Lecture 5: Postulates of QM

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QM I _ Fall 2019

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* The notion we learned from SGE

* Mathematics of the model

* Evolution

* Statement of the postulates

Summary

① State $\begin{pmatrix}\alpha \\ \beta\end{pmatrix} \in \mathbb{C}^2 \rightarrow$ A vector space with some inner-product.

Also $|\alpha|^2 + |\beta|^2 = 1 \Rightarrow$ To have normal probabilities.

② Measurement $\rightarrow$ Outcomes $\rightarrow \vec{v}_i \rightarrow$ Some basis

Probability of outcomes $\Pr(v_i) = |\vec{v}_{in}^{\,\dagger} \cdot v_i|^2$

$$= \vec{v}_{in}^{\,\dagger} \cdot \Pi_i \cdot \vec{v}_i$$

with $\Pi_i = \vec{v}_i \cdot \vec{v}_i^{\,\dagger}$

③ Physical quantities

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For a quantity $\hat{A}$ that has possible outcomes of $\vec{\nu}_i$ and that each $\vec{\nu}_i$ would point that A is a_i we get

$$\langle A \rangle = \sum_i \Pr(\nu_i)\, a_i =$$

$$\vec{\nu}_i^{\,\dagger} \cdot \left(\sum_i a_i \Pi_i \right) \cdot \vec{\nu}_i$$

We use $\sum a_i \Pi_i$ as the representation of the physical quantity A.

$$A \longrightarrow \text{An operator} \quad \sum_i a_i \Pi_i$$

Also note that a_i are real so A makes a Hermitian Operator.

Some notation

$$\vec{\nu} = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \longrightarrow |\nu\rangle$$

$$\vec{\nu}^{\dagger} = (\alpha^* \;\; \beta^*) \longrightarrow \langle \nu |$$

$$\vec{\nu}^{\dagger} \cdot \vec{\omega} \longrightarrow \langle \nu | \omega \rangle$$

$$\vec{\nu} \cdot \vec{\omega}^{\dagger} \longrightarrow |\nu\rangle\langle\omega|$$

$$\Pi_i \longrightarrow |\nu_i\rangle\langle\nu_i|$$

$$Pr(\nu_i) \longrightarrow \langle\nu_{in}|\Pi_i|\nu_{in}\rangle$$

$$\hat{A} = \sum_i a_i \Pi_i \longrightarrow \hat{A} = \sum_i a_i |\nu_i\rangle\langle\nu_i|$$

Basis

$\{\vec{\nu}_i\}$:

$$\vec{\nu}_i^{\dagger} \cdot \nu_j = \delta_{ij} \qquad \text{Orthonormal} \quad (1$$

$$\sum \vec{\nu}_i^{\dagger} \cdot \nu_i = \mathbb{1} \qquad \text{Complete} \quad (2$$

Show that: $\forall \vec{\varphi} \in \mathcal{H}$

$$\vec{\varphi} = \sum \alpha_i \vec{\nu}_i \qquad \alpha_i = \vec{\nu}_i^{\dagger} \cdot \vec{\varphi}$$

$\Rightarrow \{|\nu_i\rangle\}$

$$\langle\nu_i|\nu_j\rangle = \delta_{ij} \quad (1$$

$$\sum |\nu_i\rangle\langle\nu_j| = \mathbb{1} \quad (2$$

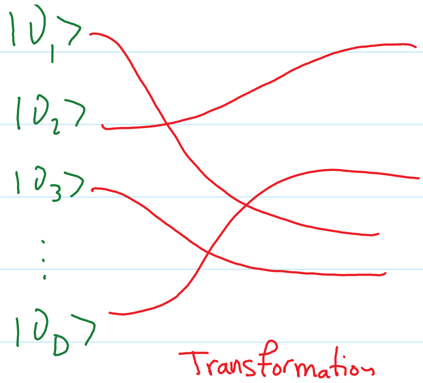
$$|\varphi\rangle = \sum_i |\nu_i\rangle\langle\nu_i|\varphi\rangle = \sum \alpha_i |\nu_i\rangle$$

Change of basis

$$\{|\nu_i\rangle\} \longrightarrow \{|\omega_i\rangle\}$$

$|\nu_1\rangle$

$$|\omega_1\rangle = \sum_i \langle\nu_i|\omega_1\rangle |\nu_i\rangle$$



Transformation

$$|w_1\rangle = \sum_i \langle v_i|w_1\rangle |v_i\rangle$$

$$|w_2\rangle = \sum_i \langle v_i|w_2\rangle |v_i\rangle$$

$$\vdots$$

$$|w_D\rangle = \sum_i \langle v_i|w_D\rangle |v_i\rangle$$

$$S = \sum_i |w_{f(i)}\rangle\langle v_i| \longrightarrow S = \sum_i |w_i\rangle\langle v_i|$$

Reorder $\{w_i\}$

$$\begin{pmatrix} |w_1\rangle \\ |w_2\rangle \\ \vdots \\ |w_D\rangle \end{pmatrix} = S \cdot \begin{pmatrix} |v_1\rangle \\ |v_2\rangle \\ \vdots \\ |v_D\rangle \end{pmatrix} \longrightarrow |w_j\rangle = \sum_i \langle v_i|w_j\rangle |v_i\rangle$$

$$S_{ij} = \langle v_i|w_j\rangle$$

Show that S defined here changes the basis:

Method 1

$$S = \sum_{ij} \langle v_i|w_j\rangle |v_i\rangle\langle v_j|$$

$$S|v_\ell\rangle = \sum_{ij} \langle v_i|w_j\rangle |v_i\rangle \underbrace{\langle v_j|v_\ell\rangle}_{\delta_{\ell j}}$$

$$= \sum_i \langle v_i|w_\ell\rangle |v_i\rangle = |w_\ell\rangle$$

Method 2

$$|w_j\rangle = \Big(\overbrace{\sum_i |w_i\rangle\langle v_i|}^{S}\Big)|v_j\rangle$$

$$S = \sum_i |w_i\rangle\langle v_i| = \sum_{ij} \langle v_j|w_i\rangle |v_j\rangle\langle v_i|$$

$$S = \sum_i |\omega_i\rangle\langle v_i| = \sum_{ij} \langle v_j|\omega_i\rangle |v_j\rangle\langle v_i|$$

$$S_{ji} = \langle v_j|\omega_i\rangle$$

Matrix Representation

$$M = \begin{bmatrix} m_{11} & m_{12} & \cdots \\ m_{21} & \ddots & \\ \vdots & & \ddots \end{bmatrix} \qquad m_{ij} = \vec{v}_i^{\,\dagger} \cdot M \cdot \vec{v}_j$$

$$= \langle v_i | M | v_j\rangle$$

$$M = \sum_{ij} |v_i\rangle\langle v_i| M |v_j\rangle\langle v_j| = \sum_{ij} m_{ij} |v_i\rangle\langle v_j|$$

Change of basis

$$M = \sum m_{ij} |v_i\rangle\langle v_j| \stackrel{S}{\Rightarrow} \sum m_{ij} |\omega_i\rangle\langle\omega_j| = \sum \underbrace{m_{ij}}_{\langle\omega_i|M|\omega_j\rangle} \sum_{k,l} \langle\omega_j|v_l\rangle\langle v_k|\omega_i\rangle |v_k\rangle\langle v_l|$$

$$= \sum_{k,l} \langle v_k|M|v_l\rangle |v_k\rangle\langle v_l|$$

$\rightarrow$ This is know as change of Basis.

Example. $S = |z+\rangle\langle x+| + |z-\rangle\langle x-|$

$$M_x = |x+\rangle\langle x+| - |x-\rangle\langle x-|$$

Change the basis :

Evolution

$$|\psi\rangle \longrightarrow |\phi\rangle = \hat{O}|\psi\rangle$$

Going from one state to another, we need to make sure that

Starting from $\langle\psi|\psi\rangle = 1$ we get

$$\langle\phi|\phi\rangle = 1 \Rightarrow \langle\psi|\hat{O}^\dagger \hat{O}|\psi\rangle = \langle\psi^\dagger|\psi\rangle = 1$$

If this were to be true for any state $\hat{O}^\dagger \hat{O} = \mathbb{1}$

$\hat{O}$ has to be unitary

We often use $\hat{U}$ to show unitary operators.

$$UU^\dagger = U^\dagger U = \mathbb{1}$$

Unitary operators are normal operators and therefore

there exists $U = \sum_i u_i |i\rangle\langle i|$

A1 Show that $u_i = e^{i\phi}$.

$\longrightarrow$ This means that we can write U as

$$U = e^{i\hat{A}}$$ for some Hermitian operator $\hat{A}$.

A2 Find $\hat{A}$.

Inspired by classical mechanics

Where Hamiltonian is the generator for time evolution, we can take $A = H(t)$

$$\Rightarrow U(t) = e^{-iHt\alpha} \rightarrow e^{-iHt/\hbar}$$

$$|\psi(t)\rangle = e^{-iHt/\hbar} |\psi(0)\rangle$$

$$\Rightarrow \frac{d}{dt}|\psi(t)\rangle = \frac{-iH}{\hbar} e^{-iHt/\hbar} |\psi(0)\rangle$$

$$\Rightarrow \frac{d}{dt}|\psi(t)\rangle = -\frac{iH}{\hbar}|\psi(t)\rangle$$

* Remark $\Rightarrow$ $A = Ht/\hbar$ would only works for time independant Hamiltonians.

* This is not a proof for the Schrodinger's equation. It is only meant to help you gain some intuition

* More formaly, one can take infinitesimal transformations and find the generators more formally.

$$U_\varepsilon = \mathbb{1} + i\varepsilon A \qquad \varepsilon \rightarrow 0$$

$$U_\varepsilon^\dagger U_\varepsilon = \mathbb{1} = \mathbb{1} + i\varepsilon\left(A - A^\dagger\right) + O(\varepsilon^2)$$

$$\Rightarrow A = A^\dagger \longrightarrow \text{Should be Hermitian.}$$

This is called the Generator of the transformation.

Transformation from infinitesimal transformation

$$\forall\theta: \quad U_\theta = \left(U_{\theta/N}\right)^N : \lim_{N\to\infty} \left(U_{\frac{\theta}{N}}\right)^N = \left(\mathbb{1} + i\frac{\theta}{N}A\right)^N$$

$$= e^{i\theta A}$$

Transformation of operations

$$|i\rangle \longrightarrow U|i\rangle = |\tilde{i}\rangle$$

$$\hat{A} \longrightarrow ?$$

$\langle \hat{A} \rangle$ should be the same:

$$\langle i|A|i\rangle \longrightarrow \langle \tilde{i}|A|\tilde{i}\rangle = \langle i|U^\dagger A U|i\rangle$$

So equivalently, we could

$$\begin{cases} |i\rangle \longrightarrow |i\rangle \\ \hat{A} \longrightarrow U^\dagger A U \end{cases}$$

Note: For a transformation, either the state or the operator is transformed, not both.

Infinitesimal transformation of operators

$\varepsilon \to 0$

$$\hat{O} \longrightarrow \tilde{O} = U_\varepsilon^\dagger \hat{O} U_\varepsilon = \left(\mathbb{1} - i\frac{\varepsilon}{\hbar}\hat{A}\right)\hat{O}\left(\mathbb{1} + i\frac{\varepsilon}{\hbar}\hat{A}\right) \quad \text{for } U_\varepsilon = e^{i\frac{\varepsilon}{\hbar}\hat{A}}$$

$$= \hat{O} + \underbrace{\frac{i\varepsilon}{\hbar}\left[\hat{O}, A\right]} + O(\varepsilon^2)$$

where

$$[A, B] = AB - BA$$

This is similar to what we have in classical mechanics.

Generators of transformation and $\{\ \} \to [\ ]$

Reminder: Transformation in classical mechanics

$$X = \{q_i, p_i\}$$

$$X(0) = \{q_i(0), p_i(0)\} \to X(t)$$

$$\frac{\partial X_i(t)}{\partial t} = \{X_i(t), H\}$$

$\{\ ,\ \}$ is the Poisson Bracket

The constraint is that

$$\{\underset{Y_i}{X_i(t)}, X_j(t)\} = \{X_i(0), X_j(0)\} = \Omega_{ij}$$

See Symplectic transformations in Hamiltonian mechanics.

$$Y_i = X_i + \varepsilon f_i(X) \to \text{Infinitesimal transformation}$$

after some calculations

$$\to \varepsilon f_i(X) = \varepsilon \Omega_{ij} \frac{\partial G}{\partial X_j} = \varepsilon \{X, G\}$$

G is referred to as the generator of the transformation.

$$\frac{i}{\hbar}[\hat{O}, \hat{A}] = \{O, A\}$$

Postulates

① State of the system

At each point in time, the state of the system is vector in a Hilbert space, ie $|\psi(t)\rangle \in \mathcal{H}$

such that $\langle\psi(t)|\psi(t)\rangle = 1$.

This means that $|\psi(t)\rangle$ contains all the information of the system.

② Observables & Physical Quantities

Any observable is assocciated/described by a Hermitian operator.

$$\hat{O} = \sum_i o_i |i\rangle\langle i| \rightarrow$$ o_i are the possible outcomes and $|i\rangle$ form a complete orthonormal

basis.

③ Measurement

Measurement of $\hat{O}$ on some state $|\psi\rangle$ gives some outcomes in each try. The probability of each outcome i is given by

$$Pr(i) = |\langle\psi|i\rangle|^2 = \langle\psi|\Pi_i|\psi\rangle$$

Also the state after the measurement is given by

$$|\varphi\rangle = \frac{\Pi_i|\psi\rangle}{\sqrt{Pr(i)}}$$

④ Evolution

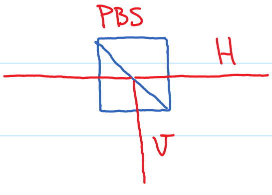
The evolution of the state $|\psi(t)\rangle$ is given by

$$\frac{d}{dt}|\psi(t)\rangle = \frac{1}{i\hbar} H(t)|\psi(t)\rangle$$

with $H(t)$ the Hamiltonian of the system.

A couple of more examples:

① Polarization of light

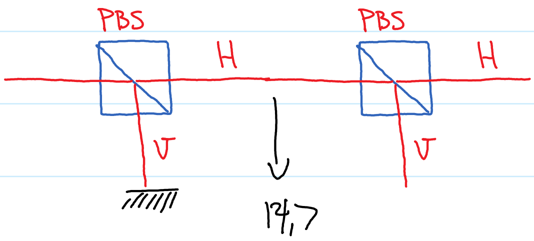


$$H \rightarrow |H\rangle = \begin{pmatrix}1\\0\end{pmatrix}$$

$$V \to |V\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Measurement $\to$ $\Pi_H = |H\rangle\langle H|$

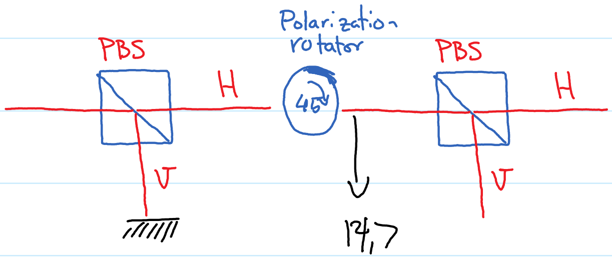
$$\Pi_V = |V\rangle\langle V|$$



$$|\psi_1\rangle = |H\rangle$$

$$Pr(H) = \langle\psi_1|\,\Pi_H\,|\psi_1\rangle = 1$$

$$Pr(V) = \langle\psi_1|\,\Pi_V\,|\psi_1\rangle = 0$$



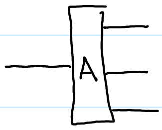
$$|\psi_1\rangle = R(45^\circ)\,|H\rangle = \begin{pmatrix} 1 \\ 1 \end{pmatrix}/\sqrt{2} = |45^\circ\rangle$$

$$Pr(H) = \langle\psi_1|\,\Pi_H\,|\psi_1\rangle = \frac{1}{2}\left|(1 \ \ 1)\cdot\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right|^2 = 1/2$$

$$Pr(V) = \langle\psi_1|\,\Pi_V\,|\psi_1\rangle = \frac{1}{2}\left|(1 \ \ 1)\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right|^2 = 1/2$$

Example 2

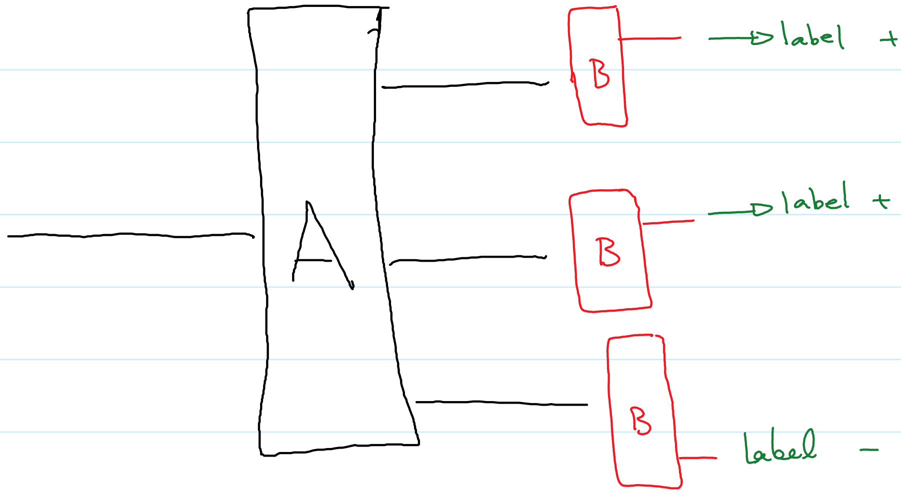
Consider the following setting



$$\{|1\rangle, |2\rangle, |3\rangle\}$$

$$|\psi\rangle = \alpha|1\rangle + \beta|2\rangle + \gamma|3\rangle$$

Also



Take input state to be $|\psi\rangle = \frac{1}{\sqrt{3}}\left[|1\rangle + |2\rangle + |3\rangle\right]$

What happens if $\underline{B}$ is measured?

Probabilities

$+ \qquad \Pr(+) = |\langle\psi|0\rangle|^2 + |\langle\psi|1\rangle|^2 = \langle\psi|\Pi_0|\psi\rangle + \langle\psi|\Pi_1|\psi\rangle$

$$= \langle\psi|\Pi_+|\psi\rangle = \frac{2}{3}$$

$$\Pi_+ = \Pi_0 + \Pi_1$$

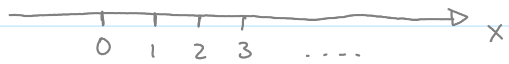
$- \qquad \Pr(-) = |\langle\psi|2\rangle|^2 = \frac{1}{3}$

States

$- \quad \Rightarrow \quad \frac{\Pi_-|\psi\rangle}{\sqrt{\Pr(-)}} = |2\rangle$

$+ \quad \Rightarrow \quad \frac{\Pi_+|\psi\rangle}{\sqrt{\Pr(+)}} = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$ → Need to check something.

Example 3



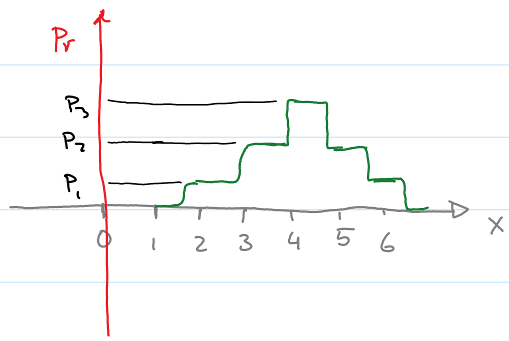
What's the corresponding Hilbert space ?

What's the basis ?

$$\mathcal{H} = \text{Span}\{ |x_1\rangle , |x_2\rangle \ \cdots \ \} \rightarrow \text{Infinite dimensional Hilbert space.}$$

$$\Pi_x = |x\rangle\langle x|$$

→ Some initial state



$$|\psi_{in}\rangle = \sqrt{P_1}\ |x_2\rangle + \sqrt{P_2}\ |x_3\rangle + \sqrt{P_3}\ |x_4\rangle + \sqrt{P_2}\ |x_5\rangle + \sqrt{P_1}\ |x_6\rangle$$

$$\Pr(x_3) = \langle \psi_{in} |\ \Pi_{x_3}\ | \psi_{in}\rangle = \sqrt{P_2}\sqrt{P_2}^{\,*} = P_2$$

A Is $|\psi_{in}\rangle$ unique ? or

Is this the only $|\psi_{in}\rangle$ that can give the probability distribution above ?

If not, what freedoms are there ?

Make one more $|\psi_{in}\rangle$ that is compatible with the Probability distribution.

What's the prob. of getting $x \in \{x_2, x_3\}$?

$$\Pi_{23} = |x_2\rangle\langle x_2| + |x_3\rangle\langle x_3|$$

$$Pr = \langle\psi|\Pi_{23}|\psi\rangle$$

Also note that $\sum_i Pr(x_i) = 1$

<u>What if $\Delta x \neq 1$?</u>

* <u>Continous limit</u>

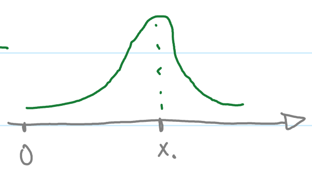
$x \in \mathbb{R}$

$\mathcal{H} = \text{span}\{|x\rangle : x \in \mathbb{R}\}$ → We'll come back to this, there's a problem with these states.

$\Pi_x = |x\rangle\langle x|$

What's the analouge of the initial state above?

Gaussian distribution ←



$$|\psi_{in}\rangle = \int_{-\infty}^{\infty} dx \sqrt{e^{-\frac{(x-x_0)^2}{2\sigma^2}}} \; |x\rangle$$

- Check that this gives the right probability distribution

- Again, the state compatible with the " "
  is not unique.

What's the prob. distribution?

$$|\psi(t)\rangle = \int dx \; |x\rangle\langle x|\psi(t)\rangle = \int \psi(x,t)\, |x\rangle \, dx$$

Normalization: $\langle \psi(t)|\psi(t)\rangle = \int dx \int dx' \; \psi(x,t)\,\psi^*(x',t) \; \langle x'|x\rangle$

$\langle x|x'\rangle = \delta(x-x') \longrightarrow$ Orthonormality of the basis elements.

$$= \int dx \; |\psi(x,t)|^2 \quad = 1 \longrightarrow \text{Square-integrable functions. (SI)}$$

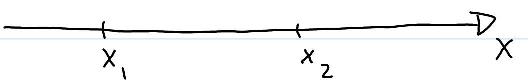
$|x'\rangle \longrightarrow$ Are the basis elements SI ?

$$|x'\rangle = \int dx \; \underbrace{\langle x|x'\rangle}_{\psi_{x'}(x) \, = \, \delta(x-x')} \; |x\rangle$$

$$\int dx \, |\delta(x-x')|^2 \quad ? \longrightarrow \text{Not SI.}$$

Probability vs Probability density

What's the prob. of getting some outcome between x_1 & x_2?



$\int^{x_2}$

$$\Pi_{x_1 - x_2} = \int_{x_1}^{x_2} |x\rangle\langle x| \, dx$$

$$Pr([x_1, x_2]) = \dots = \int_{x_1}^{x_2} |\psi(x,t)|^2 \, dx$$

Normalization

$\int_{-\infty}^{\infty} dx \, |\psi(x,t)|^2 = 1 \longrightarrow$ So $|\psi(x,t)|^2$ cannot be probability. It has units of $\frac{1}{x}$.

$Pr(x_1) \longrightarrow$ This is problematic. $\longrightarrow$ Why?

$$x \in [x, x+dx]$$

$Pr([x, x+dx]) = |\psi(x,t)|^2 dx \longrightarrow$ Check with the discrete limit.

$|\psi(x,t)| \longrightarrow$ Probability density.

Change of basis

$$|\psi(t)\rangle = \int |p\rangle \underbrace{\langle p|\psi(t)\rangle}_{\tilde{\psi}(p,t)} \, dp$$

$$\tilde{\psi}(p,t)$$

$$|\psi\rangle = \sum C_i |i\rangle \longrightarrow |\psi\rangle = \sum \tilde{C}_i |\tilde{i}\rangle$$

$$C_i \xrightarrow{?} \tilde{C}_i$$

$$\tilde{\psi}(p,t) \longrightarrow \psi(x,t)$$

$$\psi(x,t) = \langle x|\psi(t)\rangle = \langle x|\mathbb{1}|\psi(t)\rangle =$$

$$\int dp \; \langle x|p\rangle \langle p|\psi(t)\rangle = \int dp \; \underbrace{\langle x|p\rangle}_{\text{We need this.}} \tilde{\psi}(p,t)$$