Problem 2 from set 3 2M3

Solution
Problem 2, Set 3 Solution

\[ V(n) = -V S(n) \]

a) \( S(E) = ? \)

b) Poles of \( S(E) \)?

and Physical significance

\[ V(n) = -\lambda S(n) \]

\[ V(n) \]

we consider complete solution in order to compare
Poles of \( S(E) \) for \( E > 0 \) and bound states for \( E < 0 \)
so first we take a look at \( E < 0 \) solution:
for region I and III the Schrödinger equation is:

\[ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial n^2} \psi(n) = E \psi(n) \]

\[ \Rightarrow \frac{\partial^2 \psi(n)}{\partial n^2} = -\frac{2mE}{\hbar^2} \psi(n) = -\frac{k^2}{\hbar^2} \psi(n) \]
equation 1 has a solution in the form \( \Psi(n) = Ae^{\lambda n} \)

putting this back to the Schrödinger equation

\[ \lambda = \pm ik, \quad k = \sqrt{\frac{2mE}{\hbar^2}} = \sqrt{-\frac{2m|E|}{\hbar^2}} = ik \]

so:

\[
\begin{align*}
\Psi_1(n) &= Ae^{\lambda n} + Be^{-\lambda n} & n < 0 \\
\Psi_2(n) &= Fe^{\lambda n} + Ge^{-\lambda n} & n > 0 \\
\end{align*}
\]

\[ \Psi_1(n) \neq \infty \quad \text{so} \quad B = 0 \]

\[ \Psi_2(n) \neq \infty \quad \text{so} \quad F = 0 \]

\[ \Psi_1(n) = Ae^{\lambda n} \quad n < 0 \\
\Psi_2(n) = Ae^{-\lambda n} \quad n > 0 \]

\( \Psi_2(n) \) for the continuity of first derivative of \( \Psi(n) \),

potential should be piecewise continuous, But delta function is not piecewise continuous.
so we integrate the Schrödinger equation with respect to \( n \) over small interval \( \Delta n \):

\[
-h^2 \int_{-\epsilon}^{\epsilon} \frac{\partial^2 \Psi(n)}{\partial n^2} \, dn + \int_{-\epsilon}^{\epsilon} -\lambda \delta(n) \Psi(n) \, dn = E \int_{-\epsilon}^{\epsilon} \Psi(n) \, dn
\]

in the limit \( \Delta n \to 0 \), the right side is zero.

\[
\lim_{\Delta n \to 0} \left( \frac{\partial \Psi(n)}{\partial n} \right)_{-\epsilon}^{\epsilon} = 0
\]

\[
\lim_{\Delta n \to 0} \frac{2m}{\hbar^2} \int_{-\epsilon}^{\epsilon} -\lambda \delta(n) \Psi(n) \, dn
\]

\[
= -\frac{2m}{\hbar^2} \lambda \frac{\Psi(0)}{A}
\]

\[
\lim_{\Delta n \to 0} \left( \Psi(n) \right)_{-\epsilon}^{\epsilon} = -\frac{2m}{\hbar^2} \lambda A
\]

\[
\lim_{\Delta n \to 0} \left( -2kA e^{-k\epsilon} \right) = -\frac{2m}{\hbar^2} \lambda A
\]

\[
\Rightarrow \kappa = \frac{m}{\hbar^2} \sqrt{2m|E|}
\]

\[
\Rightarrow \text{for } E < 0, \text{ there is only one allowed energy:}
\]

\[
E = -\frac{m \lambda^2}{2 \hbar^2}
\]
now we go to the $E>0$ solution:

Schrödinger equation: \[ \frac{\partial^2 \Psi(x)}{\partial x^2} - \frac{2mE}{\hbar^2} \Psi(x) = -k^2 \Psi(x) \]

\[ \Rightarrow \Psi(x) = A e^{\pm ikx}, \quad k = \pm i k \]

and in the case of $E>0$, $k = \sqrt{\frac{2mE}{\hbar^2}}$

\[ \Psi(x) = A e^{ikx} + B e^{-ikx}, \quad n<0 \]
\[ \text{I} \]
\[ \Psi(x) = F e^{ikx} + G e^{-ikx}, \quad n>0 \]
\[ \text{II} \]

(form of two traveling sinusoidal waves)

for $n<0$, we have both incoming and reflected part for waves, so we have both sinusoidal waves in opposite directions, but for $n>0$, we have just outgoing part, so $G=0$.

- from continuity of $\Psi(x) \Rightarrow A+B=F$
- for the second condition again because of delta function, the first derivative is not continuous.
so again we integrate the Schrödinger equation:

\[ \frac{\text{d}^2 \psi(x)}{\text{d}x^2} = -\frac{2m}{\hbar^2} \psi(x) = -\frac{2m}{\hbar^2} (A+B) \]

\[ \lim_{\Delta x \to 0} \left( \text{ik}\frac{\text{d}^2 \psi(x)}{\text{d}x^2} - (\text{ikAe}^{-\text{ik}\epsilon} - \text{ikBe}^{\text{ik}\epsilon}) \right) = -\frac{2m}{\hbar^2} (A+B) \]

\[ F - A + B = \frac{2m}{\hbar k} i (A+B) = 2iB(A+B) \]

\[ \begin{cases} F = A + B \\ F + B - A = 2iB(A+B) \end{cases} \Rightarrow B = \frac{2B}{1 - iB} A \]

\[ \begin{cases} F = \frac{A}{1 - iB} \end{cases} \]

An \text{ Probability of measuring an incoming particle.} 

\( B \sim \) reflected \( \sim \)

\( F \sim \) transmitted \( \sim \)

\[ S(E) = \frac{F}{A} = \frac{1}{1 - iB} = \frac{1}{1 - i \frac{2m}{\hbar k^2}} = S(E) \]
Poles of $S(E) \Rightarrow$ only one pole: $1 - \frac{i m v}{\hbar^2 k} = 0$

$\Rightarrow k_{\text{scat}} = \frac{i m v}{\hbar^2} = \sqrt{2 m E}/\hbar^2$

we had $k_b = \frac{m v}{\hbar^2} = k$ for $E < 0 \Rightarrow$ we see that

$|k_b| = |k_{\text{scat}}|$

so the only bound state for $E < 0$ corresponds to the pole of $S(E)$ for $E > 0$.

⚠️ actually we cannot define a pole for $S(E)$ because for the scattering problem $E$ is assumed to be positive.