Symmetries in Noncommutative Gauge Theory

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ISS2006 - April 2006

Based on:

- F. Ardalan and N.S. (2001-05)
- F. Ardalan, H. Arfaei and N.S. (work in progress)
Plan of the Talk

- Review: Different Aspects of Anomaly
- Noncommutative (NC) Field Theory and Anomalies
- Recent progress
Anomaly

Breakdown of classically conserved symmetries due to certain quantum corrections

\[ \mathcal{L}_{QED} \text{ is inv. under } \psi \rightarrow e^{i\gamma_5 \alpha} \psi \]

\[ \partial_\mu j^{\mu,5} = 0 \]

but

\[ \langle \partial_\mu j^{\mu,5} \rangle = \frac{g^2}{16\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} \]

Axial anomaly
??

Are anomalies good or bad?
What are the consequences of anomalies?
Part I. Different Aspects of Anomalies
Global Anomalies; Good Anomalies

- Phenomenological consequences
  - Axial anomaly: Prediction of Pion decay rate $\pi^0 \rightarrow 2\gamma$

Local Anomalies; Bad Anomalies

- Severe constraint on the physical content of the theory
  - In SM: Cancellation of chiral anomaly
  - Top quark prediction

Useful Anomalies

- Building consistent models
  - In SUSY: Generalized Konishi anomaly
  - Exact eff. superpotential

Witten et al. 2002
Perturbative and Nonperturbative Aspects
Perturbative Aspects

i) **UV and IR origin**
   - **UV - Regularization**
     - arising from a linearly divergent integral
     - *e.g.* $d$-Dimensional, Pauli-Villars or Point-Split Regs.
   - **IR - Regularization**
     - *e.g.* technique of dispersion relation (Doglov-Zakharov 1971)

ii) **Anomaly is one-loop exact**
   - **Adler-Bardeen non-renomalization theorem**
     - Triangle + square + pentagon diagrams
     - Singlet anomaly
       \[
       \partial_\mu j^{\mu,5} = \frac{1}{16\pi^2} \text{tr} (F_{\mu\nu} \tilde{F}^{\mu\nu})
       \]
     - Non-Abelian anomaly for ± chiral fields
       \[
       (D_{\mu} j^{\mu})^a = \pm \frac{1}{24\pi^2} \varepsilon^{\mu\nu\rho\sigma} \text{tr} T^a \partial_\mu \left( A_\nu \partial_\rho A_\sigma + \frac{1}{2} A_\nu A_\rho A_\sigma \right)
       \]
Non-Perturbative Aspects

$i$) Fujikawa’s path integral method
  ▶ Nontrivial transformation property of certain path integral measure

$ii$) Other nonperturbative approaches
  ▶ Heat Kernel and $\zeta$-function regularization
Other Fundamental Aspects
Other Fundamental Aspects

i) Topology

- **Index theorem** ▶ Atiyah - Singer 1968, 1971
- **Family index theorem** ▶ Alvarez-Gaumé - Ginsparg 1984

ii) Differential Geometry

▶ Singlet anomaly

\[
d \ast j^5 = \frac{1}{4\pi^2} \text{tr}(F F) = \frac{1}{4\pi^2} d \text{tr} \left( A d A + \frac{2}{3} A^3 \right)
\]

▶ Non-Abelian anomaly for ± chiral fields

\[
G^\alpha[A] = -(D \ast j)^\alpha = \pm \frac{1}{24\pi^2} \text{tr} T^\alpha d \left( A d A + \frac{1}{2} A^3 \right)
\]
Differential Geometry

- **Wess-Zumino (WZ) consistency condition** corresponding to cocycles and cohomologies (on the BRST level)

\[ sG(v, A) = s \int v^a G^a [A] = 0, \]

with \( v \equiv v^a T^a \) FP ghost and \( s \) BRST operator

- **Stora-Zumino (SZ) descent equations** (1984) (chain term \( Q^k \leftarrow \text{power in } v \rightarrow \text{form degree} \))

\[
\begin{align*}
P(F^n) - dQ^0_{2n-1} &= 0 & \text{If } P(F) \text{ is str } v \rightarrow \text{singlet anomaly in } 2n \text{ dim} \\
sv_{2n-1} + dQ^1_{2n-2} &= 0 & \text{G}(v, A) = \int_{M_{2n-2}} Q^1_{2n-2}(v, A) \text{ in } 2n - 2 \text{ dim} \\
\vdots & & \\
sQ^2_{2n-1} &= 0 \\
& \vdots \\
\text{singlet anomaly} & \leftrightarrow \text{non-Abelian anomaly} \\
\downarrow & \\
\text{Schwinger term of the current algebra}
\end{align*}
\]
Current Algebra

- Gell-Mann (1961) ▶ ECT commutation relations ▶ certain sum rules
  
  ▶ e.g. Current algebra of global flavor symmetry

\[
\left[ j^a_{0,5}(\vec{x}, t), j^b_{0,5}(\vec{y}, t) \right] = if^{abc} j^c_{0,5}(\vec{x}, t)\delta^3(\vec{x} - \vec{y}) + c\delta^{ab} \varepsilon^{ijk} \tilde{F}_{jk} \partial_i \delta^3(\vec{x} - \vec{y})
\]

- e.g. Current algebra in QED

\[
\left[ j_0(\vec{x}, t), j_{0,5}(\vec{y}, t) \right] = c\varepsilon^{ijk} \tilde{F}^{jk} \partial_i \delta^3(\vec{x} - \vec{y})
\]

- Schwinger term ◀▶ Anomaly
  
  ▶ Ward identity
Ward Identity

\[ \partial_{x}^{\mu} \Gamma_{\mu\nu\rho}(x, y, z) = \partial_{x}^{\mu} \langle T (j_{\mu,5}(x) j_{\nu}(y) j_{\rho}(z)) \rangle = \]
\[ = \langle T \left( (\partial^{\mu} j_{\mu,5}(x)) j_{\nu}(y) j_{\rho}(z) \right) \rangle + \delta(x^{0} - y^{0}) \langle T \left( [j_{0,5}(x), j_{\nu}(y)] j_{\rho}(z) \right) \rangle \]
\[ + \delta(x^{0} - z^{0}) \langle T \left( j_{\nu}(y) [j_{0,5}(x), j_{\rho}(z)] \right) \rangle \]

Anomaly

\[ \langle \partial_{\mu} j^{\mu,5}(x) \rangle = -\frac{g^{2}}{2} \int d^{4}y \ d^{4}z \ \partial_{\mu} \Gamma^{\mu\nu\rho}(x, y, z) \ A_{\nu}(y) A_{\rho}(z) \]
Parallel Development
Parallel Development

Gravitational Anomaly

- Gravitational anomaly ▶ Alvarez-Gaumé - Witten 1983

Relation between gauge and gravitational anomaly

- Anomaly cancellation ▶ Green-Schwarz 1984
As far as studied
all these aspects
when not even more
are gathered in
noncommutative anomalies

▶ (> 40 articles on spires-hep) on noncommutative anomalies (2000-2005) ◀
Part II: Noncommutative Field Theory
II. Noncommutative (NC) Field Theory

- Noncommutativity of space-time
  \[ [x_i, x_j] = i\Theta_{ij} \]

  - Moyal \(*\)-product
    \[ f(x) \ast g(x) \equiv f(x + \xi) \exp \left( \frac{i\Theta^{\mu\nu}}{2} \frac{\partial}{\partial \xi^\mu} \frac{\partial}{\partial \zeta^\nu} \right) g(x + \zeta) \bigg|_{\xi=\zeta=0} \]

- Action of NC-U(1)
  \[ S[A_{\mu}, \bar{\psi}, \psi] = -\frac{1}{4} \int d^4x \, F_{\mu\nu} \ast F^{\mu\nu} + \int d^4x \, \bar{\psi}(x) \ast (i\slashed{D} - m) \psi(x) \]

  - Field strength tensor
    \[ F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]_\ast \]

  - Covariant derivative
    \[ D_\mu \psi(x) \equiv \partial_\mu \psi(x) + igA_\mu(x) \ast \psi(x) \]
Noncommutative Anomalies
Different Aspects of Noncommutative (Axial) Anomalies

- Noether Theorem
  - Two (anomalous) currents \( \leftrightarrow \) one global \( U_A(1) \) transformation
    \[
    J^{\mu,5} = \psi^\beta \star \bar{\psi}^\alpha (\gamma^\mu \gamma^5)_{\alpha\beta} \quad j^{\mu,5} = \bar{\psi}^\alpha \star (\gamma^\mu \gamma^5)_{\alpha\beta} \psi^\beta
    \]

  Two classical conservation laws \( \rightarrow \) Two anomaly equations
  \[
  D_\mu J^{\mu,5} = 0 \quad \partial_\mu j^{\mu,5} = 0
  \]
  \[
  D_\mu J^{\mu,5} = A_{\text{cov.}} \neq \partial_\mu j^{\mu,5} = A_{\text{inv.}}
  \]

- Due to the properties of \( \star \)-product
  \[
  \int dx \star A_{\text{cov.}} = \int dx \star A_{\text{inv.}}
  \]

\( \implies \) \( J^{\mu,5} \) and \( j^{\mu,5} \) have the same axial charge

\[
Q^5_{\text{cov.}} \equiv \int d^3x \; J^{0,5} = \int d^3x \; j^{0,5} \equiv Q^5_{\text{inv.}}
\]
Perturbative Aspect

• Covariant anomaly ▶ Planar diagrams (F. Ardalan and N.S. 2001)

\[ D_\mu J^{\mu,5} = -\frac{g^2}{16\pi^2} F_{\mu\nu} \star \tilde{F}^{\mu\nu} \]

• Invariant anomaly ▶ Nonplanar diagrams (F. Ardalan and N.S. 2002)

▶ UV / IR -Mixing à la Minwalla et al. (1999)

\[ \partial_\mu j^{\mu,5} = \begin{cases} 0 & (\Theta p)^2 \gg \frac{1}{\Lambda^2} \quad \text{UV limit} \\ -\frac{g^2}{16\pi^2} F_{\mu\nu} \star' \tilde{F}^{\mu\nu} + \ldots & (\Theta p)^2 \ll \frac{1}{\Lambda^2} \quad \text{IR limit} \end{cases} \]

with generalized \( \star' \)-product (H. Liu et al., Garousi, · · · 2001-2003)

\[ f(x) \star' g(x) \equiv f(x + \xi) \frac{\sin \left( \frac{\Theta \mu \nu}{2} \frac{\partial}{\partial \xi} \frac{\partial}{\partial \zeta} \right)}{\left( \frac{\Theta \mu \nu}{2} \frac{\partial}{\partial \xi} \frac{\partial}{\partial \zeta} \right)} g(x + \zeta) \bigg|_{\xi=\zeta=0} \]

This is when we keep the UV cutoff \( \Lambda \) large but finite

▶ To preserve gauge invariance smear \( F \tilde{F} \) along an open NC Wilson line and to integrate over the insertion points
Nonperturbative Aspect of Invariant (nonplanar) Anomaly

- Fujikawa’s path integral formulation (F. Ardalan and N.S. 2003)

$$\partial_{\mu}j^{\mu,5} = \lim_{\Lambda \to \infty} \frac{ig^2}{\Lambda^4} \int \frac{d^4p}{(2\pi)^4} e^{-ipx} \int \frac{d^4k}{(2\pi)^4} e^{\frac{k^2}{\Lambda^2}} \times \int d^4y P_\star \left( W(y - \tilde{k}) \star \int_0^1 d\tau_1 \int_0^1 d\tau_2 F_{\mu\nu}(y - \tilde{k} + \tilde{p}\tau_1) \tilde{F}^{\mu\nu}(y - \tilde{k} + \tilde{p}\tau_2) \right) \star e^{ipy}$$

Wilson line: $W(x, \ell) = P_\star \exp \left( i \int_0^1 d\sigma d\xi d\sigma A_\mu(x + \xi(\sigma)) \right) \star$

- Expanding $W(x, \ell)$ - First term in the IR-limit $[(\Theta p)^2 \ll \frac{1}{\Lambda^2}]$

$$\mathcal{A}_{\text{inv.}}^{(1)} = -\frac{g^2}{16\pi^2} F_{\mu\nu} \star \tilde{F}^{\mu\nu}$$

- Higher order terms

“Modification of the Adler-Bardeen’s non-renormalization theorem”

- Triangle + Square + Pentagon + diagrs. with more and more $A_\mu$ insertions

Wilson line expansion
In the UV-limit \[ (\Theta p)^2 \gg \frac{1}{\Lambda^2} \] (F. Ardalan and N.S. 2000-2001)\n\[ A_{inv.} = 0 \]

**Interpretation**
(K. Intriligator and J. Kumar 2001)

NC anomaly cancellation in UV-limit

Green-Schwarz cancellation of gauge and gravitational anomaly

**Idea**

NC gauge theory is still coupled to string theory
A finite contribution from closed string modes cancels the noncommutative gauge anomaly

**BUT**
This result seems to be in conflict with the previous one (A. Armoni, S. Theisen and E. Lopez 2002)

\[ \int dx \star A_{\text{cov.}} = \int dx \star A_{\text{inv.}} \]

Remember

\[ A_{\text{cov.}} \neq 0 \quad \text{whereas} \quad A_{\text{inv.}} \begin{cases} = 0 & \text{UV limit} \\ \neq 0 & \text{IR limit} \end{cases} \]
A Puzzle ?
Our solution to this puzzle (F. Ardalan, H. Arfaei and N.S. 2005)

- **Idea**: UV + IR regularization
  - **UV**: Point splitting, $\epsilon$ is the UV cutoff
  - **IR**: Compact space coordinates, Circle radius $R$ is the IR cutoff

\[
\langle \partial_\mu j^{\mu,5} \rangle = \lim_{\epsilon \to 0} \lim_{R \to \infty} \cdots
\]
Assuming \([x_1, x_2] = i\Theta_{12}\) and denoting \(\vec{x}_* = (x_1, x_2)\) and \(\vec{x}_c = (x_0, x_3)\)

\[
\partial_\mu j_{\mu,5}(x) = -\frac{g^2}{16\pi^2} \frac{1}{(2R)^2} \int_{-R}^{+R} d^2y_* F_{\mu\nu}(y_*, \vec{x}_c) \tilde{F}^{\mu\nu}(y_*, \vec{x}_c)
\]

and

\[
\text{in } R \to \infty \text{ limit}
A_{\text{inv.}} = 0
\]

\[\text{The integrated form of the anomaly remains finite}\]

\[
\int d^2x_* A_{\text{inv.}} = \int d^2x_* A_{\text{cov.}}
\]

\[\text{Zero charge density } \leftrightarrow \text{ Finite total charge}\]
• Integrating over the noncommutative coordinates $\vec{x}_*$

$$\partial_\mu j^{\mu,5}(x) = -\frac{g^2}{16\pi^2} \frac{1}{(2R)^2} \int_{-R}^{+R} d^2 y_\ast F_{\mu\nu}(\vec{y}_\ast, \vec{x}_c) \tilde{F}^{\mu\nu}(\vec{y}_\ast, \vec{x}_c)$$

$$\int_{-R}^{+R} d^2 x_\ast \partial_\mu j^{\mu,5}(x) = -\frac{g^2}{16\pi^2} \frac{1}{(2R)^2} \left[ \int_{-R}^{+R} d^2 x_\ast \int_{-R}^{+R} d^2 y_\ast F_{\mu\nu}(\vec{y}_\ast, \vec{x}_c) \tilde{F}^{\mu\nu}(\vec{y}_\ast, \vec{x}_c) \right] \text{ independent of } \vec{x}_*$$

• Then taking $R \to \infty$ limit

$$\int_{-\infty}^{+\infty} d^2 x_\ast \partial_\mu j^{\mu,5}(x) = -\frac{g^2}{16\pi^2} \int_{-\infty}^{+\infty} d^2 y_\ast F_{\mu\nu}(\vec{y}_\ast, \vec{x}_c) \tilde{F}^{\mu\nu}(\vec{y}_\ast, \vec{x}_c) = \int_{-\infty}^{+\infty} d^2 x_\ast \partial_\mu J^{\mu,5}(x)$$
Schwinger Terms
Commutator Algebra/Schwinger Terms

- Use the relation between the anomaly and ETCs of currents
- Calculate

\[
\begin{align*}
W.I. & \quad [J^{0.5}(\vec{x}, t), J^{0}(\vec{y}, t)] & \quad \text{Planar Anomaly} \\
& \quad [j^{0.5}(\vec{x}, t), J^{0}(\vec{y}, t)] & \quad \text{Nonplanar Anomaly} \\
& \quad [j^{0.5}(\vec{x}, t), j^{0}(\vec{y}, t)] & \quad ???
\end{align*}
\]

- **Method:** UV + IR regularization
  - **UV:** Point splitting
  - **IR:** Compactifying the space coordinates
Results:

i) (Axial) Covariant-Covariant

► In coordinate space

\[
\left[ J^{0,5}(x, t), J^{0}(y, t) \right] = \left( J^{0,5}(x, t) - J^{0,5}(y, t) \right) \star \delta^3(x - y) \\
+ \frac{ig}{32\pi^2} \varepsilon^{ijk} F_{kj}(x, t) \star \partial_l \delta^3(x - y) + \partial_l \delta^3(x - y) \star F_{kj}(x, t)
\]

► In momentum space

\[
\left[ J^{0,5}(p, t), J^{0}(q, t) \right] = +2i \sin (p \wedge q) J_{0,5}(p + q, t) - \frac{ig}{8\pi^2} \cos (p \wedge q) \varepsilon^{ijk} p_k q l A_j (p + q, t)
\]

Here

\[
p \wedge q \equiv \frac{1}{2} p \mu \Theta^{\mu\nu} q _\nu
\]
ii) (Axial) Invariant-Covariant

- **In coordinate space**

\[
\left[ j^{0,5}(\vec{x}, t), J^0(\vec{y}, t) \right] = + \frac{1}{(2R)^2} \frac{ig}{8\pi^2} \partial_3^x \delta(x_3 - y_3) \epsilon_{ij3} \partial^i A^j(\vec{y}_x, x_3, t)
\]

- **In momentum space**

\[
\left[ j^{0,5}(\vec{p}, t), J^0(\vec{q}, t) \right] = - \frac{ig}{8\pi^2} \frac{1}{(2R)^{3/2}} \epsilon_{ij3} q^i p^3 A^j(\vec{p} + \vec{q}, t) \delta_{\vec{p}, 0} \delta_{\vec{q}, 0}
\]

**Schwinger term**

- **Again:** Integrating over NC coordinates and then taking \( R \to \infty \) limit

\[
[Q^{inv.}_5, Q^{cov.}_5] = [Q^{cov.}_5, Q^{cov.}_5]
\]

with \( Q^{inv.}_5 \equiv \int d^3x \; j^0_5(\vec{x}, t) \quad Q^{cov.}_5 \equiv \int d^3x \; J^0_5(\vec{x}, t) \)
A NEW PUZZLE
Invariant-Covariant

\[ [j^0(\vec{p}, t), J^0(\vec{q}, t)] = 0 \quad \leftrightarrow \quad \text{gauge invariance} \quad \partial_\mu j^\mu = 0 \]

Covariant-Covariant

\[ [J^0(\vec{p}, t), J^0(\vec{q}, t)] = \lim_{\epsilon \to 0} -8C \sin(\vec{p} \wedge \vec{q}) A_\rho(\vec{p} + \vec{q}, t) \]
\[ \times \left( \frac{Dg_\rho}{\epsilon^2} + \text{tr} (\gamma^0 \gamma^\alpha \gamma^\rho \gamma^\beta) (p + q)_\alpha (p + q)_\beta \ln \epsilon^2 \right) \]

New divergences in the limit \( \epsilon \to 0 \)
\[ \mathbf{J}_B \rightarrow \mathbf{J}_{\text{Ren.}} = \mathcal{Z}_J \mathbf{J}_B \]
\[ [J^0_B, J^0_B] \rightarrow [J^0_{\text{Ren.}}, J^0_{\text{Ren.}}] \]

What about
\[ j_B \rightarrow j_{\text{Ren.}} = \mathcal{Z}_j j_B \]
\[ [j^0_B, j^0_B] \rightarrow [j^0_{\text{Ren.}}, j^0_{\text{Ren.}}] \]
and

What about

NC axial currents and their divergences ???

Do the NC anomalies receive quantum corrections ???
In commutative gauge theories

currents and their divergences are not subject to renormalization

\[ \mathcal{J}_{\text{Ren.}} = Z \mathcal{J}_B \implies Z = 1 \]
Composite operator renormalization of NC currents

- Invariant vector current
  \[ j_{\text{Ren.}}^\mu = Z_j j_B^\mu \]
  with \( Z_j = 1 \)
  (up to some "finite" nonplanar diagrams)

- Invariant axial vector current
  \[ j_{\text{Ren.}}^{\mu,5} = Z_{j,5} j_B^{\mu,5} \]
  with \( Z_{j,5} = 1 \)
  (up to some "finite" nonplanar diagrams)
• Covariant vector current ▶ Operator Mixing

\[ J_{\text{Ren.}}^\mu = Z_J J^\mu_B + Z_F \partial_\nu F^{\nu\mu} \]

with \( Z_J = 1 - \frac{g^2}{8\pi^2\epsilon} \) and \( Z_F = \frac{g^2}{6\pi^2\epsilon} \) at one loop level; \( \epsilon = 4 - d \)

( up to some “finite” nonplanar diagrams )

• Covariant axial vector current ▶ No Operator Mixing

\[ J_{\text{Ren.}}^{\mu,5} = Z_{J,5} J_B^{\mu,5} \]

with \( Z_{J,5} = 1 - \frac{g^2}{8\pi^2\epsilon} \) at one loop level; \( \epsilon = 4 - d \)

( up to some “finite” nonplanar diagrams )
Work is in progress

The only thing I can predict at the moment is ⋯
The title of my talk at
ISS2007
Symmetries in NC Gauge Theories

To be continued