# Lecture 6: Continuous spaces and Phase space

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QM I _ Fall 2019
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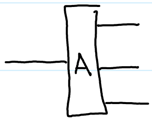
## Example 2

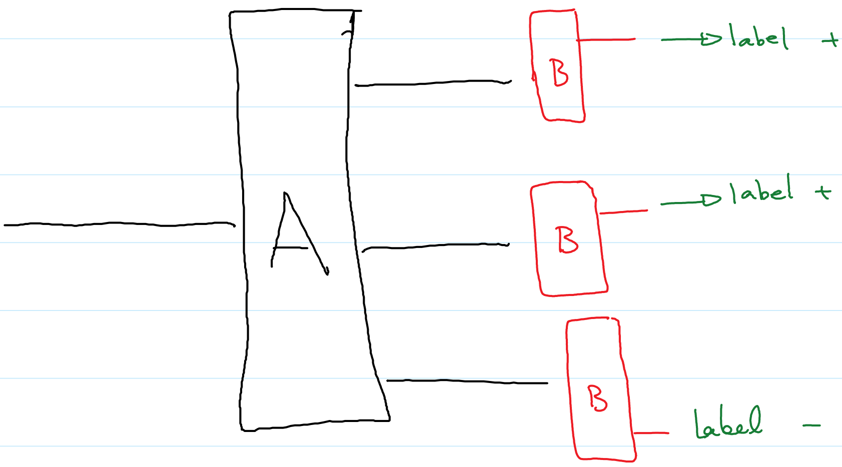
Consider the following setting




$$\{|1\rangle, |2\rangle, |3\rangle\}$$

$$|\psi\rangle = \alpha|1\rangle + \beta|2\rangle + \gamma|3\rangle$$

Also




Take input state to be $|\psi\rangle = \frac{1}{\sqrt{3}}\left[|1\rangle + |2\rangle + |3\rangle\right]$

What happens if $\underline{B}$ is measured?

Probabilities

$+$ $$Pr(+) = |\langle\psi|0\rangle|^2 + |\langle\psi|1\rangle|^2 = \langle\psi|\Pi_0|\psi\rangle + \langle\psi|\Pi_1|\psi\rangle$$
$$= \langle\psi|\Pi_+|\psi\rangle = \frac{2}{3}$$
$$\Pi_+ = \Pi_0 + \Pi_1$$

$-$ $$Pr(-) = |\langle\psi|2\rangle|^2 = \frac{1}{3}$$

States

$-$ $\Rightarrow$ $$\frac{\Pi_-|\psi\rangle}{\sqrt{Pr(-)}} = |2\rangle$$

$+$ $\Rightarrow$ $$\frac{\Pi_+|\psi\rangle}{} = |0\rangle + |1\rangle$$ → Need to

$\sqrt{Pr(-)}$

$+ \Rightarrow \dfrac{\Pi_+ |\psi\rangle}{\sqrt{Pr(+)}} = \dfrac{|0\rangle + |1\rangle}{\sqrt{2}}$ → Need to check something.

## Example 3

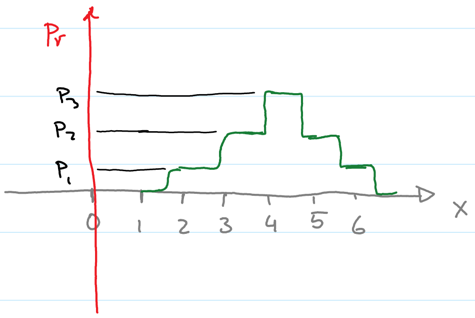

What's the corresponding Hilbert space?

What's the basis?

$\mathcal{H} = \text{Span}\{|x_1\rangle, |x_2\rangle \cdots\}$ → Infinite dimensional Hilbert space.

$\Pi_x = |x\rangle\langle x|$

→ Some initial state




$|\psi_{in}\rangle = \sqrt{P_1}\,|x_2\rangle + \sqrt{P_2}\,|x_3\rangle$
$+ \sqrt{P_3}\,|x_4\rangle + \sqrt{P_2}\,|x_5\rangle + \sqrt{P_1}\,|x_6\rangle$

$Pr(x_3) = \langle\psi_{in}|\,\Pi_{x_3}\,|\psi_{in}\rangle = \sqrt{P_2}\sqrt{P_2}^* = P_2$

A Is $|\psi_{in}\rangle$ unique? or

Is this the only $|\psi_{in}\rangle$ that can give the probability distribution above?

If not, what freedoms are there?

Make one more $|\psi_{in}\rangle$ that is compatible with the

## Probability distribution.

What's the prob. of getting $x \in \{x_2, x_3\}$ ?

$$\Pi_{2,3} = |x_2\rangle\langle x_2| + |x_3\rangle\langle x_3|$$

$$Pr = \langle \psi | \Pi_{2,3} | \psi \rangle$$

Also note that $\sum_i Pr(x_i) = 1$

**What if $\Delta x \neq 1$?**

## * Continous limit

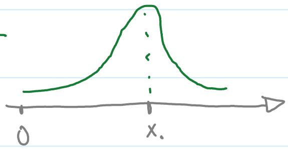
$x \in \mathbb{R}$

$$\mathcal{H} = \text{span}\{|x\rangle : x \in \mathbb{R}\}$$ → We'll come back to this, there's a problem with these states.

$$\Pi_x = |x\rangle\langle x|$$

What's the analoge of the initial state above?

Gaussian distribution ←




$$|\psi_{in}\rangle = \int_{-\infty}^{\infty} dx \sqrt{e^{-\frac{(x-x_0)^2}{2\sigma^2}}} \; |x\rangle$$

- Check that this gives the right probability distribution

- Again, the state compatible with the " "
  is not unique.

What's the prob. distribution?

$$|\psi(t)\rangle = \int dx\ |x\rangle\langle x|\psi(t)\rangle = \int \psi(x,t)\ |x\rangle\ dx$$

Normalization: $\langle\psi(t)|\psi(t)\rangle = \int dx \int dx'\ \psi(x,t)\,\psi^*(x',t)\ \langle x'|x\rangle$

$\langle x|x'\rangle = \delta(x-x') \longrightarrow$ Orthonormality of the basis elements.

$$= \int dx\ |\psi(x,t)|^2 = 1 \longrightarrow \text{Square-integrable functions. (SI)}$$

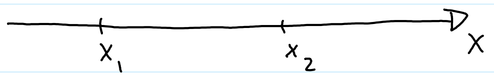
$|x'\rangle \longrightarrow$ Are the basis elements SI ?

$$|x'\rangle = \int dx\ \underbrace{\langle x|x'\rangle}_{\psi_{x'}(x) \,=\, \delta(x-x')}\ |x\rangle$$

$\int dx\ |\delta(x-x')|^2$ ? $\longrightarrow$ Not SI.

## Probability vs Probability density

What's the prob. of getting some outcome between $x_1$ & $x_2$?




$$\Pi_{x_1-x_2} = \int_{x_1}^{x_2} |x\rangle\langle x|\ dx$$

$$Pr([x_1, x_2]) = \dots = \int_{x_1}^{x_2} |\psi(x,t)|^2 \, dx$$

Normalization

$$\int_{-\infty}^{\infty} dx \, |\psi(x,t)|^2 = 1 \longrightarrow \text{So } |\psi(x,t)|^2 \text{ cannot be probability. It has units of } \frac{1}{x}.$$

$Pr(x_i) \longrightarrow$ This is problematic. $\longrightarrow$ Why?

$$x \in [x, x+dx]$$

$Pr([x, x+dx]) = |\psi(x,t)|^2 \, dx \longrightarrow$ Check with the discrete limit.

$|\psi(x,t)|^2 \longrightarrow$ Probability density.

## Change of basis

$\langle \hat{p} \rangle = \langle \psi | \hat{p} | \psi \rangle \longrightarrow$ What's the momentum?

$$|\psi(t)\rangle = \int |p\rangle \underbrace{\langle p | \psi(t) \rangle}_{\tilde{\psi}(p,t)} \, dp$$

$$|\psi\rangle = \sum c_i |i\rangle \longrightarrow |\psi\rangle = \sum \tilde{c}_i |\tilde{i}\rangle$$

$$c_i \xrightarrow{?} \tilde{c}_i$$

$$\tilde{\psi}(p,t) \longrightarrow \psi(x,t)$$

$$\psi(x,t) = \langle x | \psi(t) \rangle = \langle x | \mathbb{1} | \psi(t) \rangle =$$

$$\int dp \; \langle x | p \rangle \langle p | \psi(t) \rangle = \int dp \; \underbrace{\langle x | p \rangle}_{\text{We need this.}} \tilde{\psi}(p,t)$$

$$\hat{X} \longleftrightarrow \hat{P}$$

$$\{ x, p \} = 1 \Longrightarrow [X, P] = i\hbar \mathbb{1}$$

$$\boxed{\langle x | p \rangle \quad ?}$$

A  Prove that

$$e^{iA\lambda} B e^{-iA\lambda} = B + i[A,B] + \frac{i}{2!}[A,[A,B]] + \cdots$$

A  Prove that

$$e^{i\hat{P}a/\hbar} X e^{-i\frac{P_a}{\hbar}} = \hat{X} + a\mathbb{1}$$

How does this equality help?

Consider this state:

$$|\varphi\rangle = e^{-i\frac{a}{\hbar}\hat{P}} |x\rangle$$

$$|\varphi\rangle = e^{\hbar'}\,|x\rangle$$

$$\hat{X}|\varphi\rangle = X\, e^{-i\frac{a}{\hbar}\hat{P}}\,|x\rangle = e^{-i\frac{a}{\hbar}\hat{P}}(\hat{X}+a\mathbb{1})|x\rangle$$

$$= e^{-i\frac{a}{\hbar}\hat{P}}(x+a)|x\rangle = (x+a)\underbrace{e^{-i\frac{a}{\hbar}\hat{P}}|x\rangle}_{|\phi\rangle}$$

So, $|\phi\rangle$ is an eigenstate of $\hat{X}$ with eigenvalue $x+a$

$$\Rightarrow |\phi\rangle = |x+a\rangle$$

(A) Is this always true (for any operator)?

Now let's calculate

$$\langle p|\, e^{-i\frac{a}{\hbar}\hat{P}}\,|x\rangle = e^{-i\frac{ap}{\hbar}}\langle p|x\rangle$$

$$= \langle p|x+a\rangle$$

Let's take $\langle p|x\rangle = f(x,p)$. The equality above states that

$$e^{-iap/\hbar}\, f(x,p) = f(x+a,p)$$

This indicates

$$f(x,p) = A e^{-i\frac{x.p}{\hbar}}$$

To see this take $a \to \delta x \ll 1$

$\underline{LHS}$ : $f(x,p) + \frac{df}{dx}\,\delta x$

$\underline{RHS}$ : $f(x,p) - i\,p/\hbar\;\delta x$

$$\Rightarrow \frac{df}{dx} = -ip/\hbar \Rightarrow f(x,p) = A e^{-ip\,x/\hbar}$$

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So

$$-i/\hbar\; xp$$

So
$$\langle p|x\rangle = A\, e^{-\frac{i}{\hbar}xp}$$

For the change of basis, this implies

$$|\psi\rangle = \int dp \underbrace{\langle p|\psi\rangle}_{\tilde{\psi}(p)} |p\rangle$$

$$\tilde{\psi}(p) = \langle p|\psi\rangle = \int \langle p|x\rangle \underbrace{\langle x|\psi\rangle}_{\psi(x)} dx \quad = A\int e^{-\frac{i}{\hbar}xp}\,\psi(x)\,dx$$

So, $\tilde{\psi}(p)$ is the Fourier transform of $\psi(x)$.

Ⓐ Find the factor $A$ in transformation.

## Operators in position space

$\hat{O}|\psi\rangle = |\phi\rangle \longrightarrow$ In position space

$$\langle x|\phi\rangle = \langle x|\hat{O}|\psi\rangle$$

We sometime use this notation

$$O\,\psi(x) \longrightarrow \varphi(x)$$

or

$$O\langle x|\psi\rangle = \langle x|\hat{O}|\psi\rangle .$$

We use the $\hat{O}$ for the operator in the original $\mathcal{H}$ space and $O$ for . . acting on $\psi(x)$ i.e the representation in position.

Let's find p ? $\langle x|\hat{p}|\psi\rangle$

$$\langle x|\hat{p}|\psi\rangle = \langle x|\int dp \; p \; |p\rangle\langle p| \; |\psi\rangle =$$

$$= \int dp \; e^{ixp/\hbar} \; \tilde{\psi}(p) \, p =$$

$$= \int (-i\hbar \, \partial_x) \, e^{ixp/\hbar} \; \tilde{\psi}(p) \, dp$$

$$= (-i\hbar \, \partial_x) \; \psi(x)$$

This means

$$P \, \psi(x) = -i\hbar \, \partial_x \psi(x)$$

Similarly, we can calculate the action of any operation.

$$\hat{K} = \frac{\hat{P}^2}{2m} \; : \; K \, \psi(x) = -\frac{\hbar^2}{2m} \partial_x^2 \, \psi(x)$$

$$\hat{H} = \hat{P}^2/2m + V(\hat{X}) \; : \; H\psi(x) = \left(-\frac{\hbar^2}{2m} \partial_x^2 + V(x)\right) \psi(x)$$

A $\rightarrow$ $\hat{L} = \hat{R} \times \hat{P}$

→ Schrödinger's equation

The evolution of the state is given by:

$$\frac{d}{dt} |\psi(t)\rangle = \frac{-i}{\hbar} H(t) \, |\psi(t)\rangle$$

$$\frac{d}{dt}|\psi(t)\rangle = \frac{-i}{\hbar} H(t)\,|\psi(t)\rangle$$

Now lets see how this looks like in the position basis.

$$\langle x|\frac{d}{dt}|\psi(t)\rangle = \frac{-i}{\hbar}\langle x|\,H(t)\,|\psi(t)\rangle$$

$$\hat{H} = \frac{\hat{P}^2}{2m} + V(\hat{X})$$

$$\frac{d}{dt}\psi(x,t) = \frac{-i}{\hbar}\left[\langle x|\frac{\hat{P}^2}{2m}|\psi(t)\rangle + \langle x|V(\hat{x})|\psi(t)\rangle\right]$$

$$= \frac{-i}{\hbar}\left(-\frac{\hbar^2}{2m}\partial^2/\partial x^2\,\psi(x,t) + V(x)\,\psi(x,t)\right]$$

$$\Longrightarrow \boxed{\frac{d}{dt}\psi(x,t) = \left(\frac{i\hbar}{2m}\frac{\partial^2}{\partial x^2} - \frac{i}{\hbar}V(x)\right)\psi(x,t)}$$

Reminder:

$$|\psi(t)\rangle = \int_{-\infty}^{+\infty}\psi(x,t)\,|x\rangle\,dx$$

(A) Do the similar calculation in momentum space

→ Plane waves and wave packets

Take the state to be $|\psi\rangle = |x\rangle$ at $t=0$.

Find the $\psi(x,t)$ when there's no force (free particle)

$$\frac{d}{dt}\psi(x,t) = \frac{i\hbar}{}\,\frac{\partial^2}{}\,\psi(x,t)$$

$$\frac{d}{dt}\psi(x,t) = \frac{i\hbar}{2m}\frac{\partial^2}{\partial x^2}\psi(x,t)$$

$$\psi(x,t) = \psi(x)\,\varphi(t) \implies$$

$$\frac{i}{\varphi(t)}\frac{d}{dt}\varphi(t) = \frac{-\hbar}{2m}\frac{1}{\psi(x)}\frac{\partial^2}{\partial x^2}\psi(x) = \omega$$

$$\varphi(t) = \varphi_0 e^{-i\omega t}, \quad \psi(x) = e^{\pm i\sqrt{\frac{2m}{\hbar}\omega}\,x} = e^{\pm i k x}$$

$$\psi(x,t) = A\,e^{i(kx-\omega t)} + B\,e^{-i(kx+\omega t)}$$

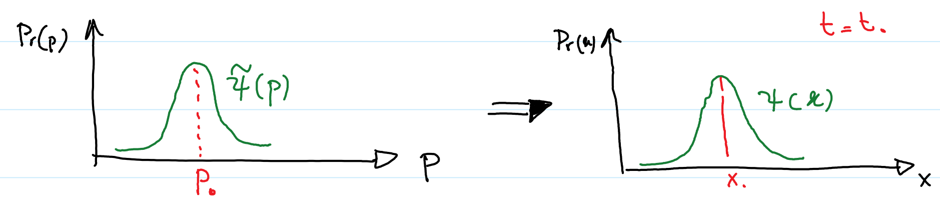
$\longrightarrow$ Solutions are plane waves moving to the left or right in time

(A) What's the dispersion relation?

(A) Find $\tilde{\psi}(p,t)$.

(A) What's the $Pr(x)$ for this state? What's wrong with it?
Does it change with time?
What does such a wave function mean (say for electrons)?

Now, consider the following distribution



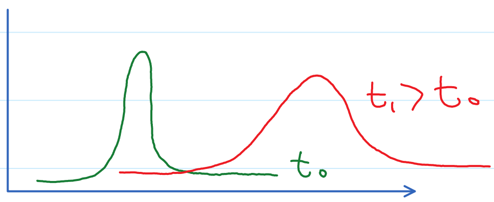

This is a combination of plane waves which is localized both in space & momentum.

(A) Take $\psi(x) = e^{-(x-x_0)^2/2\sigma^2}$ for $t=0$ and find

a) $\tilde{\psi}(p,0)$ b) $\psi(x,t)$

c) Does this have the normalization problem of plane waves?

d) Show that the probability $Pr(x)$ spreads out in time.




→ Probability current and Conservation of Probability

$$\langle\psi(t)|\psi(t)\rangle = \langle\psi(0)|U^\dagger U|\psi(0)\rangle = \langle\psi(0)|\psi(0)\rangle = 1$$

Equivalently, we can check that: $\frac{d}{dt} \qquad = 0$

$$\frac{d}{dt}\langle\psi(t)|\psi(t)\rangle = \left(\frac{d}{dt}\langle\psi(t)|\right)\left(|\psi(t)\rangle\right) + \langle\psi(t)|\left(\frac{d}{dt}|\psi(t)\rangle\right)$$

We have: $\frac{d}{dt}|\psi(t)\rangle = \frac{-i}{\hbar} H |\psi(t)\rangle$

$$\frac{d}{dt}\langle\psi(t)|\psi(t)\rangle = \dots = 0$$

$\frac{d}{dt}$ ...

↳ Check to see that if $U$ was not unitary, this wouldn't be true.

Let's take a closer look at this in the position space:

$$\langle \psi(t) | \psi(t) \rangle = \int dx \, |\psi(x,t)|^2 = 1$$

$$i\hbar \frac{\partial \psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x,t)}{\partial x^2} + V(x)\, \psi(x,t) \qquad \times \psi^*(x,t)$$

$$- \Big( \qquad \qquad '' \qquad \qquad \Big)^* \qquad \times \psi(x,t)$$

$$\Rightarrow i\hbar \frac{\partial}{\partial t} \underbrace{\left( \psi^*(x,t)\, \psi(x,t) \right)}_{\displaystyle \rho = |\psi(x,t)|^2 \text{ prob. density}} = -\frac{\hbar^2}{2m} \left( \psi^*(x,t) \frac{\partial^2}{\partial x^2} \psi(x,t) - \psi(x,t) \frac{\partial^2}{\partial x^2} \psi^*(x,t) \right)$$

We also introduce current density

$$J = \frac{i\hbar}{2m} \left[ \psi^*(x,t) \frac{\partial}{\partial x} \psi(x,t) - \psi(x,t) \frac{\partial}{\partial x} \psi^*(x,t) \right]$$

$$\Longrightarrow \quad \frac{\partial \rho(x,t)}{\partial t} + \frac{\partial}{\partial x} J = 0 \qquad (\text{eq} *)$$

(A) <u>3D</u>

a) Find P in 3 dimensional space: $(P_x, P_y, P_z)$

b) '' Schrodinger's eq. in 3D

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c) " the probability conservation above (eq *)

in 3D.