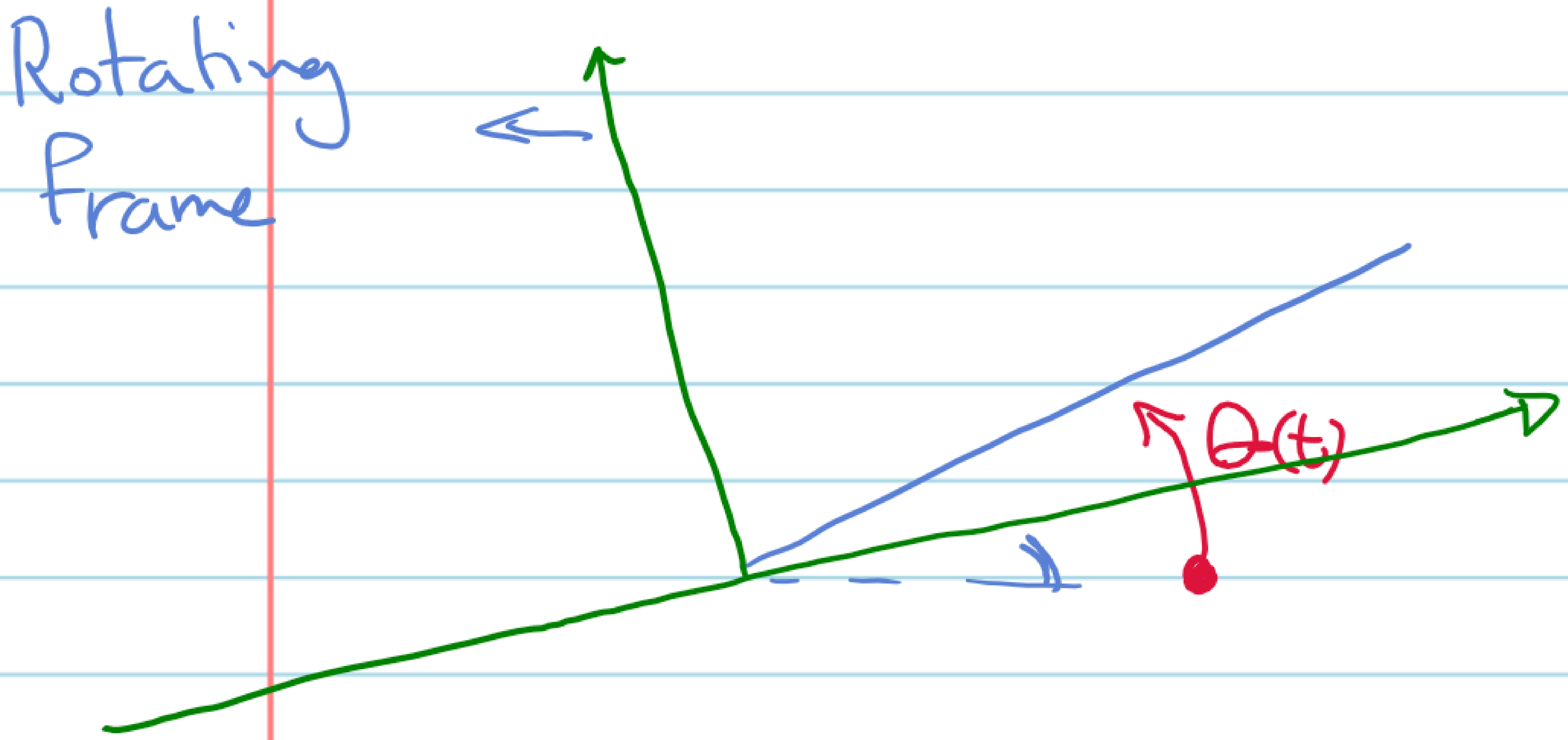
$$\mathcal{U} = R_z(\theta(t)) \qquad \theta(t) = \frac{\omega t}{2} + \frac{\delta t^2}{2}.$$

↳ This looks complicated. But, let's switch to a frame that is rotating with $\frac{\omega t}{2}$. Then





$\theta(t)$ is the sum of the two plots.



In a frame that is rotating with ω, it reduces to the extra part, $\frac{\delta t^2}{2}$.

So, we want to switch to a frame that focuses on the role of the $\lambda W(t)$ rather than H_0.

That would be given by

$$|\psi\rangle \longrightarrow |\psi\rangle_I = U_I(t)\,|\psi\rangle$$

$$U_I(t) = e^{-iH_0 t/\hbar}$$

Now, let's derive the Sch. eq in this frame:

$$\frac{d|\psi(t)\rangle_I}{dt} = \frac{d}{dt}\Big(U_I(t)\,|\psi(t)\rangle\Big) =$$

$$-\frac{iH_0}{\hbar}U_I(t)|\psi(t)\rangle + U_I(t)\underbrace{\frac{d}{dt}|\psi(t)\rangle}_{-\frac{i}{\hbar}H(t)|\psi(t)\rangle}$$

$$= -\frac{i}{\hbar}\Big(-H_0 U_I(t) + U_I(t) H U_I^\dagger(t)\, U_I(t)\Big)|\psi(t)\rangle$$

$$\Rightarrow \frac{d}{dt}|\psi(t)\rangle_I = -\frac{i}{\hbar}\underbrace{\Big(U_I(t) H U_I^\dagger(t) - H_0\Big)}_{H_{eff} \text{ or } H_{int}}|\psi(t)\rangle_I$$

The dynamics in the interaction picture is given by the

$$H_{int} = U_I(t)\big(H - H_0\big)U_I^\dagger = \boxed{\lambda\,\underbrace{U_I(t)\,W(t)\,U_I^\dagger(t)}_{W_I(t)}} \quad *$$

$$\frac{d}{dt}|\psi(t)\rangle_I = \frac{-i}{\hbar}(\lambda W_I(t))\,|\psi(t)\rangle_I \qquad (*)$$

Full evolution operator.

$$|\psi(t)\rangle_I = e^{+\frac{i}{\hbar}H_0 t}\,|\psi(t)\rangle = e^{\frac{i}{\hbar}H_0 t}\,U(t)\,|\psi(0)\rangle$$

$$= \underbrace{e^{\frac{iH_0 t}{\hbar}}\,U(t)\,e^{\frac{-iH_0 t}{\hbar}}}_{U_I(t)}\,|\psi(0)\rangle_I \qquad (**)$$

Evolution in the Int. picture.

But in the int. frame, the operators evolve too:

$$A_I(t) = e^{iH_0 t/\hbar}\,A\,e^{\frac{-iH_0 t}{\hbar}} \qquad (***)$$

$$\frac{dA_I(t)}{dt} = \frac{-i}{\hbar}\left[A_I(t), H_0\right] \qquad (****)$$

Essentially, the evolution of the operators is given by H_0 & the evolution of the state is given by the perturbation in the interaction frame.

Lets compare this to the Schrödingers & Heisenberg pictures.

Comparison of different pictures

	State	Operator
Sch.	$\vert\psi(t)\rangle = \mathcal{U}(t,t_0)\vert\psi(0)\rangle$	A const.
Heisenberg	$\vert\psi\rangle$ const.	$A(t) = U^\dagger(t,t_0)\, A\, U(t,t_0)$
Interaction	$\vert\psi(t)\rangle_I = \mathcal{U}_I(t,t_0)\vert\psi(t)\rangle$ $= U_I(t,t_0)U(t,t_0)\vert\psi(0)\rangle$	$A_I(t) =$ [Do this as an assignment.]

$$U_I(t,t_0) = e^{iH_0(t-t_0)/\hbar}$$

* It seems that the interaction picture maybe more complicated. But it is not. Here's why: (Assume $H = H_0 + \lambda W(t)$)

Evolution

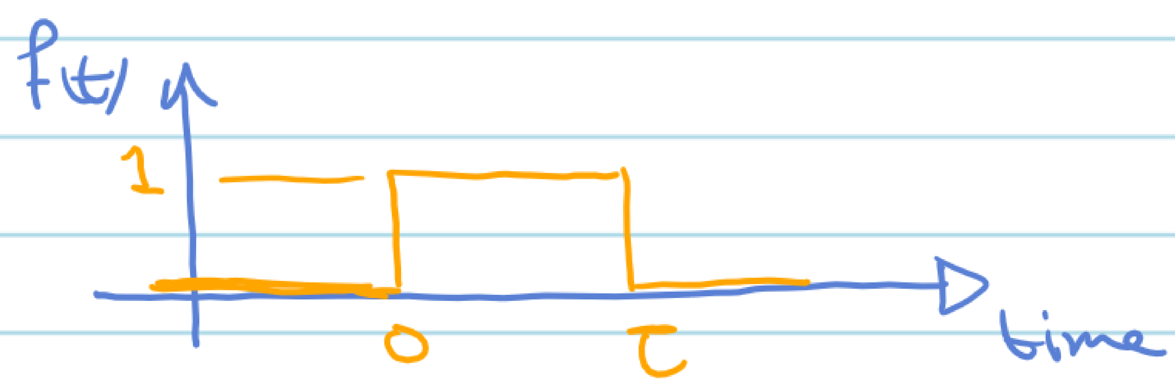
	State	Operator
Sch.	$\frac{d}{dt}\vert\psi(t)\rangle = \frac{-i}{\hbar}(H_0+\lambda W(t))\vert\psi(t)\rangle$	—
Heisenberg	—	$\frac{dA}{dt} = -\frac{i}{\hbar}[A, H_0+\lambda W(t)]$
Int.	$\frac{d}{dt}\vert\psi(t)\rangle_I = \frac{-i}{\hbar} W_I(t)\vert\psi(t)\rangle_I$	$\frac{dA_I}{dt} = -\frac{i}{\hbar}[A_I(t), H_0]$

* $A_I(t) = U_I^\dagger(t)\, A\, U_I(t)$

* $W_I(t) = U_I^\dagger(t)\, W\, U_I(t)$

Example: $H_0 = \frac{\omega}{2}\sigma_z$, $H = \frac{\omega}{2}\sigma_z + \frac{\delta}{2}\sigma_z f(t)$

$f(t)$:



Sch. picture

$[W, H_0] = 0 \rightarrow$ Easy:

$$\frac{d}{dt}|\psi(t)\rangle = -\frac{i}{\hbar}\left(\frac{\omega + \delta f(t)}{2}\sigma_z\right)|\psi(t)\rangle$$

$$|\psi(t)\rangle = e^{-i/\hbar \int \frac{\omega + \delta f(t)}{2}\sigma_z dt}|\psi(0)\rangle$$

$$= e^{-\frac{i}{\hbar}\frac{\omega t}{2}\sigma_z} e^{-\frac{i}{\hbar}\frac{\delta\tau}{2}\sigma_z}|\psi(0)\rangle$$

Int. Picture: $W(t) = f(t)\sigma_z \rightarrow W_I(t) = e^{+\frac{i}{\hbar}H_0 t}\left(f(t)\sigma_z\right)e^{-\frac{i}{\hbar}H_0 t}$

$$= f(t)\sigma_z \qquad [H_0, W] = 0$$

$$\frac{d}{dt}|\psi(t)\rangle_I = -\frac{i}{\hbar}\left(\frac{f(t)\delta}{2}\right)\sigma_z|\psi(t)\rangle_I$$

$$|\psi(t)\rangle_I = e^{-\frac{i}{\hbar}\frac{\delta\tau}{2}\sigma_z}|\psi(t)\rangle_I$$

$\rightarrow$ We only see the effect of $W(t)$ in this frame.

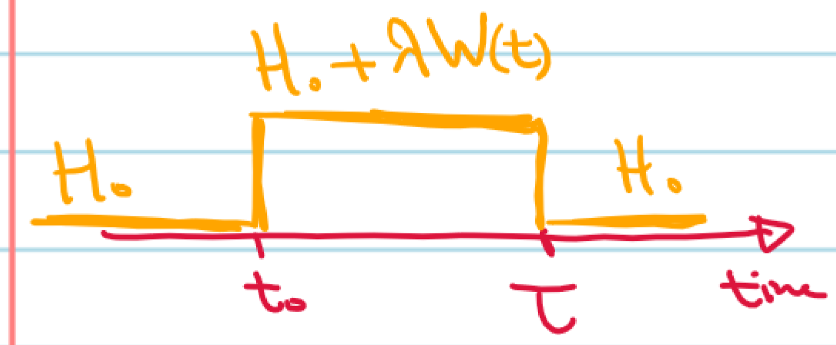
This is only this simple when $[H_0, W(t)] = 0$.

Question: Redo the example for when $W(t) = \frac{\delta f(t)}{2}\sigma_x$.

(Go as far as you can).

Transition Probabilities

Reminder



Assume that we start in an eigenstate before $W(t)$ is turned on. After the perturbation is turned back off, the state is not necessarily in the initial state.

$$|\psi_i(t_0)\rangle = |n\rangle \longrightarrow |\psi_f\rangle = |\psi(t+\tau)\rangle = \sum C_m^{(n)}(\tau)\ |m\rangle$$

This means that, $W(t)$ leads to transitions to other eigenstates.

* Note: In the absence of $\lambda W(t)$, an eigen state would remain unchanged under the evolution of H_0.

The transition probabilities are given by

$$P_{n \to m} = |\langle m| \psi_n(t)\rangle|^2 =$$

$$|\langle m| U(t,t_0)|n\rangle|^2 = |{}_I\langle m| U_I(t,t_0)|n\rangle_I|^2$$

$$= |\langle m| U_I(t,t_0)|n\rangle|^2$$

$$P_{n\to m} = \left|\langle m | \psi_n(t)\rangle\right|^2 = \left|{}_I\langle m| U_I(t,t_0)|n\rangle_I\right|^2$$

$$= \left|\langle m| U_I(t,t_0)|n\rangle\right|^2$$

* Note: $|m\rangle_I = e^{+\frac{i}{\hbar}H_0 t}|m\rangle = e^{\frac{i}{\hbar}E_m t}|m\rangle$

$\Rightarrow \left|\langle m|\hat{O}|n\rangle\right| = \left|{}_I\langle m|\hat{O}|n\rangle_I\right|$

So, to calculate the transition probabilities, we need to calculate

$$P_{n\to m} = \left|\langle m| U_I(t,t_0)|n\rangle\right|^2$$

So, we would need to calculate $U_I(t,t_0)$.

Dyson Series:

$$U_I(t) = e^{\frac{iH_0 t}{\hbar}}\, U(t)\, e^{\frac{-iH_0 t}{\hbar}}, \quad \text{Also, } \frac{d|\psi(t)\rangle_I}{dt} = \frac{-i}{\hbar}\lambda W_I(t)\,|\psi(t)\rangle_I$$

$$\Rightarrow \frac{d}{dt} U_I(t,t_0) = \frac{-i}{\hbar}\lambda W_I(t)\, U_I(t,t_0)$$

$$U_I(t,t_0) = \mathbb{1} - \frac{i}{\hbar}\lambda \int_{t_0}^{t} W_I(t')\, U_I(t',t_0)\, dt'$$

Perturbative expasion:

$$U_I(t,t_0) = U_I^{(0)}(t,t_0) + \lambda U_I^{(1)}(t,t_0) + \lambda^2 U_I^{(2)}(t,t_0) + \dots$$

$$\hookrightarrow U_I^{(0)}(t,t_0) + \lambda U_I^{(1)}(t,t_0) + \lambda^2 U_I^{(2)}(t,t_0) + \dots =$$

$$\mathbb{1} - \frac{i}{\hbar}\lambda \int_{t_0}^{t} W_I(t') \left[U_I^{(0)}(t',t_0) + \lambda U_I^{(1)}(t',t_0) + \lambda^2 U_I^{(2)}(t',t_0) + \dots \right] dt'$$

$\Rightarrow$ Solving for different powers of λ:

$$\lambda^{(0)} \quad U_I^{(0)}(t,t_0) = \mathbb{1}$$

$$\lambda \quad U_I(t,t_0) = \mathbb{1} \underbrace{- \frac{i}{\hbar}\lambda \int_{t_0}^{t} W_I(t')(\mathbb{1})\, dt'}_{U^{(1)}}$$

$$\lambda^2 \quad U^{(2)} = \left(\frac{-i}{\hbar}\right)^2 \lambda^2 \int_{t_0}^{t} W_I(t'') \int_{t_0}^{t''} W(t')\, dt'\, dt''$$

This is known as the dyson series.