Ward - Takahashi identities in QED: (WT)

WT identities are "exact" relations between 1PI vertex functions and propagators (N- point Green's functions)

منظور از "exact" این است که این اتحادها در همه مراتب بسط اختلال صحیح هستند (ولی برای آزمودن آنها باید مرتبه به مرتبه صحت آنها را آزمود).

این اتحادها از gauge invariance (ناوردایی تحت ت. پیمانه‌ای) ناشی می‌شوند

برای QED →

$$Z[\eta,\bar\eta,J_\mu] = \frac{1}{N}\int \mathcal{D}A_\mu\, \mathcal{D}\psi\, \mathcal{D}\bar\psi\; e^{iS[\bar\psi,\psi,A_\mu;\,\eta,\bar\eta,J_\mu]}$$

$$S = \int d^dx\, \mathcal{L} = \int d^dx\, \Big[\underbrace{\tfrac{-1}{4}F_{\mu\nu}F^{\mu\nu} + \bar\psi(i\not{\partial}-m)\psi}_{QED} \underbrace{-\tfrac{1}{2\xi}(\partial_\mu A^\mu)^2}_{\text{gauge fixing term}} + \underbrace{J_\mu A^\mu + \bar\psi\eta + \bar\eta\psi}_{\text{source terms}}$$

local gauge transformation:

$$\psi \to e^{-ig\alpha(x)}\psi(x) \simeq (1-ig\alpha(x))\psi(x) = \psi(x) + \delta_\alpha\psi(x) \rightsquigarrow \delta_\alpha\psi = -ig\alpha(x)\psi$$

$$\bar\psi \to \bar\psi e^{+ig\alpha(x)} \simeq \bar\psi(x)\,(1+ig\alpha(x)) = \bar\psi(x) + \delta_\alpha\bar\psi(x) \rightsquigarrow \delta_\alpha\bar\psi = +ig\alpha(x)\bar\psi$$

$$A_\mu \to A_\mu + \delta_\alpha A_\mu \quad ; \quad \delta_\alpha A_\mu = \partial_\mu\alpha(x)$$

$$\mathcal{L}_{QED} = -\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar\psi(i\not{\partial}-m)\psi \qquad \text{gauge inv.} \qquad \delta_\alpha\mathcal{L}_{QED} = 0$$

$$\mathcal{L}_{g.f.} = -\tfrac{1}{2\xi}(\partial_\mu A^\mu)^2 \qquad \delta\mathcal{L}_{g.f.} \neq 0$$

$$\mathcal{L}_{source} = J_\mu A^\mu + \bar\psi\eta + \bar\eta\psi \qquad \delta\mathcal{L}_{source} \neq 0$$

• البته می‌دانیم که فیزیک نباید به انتخاب پیمانه وابسته باشد. به عبارت دیگر می‌توان فرض کرد که $Z[J_\mu,\eta,\bar\eta]$ تحت تبدیلات پیمانه‌ای ناوردا است.

$$\delta_\alpha Z = 0$$

$$\delta_\alpha \int \mathcal{D}A_\mu\, \mathcal{D}\psi\, \mathcal{D}\bar\psi\; e^{iS} = 0 = i\int \mathcal{D}A_\mu\, \mathcal{D}\psi\, \mathcal{D}\bar\psi\; \delta_\alpha S\; e^{iS}$$

• $$\delta_\alpha S = \delta_\alpha\int d^dx\,\mathcal{L} = \int d^dx\, \delta_\alpha\left(\mathcal{L}_{QED} + \mathcal{L}_{g.f.} + \mathcal{L}_{source}\right)$$

$$= \int d^dx\, \Big\{\underbrace{0}_{\delta_\alpha\mathcal{L}_{QED}=0} + \delta_\alpha\Big(-\tfrac{1}{2\xi}(\partial_\mu A^\mu)^2\Big) + \delta_\alpha\big(J_\mu A^\mu + \bar\eta\psi + \bar\psi\eta\big)\Big\}$$

$$= \int d^dx\, \Big\{\tfrac{-1}{2\xi}\, 2\,(\partial_\mu A^\mu)(\partial_\nu\delta_\alpha A^\nu) + J_\mu\delta_\alpha A^\mu + \bar\eta\,\delta_\alpha\psi + \delta_\alpha\bar\psi\,\eta\Big\}$$

$$= \int d^dx\, \Big\{-\tfrac{1}{\xi}(\partial_\mu A^\mu)(\Box\alpha(x)) + J_\mu\partial^\mu\alpha(x) - ig\alpha(x)\bar\eta\psi + ig\alpha(x)\bar\psi\eta\Big\}$$

$$\underset{P.I.}{=} \int d^dx\, \Big\{\tfrac{-1}{\xi}\Box(\partial_\mu A^\mu) - \partial^\mu J_\mu - ig(\bar\eta\psi - \bar\psi\eta)\Big\}\alpha(x)$$

↳

$$\delta_\alpha Z = 0 = i\int \mathcal{D}A_\mu \mathcal{D}\psi \mathcal{D}\bar\psi \;\; \delta_\alpha S \; e^{iS} =$$

$$= i\int \mathcal{D}A_\mu \mathcal{D}\psi \mathcal{D}\bar\psi \qquad \int d^dx \; \alpha(x)\left[\frac{-1}{\xi}\,\Box(\partial_\mu A^\mu) - \partial_\mu J^\mu - ig(\bar\eta\psi - \bar\psi\eta)\right] e^{iS}$$

$$A_\mu = \frac{1}{i}\frac{\delta}{\delta J^\mu} \qquad \bar\psi = \frac{-1}{i}\frac{\delta}{\delta\eta} \qquad \psi = \frac{1}{i}\frac{\delta}{\delta\bar\eta}$$

$$0 = i\int d^dx \;\alpha(x)\left[\frac{+i}{\xi}\,\Box\left(\partial_\mu\frac{\delta}{\delta J_\mu(x)}\right) - \partial_\mu J^\mu(x) - ig\left(\frac{1}{i}\,\bar\eta\frac{\delta}{\delta\bar\eta} - \frac{1}{i}\,\eta\frac{\delta}{\delta\eta}\right)\right] Z$$

از آنجا که این معادله برای هر α دلخواهی درست است خواهیم داشت:

$$\boxed{(a)\quad \left[\frac{1}{\xi}\,\Box\left(\partial_\mu\frac{\delta}{\delta J_\mu(x)}\right) + i\,\partial_\mu J^\mu(x) + ig\left(\bar\eta\frac{\delta}{\delta\bar\eta} - \eta\frac{\delta}{\delta\eta}\right)\right] Z[\eta,\bar\eta,J_\mu] = 0}$$

Ward-Takahashi identity for Z.

از طرفی $\quad Z = e^{iW} \qquad \dfrac{\delta}{\delta J_\mu} Z = i\,\dfrac{\delta W}{\delta J_\mu}\, e^{iW} \qquad W[J_\mu,\eta,\bar\eta] = W$

$$\boxed{(b)\quad \frac{i}{\xi}\,\Box\,\partial_\mu\frac{\delta W}{\delta J_\mu} + i\,\partial_\mu J^\mu(x) - g\left(\bar\eta\frac{\delta}{\delta\bar\eta} - \eta\frac{\delta}{\delta\eta}\right) W[\eta,\bar\eta,J_\mu] = 0}$$

از طرف دیگر $\quad \Gamma[\psi,\bar\psi,A_\mu] = W[\eta,\bar\eta,J_\mu] - \int d^dx\,(\bar\eta\psi + \bar\psi\eta + J_\mu A^\mu)$

$$\frac{\delta W}{\delta J_\mu} = A^\mu \qquad \frac{\delta W}{\delta\eta} = -\bar\psi \qquad \frac{\delta W}{\delta\bar\eta} = \psi$$

$$\frac{\delta\Gamma}{\delta A_\mu} = -J^\mu \qquad \frac{\delta\Gamma}{\delta\psi} = +\bar\eta \qquad \frac{\delta\Gamma}{\delta\bar\psi} = -\eta$$

$$\rightarrow (c)\qquad \frac{i}{\xi}\,\Box\,\partial_\mu A^\mu - i\,\partial_\mu\frac{\delta\Gamma}{\delta A_\mu} - g\left(\left(\frac{\delta\Gamma}{\delta\psi}\right)\psi - \left(\frac{-\delta\Gamma}{\delta\bar\psi}\right)(-\bar\psi)\right) = 0$$

$$\boxed{\frac{i}{\xi}\,\Box\,\partial_\mu A^\mu - i\,\partial_\mu\frac{\delta\Gamma}{\delta A_\mu} - g\left(\frac{\delta\Gamma}{\delta\psi}\,\psi + \bar\psi\,\frac{\delta\Gamma}{\delta\bar\psi}\right) = 0}$$

موارد کاربرد:

الف) از (b) نسبت به $J_\nu(y)$ مشتق می‌گیریم و در نهایت $\eta = \bar\eta = J_\mu = 0$ قرار می‌دهیم:

$$\frac{i}{\xi}\,\Box\,\partial_\mu\frac{\delta^2 W}{\delta J_\mu(x)\,\delta J_\nu(y)}\bigg|_{J^\mu=\eta=\bar\eta=0} = -\,i\,\partial_\mu\frac{\delta J^\mu(x)}{\delta J_\nu(y)}\bigg|_{J^\mu=\eta=\bar\eta=0}$$

$\rightarrow$

$$\frac{i}{\xi}\Box\partial_\mu \frac{\delta^2 W}{\delta J_\mu(x)\delta J_\nu(y)}\bigg|_{J^\mu=\eta=\bar\eta=0} = -i\partial_\mu\,\delta(x-y)\,g^{\mu\nu} = -i\partial^\nu\delta^4(x-y)$$

از طرفی $$\frac{\delta^2 W}{\delta J_\mu(x)\delta J_\nu(y)}\bigg|_{J^\mu=\eta=\bar\eta=0} = iD^{\mu\nu}(x-y)$$ انتشارگر کامل (exact) فوتون

$$\frac{i}{\xi}\Box\partial_\mu\left(iD^{\mu\nu}(x-y)\right) = -i\partial^\nu\delta^4(x-y)$$

$$\longrightarrow \boxed{\frac{1}{\xi}\Box\partial_\mu D^{\mu\nu}(x-y) = i\partial^\nu\delta^4(x-y)}$$

از فضای فوریه $$\frac{1}{\xi}(-k^2)(ik_\mu)\tilde D^{\mu\nu}(k) = i(ik^\nu)$$

$$\boxed{\frac{i}{\xi}k^2 k_\mu\tilde D^{\mu\nu}(k) = k^\nu}$$

$\times k_\nu$ $$\frac{i}{\xi}k^2 k_\nu k_\mu\tilde D^{\mu\nu}(k) = k^2 \rightarrow \boxed{k_\mu k_\nu\tilde D^{\mu\nu}(k) = -i\xi}$$

که همان اتحادی بود که قبلاً هم معرفی شده بود.

فرم کلی انتشارگر کامل فوتون $$\tilde D_{\mu\nu}(k) = D^T_{\mu\nu}(k) - \frac{i\xi k_\mu k_\nu}{(k^2)^2} = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)f(k^2) - \frac{i\xi k_\mu k_\nu}{(k^2)^2}$$

$$k_\mu k_\nu\tilde D^{\mu\nu}(k) = (k^2 - k^2)f(k^2) - i\xi = -i\xi$$

که البته این رابطه باید مرتبه به مرتبه در بسط اختلال چک شود.

وجود این رابطه و صحت آن عملاً تضمین می‌کند که تقارن پیمانه‌ای در همه مراتب وجود دارد.

ب) رابطه بین تابع رأس و انتشارگر فرمیون:

رابطه (b) را در نظر بگیرید آن را در i ضرب کنید. بعد $\frac{\delta^2}{\delta\bar\eta(x)\delta\eta(z)}\Big|_{\eta=\bar\eta=J^\mu=0}$ را اعمال کنید؛

$$\frac{1}{\xi}\Box\partial_\mu\frac{\delta^3 W}{\delta J_\mu(y)\delta\bar\eta(x)\delta\eta(z)}\bigg|_{\bar\eta=\eta=J^\mu=0} + 0$$

$$-g\left(-\delta^4(x-y)\frac{\delta^2}{\delta\eta(z)\delta\bar\eta(y)} - \delta^4(z-y)\frac{\delta^2}{\delta\bar\eta(x)\delta\eta(y)}\right)W[0] = 0$$

$$\boxed{0 = \frac{1}{\xi}\Box\partial_\mu\frac{\delta^3 W[0]}{\delta J_\mu(y)\delta\bar\eta(x)\delta\eta(z)} = +g\left(\delta^4(x-y)\frac{\delta^2}{\delta\eta(z)\delta\bar\eta(y)} + \delta^4(z-y)\frac{\delta^2}{\delta\bar\eta(x)\delta\eta(y)}\right)W[0]}$$

Use now:

$$\frac{i\,\delta^2 W[0]}{\delta\bar\eta(x)\,\delta\eta(y)} = \langle\Omega|T(\psi(x)\bar\psi(y))|\Omega\rangle$$

$$\frac{\delta^3 W[0]}{\delta\bar\eta(x)\,\delta\eta(z)\,\delta J_\mu(y)} = \langle\Omega|T(\psi(x)\bar\psi(z)A^\mu(y))|\Omega\rangle$$

$$\boxed{\begin{aligned}&\frac{1}{\xi}\,\Box\partial_\mu\langle\Omega|T(\psi(x)\bar\psi(z)A^\mu(y))|\Omega\rangle\\&= ig\Big(i\delta^4(x-y)\,\langle\Omega|T(\psi(y)\bar\psi(z))|\Omega\rangle\\&\qquad - i\delta^4(z-y)\,\langle\Omega|T(\psi(x)\bar\psi(y))|\Omega\rangle\Big)\end{aligned}}$$

Ward identity in momentum space

با توجه به تقارن انتقالی خواهیم داشت:

$$* \quad \boxed{\begin{aligned}&-\frac{1}{\xi}\,\Box\partial_\mu\langle\Omega|T(A^\mu(y-z)\,\psi(x-z)\,\bar\psi(0))|\Omega\rangle\\&= g\,\delta^4(x-y)\,\langle\Omega|T(\psi(y-z)\bar\psi(0))|\Omega\rangle\\&\quad -g\,\delta^4(z-y)\,\langle\Omega|T(\psi(x-y)\bar\psi(0)|\Omega\rangle\end{aligned}}$$

Fourier space:

تعریف a) $\langle\Omega|T(A^\mu(y-z)\,\psi(x-z)\,\bar\psi(0))|\Omega\rangle \equiv \int\frac{d^4p}{(2\pi)^4}\frac{d^4q}{(2\pi)^4}\,e^{-ip(y-z)}\,e^{-iq(x-z)}\,V^\mu(p,q)$

طرف چپ رابطه * عکس تبدیل فوریه

$$\int d^4x\, d^4y\, d^4z\,\Big\{\Big(\frac{-1}{\xi}\Box\partial^\gamma_{\mu\,\gamma}\Big)\langle\Omega|T(A^\mu(y-z)\psi(x-z)\bar\psi(0))|\Omega\rangle\; e^{ip'x}\,e^{iq'y}\,e^{-ik'z}$$

$$=\int_{p,q} d^4x\,d^4y\,d^4z\,\Big(-\frac{1}{\xi}\Box_y\,\partial^y_\mu\Big)\,e^{-ip(y-z)}e^{-iq(x-z)}\,V^\mu(p,q)\,e^{ip'x}e^{iq'y}e^{-ik'z}$$

$$=\int_{p,q} d^4x\,d^4y\,d^4z\,\Big(-\frac{1}{\xi}(-p^2)(-ip_\mu)\Big)\quad V^\mu(p,q)\;e^{ix(p'-q)}\,e^{iy(q'-p)}\,e^{iz(q-k'+p)}$$

$$=\int_{p,q}(2\pi)^{12}\Big(\frac{-1}{\xi}(-p^2)(-ip_\mu)\Big)\,\delta^4(p'-q)\,\delta^4(q'-p)\,\delta^4(q+p-k')\;V^\mu(p,q)$$

$$= \boxed{(2\pi)^4\;\Big(\frac{-i}{\xi}\Big)\,q'^2\,q'_\mu\;V^\mu(q',p')\;\delta^4(p'+q'-k')}$$

* طرف راست رابطه جمله اول b)

$$g\,\delta^4(x-y)\langle\Omega|T(\psi(y-z)\bar\psi(0))|\Omega\rangle \overset{\text{تعریف!}}{=} g\int_q e^{-iq(x-y)}\int_k i\tilde S_f(k)\,e^{-ik(y-z)}$$

عکس تبدیل فوریه

$$g\int_{q,k} d^4x\, d^4y\, d^4z\; e^{ip'x} e^{iq'y} e^{-ik'z}\, e^{-iq(x-y)}\, e^{-ik(y-z)}\, i\tilde{S}_f(k)$$

$$= ig\int_{q,k} d^4x\, d^4y\, d^4z\; e^{ix(p'-q)} e^{iy(q'+q-k)} e^{iz(k-k')}\, \tilde{S}_f(k)$$

$$= ig\int_{q,k} (2\pi)^{12}\, \delta^4(p'-q)\, \delta^4(q'+q-k)\, \delta^4(k-k')\, \tilde{S}_f(k)$$

$$= \boxed{ig(2\pi)^4\, \delta^4(q'+p'-k')\, \tilde{S}_f(q'+p')}$$

* c) طرف راست رابطه (جمله دوم)

$$-g\delta^4(z-y)\langle\Omega|T(\psi(x-y)\bar{\psi}(0))|\Omega\rangle \overset{\text{تعریف}}{=} -g\int_{q,k} e^{-iq(z-y)}\, i\tilde{S}_f(k)\, e^{-ik(x-y)}$$

عکس تبدیل فوریه

$$-ig\int_{q,k} d^4x\, d^4y\, d^4z\; e^{ip'x} e^{iq'y} e^{-ik'z}\, e^{-iq(z-y)}\, e^{-ik(x-y)}\, \tilde{S}_f(k)$$

$$= -ig\int_{q,k} d^4x\, d^4y\, d^4z\; e^{ix(p'-k)} e^{iy(q'+q+k)} e^{iz(-k'-q)}\, \tilde{S}_f(k)$$

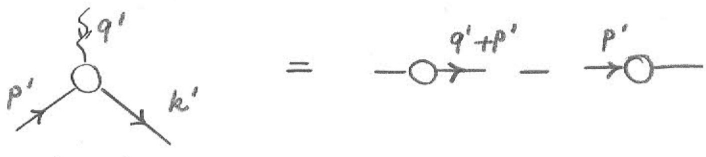
$$= -ig\int_{q,k} (2\pi)^{12}\, \delta^4(p'-k)\, \delta^4(q'+q+k)\, \delta^4(k'+q)\, \tilde{S}_f(k)$$

$$\boxed{= -ig\,(2\pi)^4\, \delta^4(q'-k'+p')\, \tilde{S}_f(p')}$$

نتیجه اینکه:

$$\delta^4(p'+q'-k')\left(\frac{-i}{\xi}\, q'^2 q'_\mu V^\mu(q';p')\right) = ig\left(\tilde{S}_f(q'+p') - \tilde{S}_f(p')\right)\delta^4(q'+p'-k')$$

$$\rightarrow \boxed{\frac{-i}{\xi}\, q'^2 q'_\mu V^\mu(q',p') = ig\left(\tilde{S}_f(q'+p') - \tilde{S}_f(p')\right)}$$

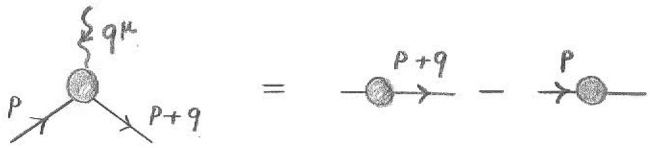


$q' = k' - p'$

به همین ترتیب خواهیم داشت:

$$-\frac{1}{\xi}\Box\partial_\mu A^\mu+\partial^\mu\frac{\delta\Gamma}{\delta A^\mu}-ig\left(\frac{\delta\Gamma}{\delta\psi}\psi+\bar\psi\frac{\delta\Gamma}{\delta\bar\psi}\right)=0$$

$$\rightarrow\quad\boxed{iq_\mu\,\tilde\Gamma^{\mu(3)}(p,q\,;k=p+q)=g\,\tilde\Gamma^{(2)}(q+p)-g\,\tilde\Gamma^{(2)}(p)}$$



یادآوری: $\tilde\Gamma^{(2)}(p)=i\left(\tilde G_c^{(2)}(p)\right)^{-1}$ که در مرتبه درختی (tree level) برابر است با:

$$\tilde\Gamma^{(2)}(p)=i\left(\tilde G_c^{(2)}(p)\right)^{-1}\Big|_{\text{treelevel}}=(\not p-m)$$

حکم: * اولاً در حد $q\to 0$ میتوان نشان داد که:

$$\tilde\Gamma^{(3)}_\mu(p,0;p)=g\,\frac{\partial\left(\tilde G_c^{(2)}(p)\right)^{-1}}{\partial p^\mu}$$

← این رابطه را میتوان در حد treelevel چک کرد:

اثبات:

$$iq_\mu\,\tilde\Gamma^{\mu(2)}(p,q;p+q)=ig\left(\left(\tilde G_c^{(2)}(p+q)\right)^{-1}-\left(\tilde G^{(2)}(p)\right)^{-1}\right)$$

در حد $q\to 0$

$$\boxed{\tilde\Gamma^{\mu(2)}(p,0,p)=g\,\frac{\partial\left(\tilde G_c^{(2)}(p)\right)^{-1}}{\partial p_\mu}}\quad\checkmark\quad\text{q.e.d.}$$

در مرتبه treelevel

طرف چپ $\quad\tilde\Gamma_\mu^{(3)}(p,0;p)=-ig\gamma_\mu$

طرف راست $\quad -ig\frac{\partial}{\partial p_\mu}(\not p-m)=g\frac{\partial}{\partial p_\mu}\left(\tilde G_c^{(2)}(p)\right)^{-1}$

$$=-ig\gamma^\mu\quad\checkmark$$

حکم: ثانیاً: میتوان نشان داد که مشتق گرین انتشارگر فرمیونی نسبت به p^μ معادل است با insertion یک فوتون با تکانه صفر به یک خط فرمیونی:

اثبات:

$$\left(\hat G_c^{(2)}(p)\right)\left(\tilde G_c^{(2)}(p)\right)^{-1}=1$$

$$\frac{\partial}{\partial p_\mu}\left(\tilde G_c^{(2)}(p)\right)\left(\tilde G_c^{(2)}(p)\right)^{-1}+\tilde G_c^{(2)}(p)\,\frac{\partial}{\partial p_\mu}\left(\tilde G_c^{(2)}(p)\right)^{-1}=0$$

$$\longrightarrow$$

a) $$\frac{\partial}{\partial p_\mu}\left(G_c^{(2)}(p)\right) = -\tilde{G}_c^{(2)}(p)\,\frac{\partial}{\partial p_\mu}\left(\tilde{G}_c^{(2)}(p)\right)^{-1}\tilde{G}_c^{(2)}(p)$$

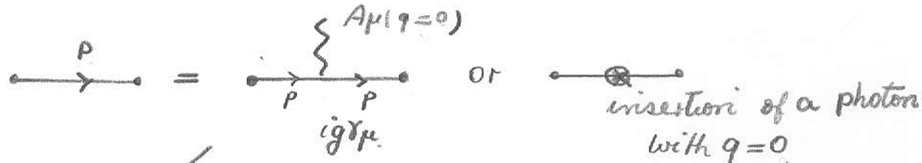
or

b) $$\frac{\partial}{\partial \mu}\left(G_c^{(2)}(p)\right)^{-1} = -\left(\tilde{G}_c^{(2)}(p)\right)^{-1}\frac{\partial}{\partial p_\mu}\tilde{G}_c^{(2)}(p)\left(\tilde{G}_c^{(2)}(p)\right)^{-1}$$

Tree level

(a) رابطه الف

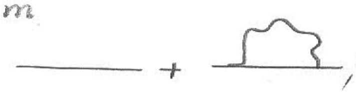
$$g\frac{\partial}{\partial \mu}S_f^{(0)}(p) = ig\, S_f^{(0)}(p)\,\frac{\partial}{\partial p_\mu}(\not p - m)\, S_f^{(0)}(p)$$
$$= ig\, S_f^{(0)}(p)\,\gamma^\mu\, S_f^{(0)}(p)$$

$\curvearrowright$ $g\dfrac{\partial}{\partial p_\mu}$ 

توجه کنید که هر دو است ورتکس‌هایی که دارند، به علت پایستگی انرژی و تکانه $q=0$ باید باشد.

حکم: میتوان رابطه الف $\Gamma^\mu(p,0,p) = g\dfrac{\partial\,(G_c^{(2)}(p))^{-1}}{\partial p_\mu}$ را در مرتبه تک حلقه اثبات کرد:

$$S_F^{(0)} = \frac{i}{\not p - m}$$

$iG_c^{(2)}$ =  + ... at one-loop level.

$$i\tilde{G}_c^{(2)} = iS_F^{(0)} + (iS_F^{(0)})(-i\Sigma(p))(iS_F^{(0)}) + \dots$$

از طرفی

$$(i\tilde{G}_c^{(2)})^{-1} = (S_F^{(0)})^{-1} - \Sigma(p) = \not p - m - \Sigma(p)$$ $\curvearrowleft$ Resummation of 1PI diagr.

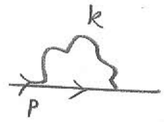
Now compute:

(b) طرف چپ گزاره دوم $\curvearrowright$ $$g\frac{\partial}{\partial p_\mu}\left(\tilde{G}_c^{(2)}(p)\right)^{-1} = g\frac{\partial}{\partial p_\mu}\left(S_F^{(0)}\right)^{-1} - g\frac{\partial}{\partial p_\mu}\Sigma(p)$$
$$\|$$

$$\Longrightarrow \boxed{\Gamma^\mu(p;0,p) = -ig\gamma^\mu - g\frac{\partial}{\partial p_\mu}\Sigma(p)}$$

و این رابطه (exact) است یعنی در همه مراتب نظریه اختلال (بصورت مرتبه به مرتبه) صدق می‌کند

سوال $\dfrac{\partial}{\partial p_\mu}\Sigma(p) = ?$

جواب آن را در مرتبه اول نظریه اختلال بدست می‌آوریم. $\curvearrowright$ $-i\Sigma(p)\big|_{1\text{-loop}}$ = 

$$= (-ig)^2\int\frac{d^4k}{(2\pi)^4}\left(\frac{-ig_{\mu\lambda}}{k^2}\right)\gamma^\mu\frac{i}{(\not p - \not k) - m}\gamma^\lambda$$
$$= -g^2\int\frac{d^4k}{(2\pi)^4}\,\gamma^\mu S_F^{(0)}(p-k)\,\gamma_\mu\,\frac{1}{k^2}$$

$\longrightarrow$

$$-\frac{i\,\partial\Sigma(p)}{\partial p_\mu}\bigg|_{1-loop} = -g^2\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2}\,\gamma^\rho\,\frac{\partial}{\partial p_\mu}S_F^{(0)}(p-k)\,\gamma_\rho$$

در مرتبه‌ی یک حلقه
$$\frac{\partial}{\partial p_\mu}G_c^{(2)} = -G_c^{(2)}\frac{\partial}{\partial p_\mu}\left(G_c^{(2)}(p)\right)^{-1}G_c^{(2)}$$

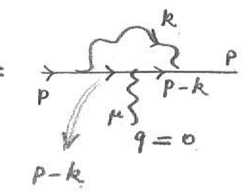
حالا باید از همین رابطه استفاده کرد تا $\frac{\partial}{\partial p_\mu}S_F^{(0)}(p-k)$ را بدست آورد:

$$\frac{\partial}{\partial p_\mu}S_F^{(0)}(p-k) = -S_F^{(0)}(p-k)\underbrace{\frac{\partial}{\partial p_\mu}\underbrace{(\not p-\not k-m)}_{(S_F^{(0)}(p-k))^{-1}}}_{\gamma^\mu}S_F^{(0)}(p-k)$$

$$= -iS_F^{(0)}(p-k)\,\gamma^\mu\,S_F^{(0)}(p-k)$$

$$\longrightarrow\ -\frac{i\,\partial\Sigma(p)}{\partial p_\mu}\bigg|_{1-loop} = +ig^2\int\frac{d^4k}{(2\pi)^4}\,\frac{1}{k^2}\,\gamma^\rho S_F^{(0)}(p-k)\gamma^\mu S_F^{(0)}(p-k)\,\gamma_\rho$$

$$\frac{\partial\Sigma(p)}{\partial p_\mu}\bigg|_{1-loop} =$$



$$= i\Lambda^\mu(p, q=0, p)\Big|_{1-loop}$$

$$\longrightarrow\ \Gamma^\mu(p,0,p) = -ig\gamma^\mu - g\frac{\partial}{\partial p_\mu}\Sigma(p) =$$
$$= -ig\gamma^\mu - ig\,\Lambda^\mu(p,0,p)\Big|_{1-loop}$$
در مرتبه اول بسط اختلال

$$\longrightarrow\ \boxed{\Gamma^\mu(p,0,p) = -ig\left(\gamma^\mu + \Lambda^\mu(p,0,p)\right)}$$

Ward - identity and Renormalization $(Z_1 = Z_2)$:

ادعا: از رابطه $\Gamma^\mu(p,0,p) = g\frac{\partial}{\partial p_\mu}\left(G_c^{(2)}(p)\right)^{-1}$ می‌توان استفاده کرد نشان داد که $Z_1 = Z_2$ است

(این رابطه هم exact است، می‌توان آن را مرتبه به مرتبه در بسط اختلال چک کرد.)

(در مرتبه اول بسط اختلال در QED صحت این رابطه را دیده بودیم)

یادآوری:
$$\psi_B = Z_2^{1/2}\psi_R$$
$$A_{\mu B} = Z_3^{1/2}A_{\mu R}$$
$$g_B = g_R\,Z_1 Z_2^{-1} Z_3^{-1/2}$$

اثبات:

Use: $$\Gamma^{\mu\,(3)}_B = \left(\frac{\delta\Gamma[0]}{\delta\psi\,\delta\bar\psi\,\delta A_\mu}\right)_{bare} = \left(\frac{\delta\Gamma[0]}{\delta\psi\,\delta\bar\psi\,\delta A_\mu}\right)_{Ren.}Z_2^{-1}Z_3^{-1/2} = Z_2^{-1}Z_3^{-1/2}\,\Gamma^{\mu\,(3)}_R$$

از طرفی
$$G_B^{(2)} = Z_2\,G_R^{(2)}$$

$$g_B \frac{\partial}{\partial p_\mu} \left(\tilde{G}_c^{(2)}(p)\right)_B^{-1} = \tilde{\Gamma}^{\mu(3)}_B (p, 0; p)$$

$$g_R \; Z_1 \cancel{Z_2^{-1}} \cancel{Z_3^{-1/2}} \frac{\partial}{\partial p_\mu} \left(Z_2 \; \tilde{G}_R^{(2)}(p) \right)^{-1} = \cancel{Z_2^{-1}} \cancel{Z_3^{-1/2}} \; \tilde{\Gamma}^{\mu(3)}_R (p, 0; p)$$

$$Z_1 Z_2^{-1} g_R \frac{\partial}{\partial p_\mu}\left(\tilde{G}_R^{(2)}(p)\right)^{-1} = \tilde{\Gamma}^{\mu(3)}_R (p, 0, p) \quad \curvearrowright \quad Z_1 = Z_2$$

رابطه‌ای که در دو داشتیم باید برای توابع چند نقطه‌ای bare , renormalized صحیح باشد

یعنی

$$g_R \frac{\partial}{\partial p_\mu}\left(\tilde{G}_R^{(2)}(p)\right)^{-1} = \tilde{\Gamma}^{\mu(3)}_R (p, 0, p)$$

و این تساوی exact است

و می‌توان آن را مرتبه به مرتبه در بسط اختلال چک کرد.