

تبدیل: $y = x + \xi$ $\eta(x)$ (تبدیل) $\eta(y)$ $\eta(x)$ $\eta(y)$ $\eta(x)$

$$\eta(y = x + \xi) = \eta(x) + \xi_\mu \partial^\mu \eta(x) + \xi^\mu \xi^\nu \partial_\mu \partial_\nu \eta(x) + \dots$$

با جایگزینی در رابطه (A) داریم:

$$\Gamma[\phi_{cl}] = \Gamma[\phi_{cl}^{(0)}] - \frac{1}{2} \int d^4x M^2[\phi_{cl}] \eta^2(x) + \frac{1}{2} \int \chi^{\mu\nu}[\phi_{cl}^{(0)}] \partial_\mu \eta(x) \partial_\nu \eta(x) d^4x + \dots$$

with $M^2 = - \int \frac{\delta^2 \Gamma[\phi]}{\delta \phi^{(0)} \delta \phi^{(0)}} \Big|_{\phi_{cl}^{(0)}} d^4x$

$$\chi^{\mu\nu} = - \frac{1}{2} \int \frac{\delta^2 \Gamma[\phi]}{\delta \phi^{(0)} \delta \phi^{(0)}} \Big|_{\phi_{cl}^{(0)}} \xi^\mu \xi^\nu$$

$\Gamma[\phi] = \int d^4x \left\{ \frac{1}{2} \xi_\mu \partial^\mu \phi \partial^\mu \phi - \mathcal{U}(\phi) \right\} + \dots$ نشان این طور می

$$\phi(x) = \phi_{cl}^{(0)} + \eta(x)$$

$$\partial_\mu \phi = \partial_\mu \eta(x)$$

$$\phi_{cl}^{(0)} = const$$

$$\mathcal{U}(\phi_{cl}^{(0)})$$

توانیله

(در ارتباط با حالت خورجی در آن مورد استفاده زیاد دارد)

در اغلب موارد فرض می کنند min کنش نورمید میسرند ثابت است $\phi_{cl}^{(0)} = const$ (نشان) $\eta(x) = 0$ در این صورت:

$$\Gamma[\phi_{cl}^{(0)}] = - \int d^d x \mathcal{U}(\phi_{cl}^{(0)}) = - \Omega_d \mathcal{U}(\phi_{cl}^{(0)})$$

توانیله \int حجم d بعدی

(\rightarrow later Spontaneous Symmetry Breaking)

$$\Gamma[\phi_{cl}] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^d x_1 \dots d^d x_n$$

نوع دیگری از رابطه برای توانیله

$$\times \Gamma^{(n)}(x_1, \dots, x_n) \phi_{cl}(x_1) \dots \phi_{cl}(x_n)$$

$$\phi_{cl}^{(0)} = const \rightarrow \Gamma[\phi_{cl}^{(0)}] = - \Omega_d \mathcal{U}(\phi_{cl}^{(0)})$$

اگر $\phi_{cl} = \phi_{cl}^{(0)}$ ثابت باشد

$$\Gamma[\phi_{cl}^{(0)}] = \sum_{n=0}^{\infty} \frac{1}{n!} \int d^d x_1 \dots d^d x_n \Gamma^{(n)}(x_1, \dots, x_n) (\phi_{cl}^{(0)})^n$$

$$= \tilde{\Gamma}^{(n)}(p_1, \dots, p_n) \Big|_{p_i=0}$$

\Rightarrow

zero mode of $\Gamma^{(n)}$ in mom-space.

$$\Gamma[\phi_c] = \sum_{n=0}^{\infty} \frac{1}{n!} \tilde{\Gamma}^{(n)}(p_1, \dots, p_n) \Big|_{p_i=0} (\phi_c^{(0)})^n = -\Omega U(\phi_c^{(0)})$$

$$\rightarrow \boxed{U(\phi_c^{(0)}) = -\frac{1}{\Omega} \sum_{n=0}^{\infty} \frac{1}{n!} \tilde{\Gamma}^{(n)}(p_1, \dots, p_n) \Big|_{p_i=0}}$$

One-loop effective action (۲) *کنش موثر تا مرتبه یک حلقه*

Saddle point expansion:

$$f(x) = f(x_0) + (x-x_0) \frac{df(x)}{dx} \Big|_{x_0} + \frac{1}{2!} (x-x_0)^2 \frac{d^2f}{dx^2} \Big|_{x_0} + \dots$$

Stationary $\frac{df(x)}{dx} \Big|_{x_0} = 0$ *بشد، یعنی اصطلاحاً x_0 یک نقطه ایستادگی است (برای $f(x)$)*

$$\rightarrow f(x) = f(x_0) + \frac{1}{2!} (x-x_0)^2 \frac{d^2f}{dx^2} \Big|_{x_0} + \dots$$

$$I = \int dx e^{-f(x)} = \int dx e^{-f(x_0) - \frac{1}{2!} (x-x_0)^2 f''(x_0) + \dots}$$

$$= e^{-f(x_0)} \int dx e^{-\frac{1}{2} (x-x_0)^2 f''(x_0) + \dots}$$

Gaussian integral

ما از این ایده برای سبک کردن کنش موثر در حد یک حلقه (رتبه \hbar^0) استفاده می‌کنیم؛

For $J=0$:

$$Z[0] = \int \mathcal{D}\varphi e^{iS[\varphi]} = \int \mathcal{D}\varphi e^{i \int d^d x \mathcal{L}[\varphi]}$$

$\rightarrow \varphi = \varphi_c + \eta(x)$ with $\varphi_c = \text{solution of classical EOM.}$

i.e. $\frac{\delta \mathcal{L}}{\delta \varphi} \Big|_{\varphi_c} = 0$

$$\rightarrow \int d^d x \mathcal{L}[\varphi] = \int d^d x \mathcal{L}[\varphi_c + \eta(x)]$$

$$= \int d^d x \mathcal{L}[\varphi_c] + \int d^d x \eta(x) \frac{\delta \mathcal{L}}{\delta \varphi(x)} \Big|_{\varphi_c} + \dots$$

$$+ \frac{1}{2!} \int \frac{\delta^2 \mathcal{L}}{\delta \varphi(x) \delta \varphi(y)} \Big|_{\varphi_c} \eta(x) \eta(y) d^d x d^d y + \dots$$

نیاز فرض

$$\Rightarrow \boxed{S[\varphi] = S[\varphi_c] + \frac{1}{2!} \int d^d x d^d y \frac{\delta^2 \mathcal{L}}{\delta \varphi(x) \delta \varphi(y)} \Big|_{\varphi_c} \eta(x) \eta(y) + \dots}$$

$$Z[0] = e^{iS[\varphi_c]} \int \mathcal{D}\eta \exp\left(\frac{i}{2} \int d^d x d^d y \frac{\delta^2 \mathcal{L}}{\delta \varphi(x) \delta \varphi(y)} \Big|_{\varphi_c} \eta(x) \eta(y) + \dots\right)$$

آنندال روی $\eta(x)$ این توان گرفت.

$$Z[0] = e^{iW[0]} = e^{i\Gamma[\varphi_0]} \approx e^{iS[\varphi_0]} \left[\det \left(\frac{-\delta^2 L}{\delta\varphi(x)\delta\varphi(y)} \Big|_{\varphi_0} \right) \right]^{-1/2}$$

$\det A = e^{\ln \det A}$

$$Z[0] \approx e^{iS[\varphi_0]} \exp \left(-\frac{1}{2} \ln \det \left(\frac{-\delta^2 L}{\delta\varphi(x)\delta\varphi(y)} \Big|_{\varphi_0} \right) \right) = e^{i\Gamma[\varphi_0]}$$

$\rightarrow \Gamma[\varphi_0] = S[\varphi_0] + \frac{i}{2} \ln \det \left(\frac{-\delta^2 L}{\delta\varphi(x)\delta\varphi(y)} \Big|_{\varphi_0} \right)$ One-loop Effective Action.

\hbar^0 تا مرتبه تک حلقه

به بزرگی سال به بزرگی بزرگی بزرگی.

Functional Determinant (Background Field) (۲)

Fermion Determinant:

$I[A_\mu] = \int D\psi D\bar{\psi} \exp \left(i \int d^4x \bar{\psi}(x) (i\not{\partial} - g\not{A} - m) \psi(x) \right)$
 A_μ میدان است و در انتگرال سردی A_μ انتگرال نمی شود و تنها میدانهای ψ و $\bar{\psi}$ است که در انتگرال سردی قرار می گیرند.

$$\begin{aligned} I[A_\mu] &= \det (i\not{\partial} - m) = \det (i\not{\partial} - m - g\not{A}) \\ &= \det \left((i\not{\partial} - m) \left[1 - \frac{g\not{A}}{i\not{\partial} - m} \right] \right) = \det (i\not{\partial} - m) \det \left(1 - \frac{g\not{A}}{i\not{\partial} - m} \right) \\ &= \det (i\not{\partial} - m) \det \left(1 - \frac{i(-ig\not{A})}{i\not{\partial} - m} \right) \end{aligned}$$

is not important

$$\sim \det \left(1 - \frac{i}{i\not{\partial} - m} (-ig\not{A}) \right)$$

این قسمت از در میان فرمیون را می توان به صورت یک لپت در حساب توانی A_μ نوشت:

$$\log \det (i\not{\partial} - m) \sim \log \det \left(1 - \frac{i}{i\not{\partial} - m} (-ig\not{A}) \right) = \text{Tr} \log \left(1 - \frac{i}{i\not{\partial} - m} (-ig\not{A}) \right)$$

$$\begin{aligned} &= \text{Tr} \sum_{n=1}^{\infty} \frac{1}{n} (-1)^{n+1} \left(\frac{-i}{i\not{\partial} - m} (-ig\not{A}) \right)^n \\ \log(1+x) &= \sum_{n=1}^{\infty} \frac{1}{n} (-1)^{n+1} x^n \\ &= - \text{Tr} \sum_{n=1}^{\infty} \frac{1}{n} \left(\frac{i}{i\not{\partial} - m} (-ig\not{A}) \right)^n \end{aligned}$$

→

$$\det (i\not{D} - m) \sim \exp \left(- \sum_{n=1}^{\infty} \frac{1}{n} \text{Tr} \left(\frac{i}{i\not{D} - m} (-ig\not{A}) \right)^n \right)$$

نمایش نموداری صفت بالا: $-ig \gamma^\mu \int d^d x A_\mu(x)$

$$I[A_\mu] = \det (i\not{D} - m) = 1 + \text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \dots$$

See Peskin - Chapter 9

$$\stackrel{!}{=} \exp \left(\text{diagram 1} + \text{diagram 2} + \text{diagram 3} + \dots \right)$$

Note : 1)

تعداد ذرات در نمودارها

2)

n -insertion

n نمودار در رابطه:

- (-) for each fermion loop
- $(\frac{1}{n})$ for n -insertion of the external field A_μ

$\frac{1}{n}$ ضرب تعداد آن است. ← عبارت را می توان به نمودار در n از من رقیب:

$$- \frac{1}{n} \text{Tr} \left(\frac{i}{i\not{D} - m} (-ig\not{A}) \right)^n$$

$$= \frac{1}{n} \int d^d x_1 \dots d^d x_n \text{tr} \left[(-ig\not{A}(x_1)) S_F(x_1, x_2) (-ig\not{A}(x_2)) S_F(x_2, x_3) \right. \\ \left. \times \dots (-ig\not{A}(x_n)) S_F(x_n, x_1) \right]$$

A Toy Model - The Gross - Neveu Model

Aim: Computing the effective potential of this model.

- ✓ This model exhibits spontaneous symmetry breaking (Chiral)
- ✓ Asymptotically free theory (like QCD \rightarrow negative 1-loop β -funct.)
- ✓ A $d=2$ dimensional theory:

$$\mathcal{L} = \bar{\Psi}_i (i\not{\partial}) \Psi_i + \frac{g^2}{2} (\bar{\Psi}_i \Psi_i)^2 \quad i=1, \dots, N_f$$

number of flavors

$$\gamma^0 = \sigma_2$$

$$\gamma^1 = i\sigma_1$$

✓ Symmetry properties:

Classical \mathcal{L} has a discrete chiral symmetry:

$$\Psi_i \rightarrow \gamma_5 \Psi_i \quad \bar{\Psi}_i \rightarrow -\bar{\Psi}_i \gamma_5$$

$$\begin{aligned} \mathcal{L}' &= \bar{\Psi}'_i (i\not{\partial}) \Psi'_i + \frac{g^2}{2} (\bar{\Psi}'_i \Psi'_i)^2 \\ &= \underbrace{(-\bar{\Psi}_i \gamma_5)}_{=1} (i\not{\partial}) \underbrace{\gamma_5 \Psi_i}_{=1} + \frac{g^2}{2} (-\bar{\Psi}_i \gamma_5 \gamma_5 \Psi_i)^2 \\ &= \bar{\Psi}_i (i\not{\partial}) \Psi_i + \frac{g^2}{2} (\bar{\Psi}_i \Psi_i)^2 = \mathcal{L} \end{aligned}$$

$$\{\partial_\mu, \gamma_5\} = 0$$

(s.b.)

Semi-bosonized Lagrangian:

Introduce $\sigma = -g^2 \bar{\Psi}_i \Psi_i$ and rewrite \mathcal{L} :

$$\boxed{\mathcal{L}_{s.b.} = \bar{\Psi}_i (i\not{\partial}) \Psi_i - \frac{\sigma^2}{2g^2} - \sigma \bar{\Psi}_i \Psi_i}$$

$\sigma(x)$ is an auxilliary field \rightarrow

$$\frac{\delta \mathcal{L}_{s.b.}}{\delta \sigma(x)} = -\frac{\sigma}{g^2} - \bar{\Psi}_i \Psi_i = 0 \rightarrow \sigma = -g^2 \bar{\Psi}_i \Psi_i \quad \checkmark$$

Effective action of the model:

$$e^{iW[\mathcal{J}]} = \mathcal{Z}[\mathcal{J}] = \int \mathcal{D}\sigma \mathcal{D}\Psi \mathcal{D}\bar{\Psi} \exp\left(i \int d^d x (\mathcal{L}[\sigma, \Psi, \bar{\Psi}] + \mathcal{J}\sigma)\right)$$

$$\mathcal{D}\Psi = \prod_{i,x} d\Psi_i(x)$$

Define:

$$\frac{\delta W[\mathcal{J}]}{\delta \mathcal{J}(x)} = \langle \sigma(x) \rangle_{\mathcal{J}} = \sigma_{\mathcal{J}}(x)$$

نیمه کلاسیک

Legendre \bar{J}

$$W[J] = \Gamma[\sigma_J] + \int \mathcal{J}(x) \sigma_J(x) d^4x$$

for $J=0$ $\langle \sigma(x) \rangle_{J=0} = \sigma_{cl}(x)$

فرض $\lim_{J \rightarrow 0} \sigma_J = \sigma_{cl} = a = \text{const}$

$$\Gamma[\sigma_{cl}] = - \int \Omega_d V_{eff}(\sigma_{cl})$$

$$V_{eff}(\sigma_{cl}) = V_{eff}(a)$$

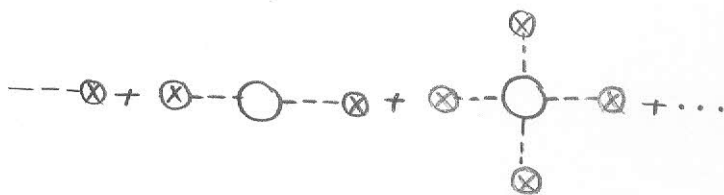
حالتی که در آن σ_{cl} است

propagator $V_{eff}(\sigma_{cl}) = \sum_{n=0}^{\infty} \frac{1}{n!} \sigma_{cl}^n \Gamma^{(n)}(\rho_i=0)$

$-\frac{1}{2g^2} \sigma_{cl}^2$ $\sim (2g^2)$
external field

$-\sigma_{cl} \bar{\psi}_i \psi_i$ \rightarrow $\begin{matrix} \bar{\psi}_i \\ \psi_i \end{matrix}$ -1

$\bar{\psi}_i (i\not{\partial}) \psi_i \rightarrow \frac{i}{\not{p}} \sim \frac{i\not{p}}{p^2}$



$$\begin{aligned} e^{i\Gamma[\sigma_{cl}]} &= \int \mathcal{D}\bar{\psi} \mathcal{D}\psi e^{iS_{s.b.}[\psi, \bar{\psi}, \sigma_{cl}]} \\ &= \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp\left(i \int d^d x [\bar{\psi}_i (i\not{\partial} - \sigma_{cl}) \psi_i - \frac{\sigma_{cl}^2}{2g^2}]\right) \\ &= \int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp\left(i \int d^d x \bar{\psi}_i (i\not{\partial} - \sigma_{cl}) \psi_i\right) e^{-i\Omega \frac{\sigma_{cl}^2}{2g^2}} \\ &= \underbrace{\int \mathcal{D}\bar{\psi} \mathcal{D}\psi \exp\left(i \int d^d x \bar{\psi}_i (i\not{\partial} - \sigma_{cl}) \psi_i\right)}_{(\det(i\not{\partial} - \sigma_{cl}))^{N_f}} \end{aligned}$$

$\psi = (\psi_1, \dots, \psi_{N_f})$

flavors ψ ψ

$$e^{i\Gamma[\sigma_{cl}]} = (\det(i\not{\partial} - \sigma_{cl}))^{N_f} e^{-i\Omega \frac{\sigma_{cl}^2}{2g^2}}$$

$$\Gamma[\sigma_{cl}] = -i \ln(\det(i\not{\partial} - \sigma_{cl}))^{N_f} - \Omega \frac{\sigma_{cl}^2}{2g^2}$$

$$\Gamma[\sigma_{cl}] = -i N_f \ln \det(i\not{\partial} - \sigma_{cl}) - \Omega \frac{\sigma_{cl}^2}{2g^2}$$

Question: $\ln \det(i\not{\partial} - \sigma_{cl}) = ?$

Use: $\frac{\partial}{\partial \alpha} x^{-\alpha} = \frac{\partial}{\partial \alpha} (e^{\ln x^{-\alpha}}) = \frac{\partial}{\partial \alpha} \frac{e^{-\alpha \ln x}}{e^{-\alpha \ln x}} = -\ln x e^{-\alpha \ln x} = -x^{-\alpha} \ln x$

$-\frac{\partial}{\partial \alpha} x^{-\alpha} \Big|_{\alpha=0} = (x^{-\alpha} \ln x) \Big|_{\alpha=0} = \ln x$
 $\ln x = -\frac{\partial}{\partial \alpha} x^{-\alpha} \Big|_{\alpha=0}$

$\rightarrow i \ln(\sigma^2 + p_E^2) = -i \frac{\partial}{\partial \alpha} \frac{1}{(\sigma^2 + p_E^2)^\alpha} \Big|_{\alpha=0}$
 $\rightarrow i \int \frac{d^d p}{(2\pi)^d} \ln(\sigma^2 + p_E^2) = -i \int \frac{d^d p}{(2\pi)^d} \frac{\partial}{\partial \alpha} \frac{1}{(\sigma^2 + p_E^2)^\alpha} \Big|_{\alpha=0}$

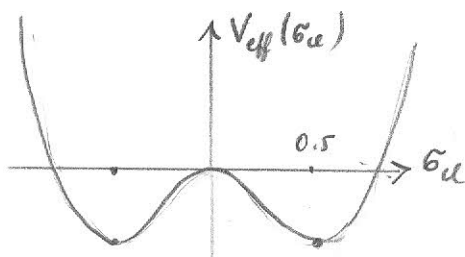
$= -i \frac{\partial}{\partial \alpha} \left\{ \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(\alpha - \frac{d}{2})}{\Gamma(\alpha)} \frac{1}{(\sigma^2)^{\alpha - \frac{d}{2}}} \right\} \Big|_{\alpha=0}$
 $= -\frac{i \sigma^d}{(4\pi)^d} \Gamma(-\frac{d}{2})$
 $= -i \frac{\Gamma(1 - \frac{d}{2})}{(1 - \frac{d}{2})} \frac{\sigma^2}{4\pi} \left(\frac{4\pi}{\sigma^2}\right)^{1 - \frac{d}{2}}$
 $= +i \frac{\sigma^2}{4\pi} \left(\frac{2}{\epsilon} - \gamma_E\right) \left(1 - \frac{\epsilon}{2} \ln \frac{\sigma^2}{4\pi}\right) \quad \epsilon = 2-d$
 $= \frac{i \sigma^2}{4\pi} \left(\frac{2}{\epsilon} - \gamma_E - \ln \frac{\sigma^2}{4\pi}\right) \stackrel{\text{MS-Scheme}}{=} \frac{i \sigma^2}{4\pi} \left(-\ln \frac{\sigma^2}{\mu^2}\right) \star$
 OR $\frac{i \sigma^2}{4\pi} \left(1 - \ln \frac{\sigma^2}{\mu^2}\right)$

$\rightarrow \Gamma[\sigma] = -i \frac{N_f}{2} \ln \det(\partial^2 + \sigma^2) - \frac{\Omega \sigma^2}{2g^2} =$
 $= -i \frac{N_f}{2} \Omega \int \frac{d^d p}{(2\pi)^d} \ln(\sigma^2 - p^2) - \frac{\Omega \sigma^2}{2g^2}$
 $= \left[-\frac{\sigma^2}{2g^2} - \frac{\sigma^2}{8\pi} N_f \ln \frac{\sigma^2}{\mu^2} \right] \Omega = -\Omega V_{\text{eff}} \quad \text{for } \sigma = \text{const.}$

$\rightarrow V_{\text{eff}}(\sigma_c) = \frac{\sigma_c^2}{2g^2} + \frac{N_f}{8\pi} \sigma_c^2 \ln \frac{\sigma_c^2}{\mu^2}$

OR $\frac{\sigma_c^2}{2g^2} + \frac{N_f}{8\pi} \sigma_c^2 \ln \left(\frac{\sigma_c^2}{\mu^2} - 1\right)$

For $2g^2 = 20$ $N_f = 1$ $\mu^2 = 1$, I got



Mexican Hat potential
 \rightarrow Spontaneous Symmetry Breaking!