Lusternik-Schnirelmann Theory for a Morse Decomposition

M.R. Razvan
Institute for Studies in Theoretical Physics and Mathematics
P.O.Box: 19395 − 5746*, Tehran, IRAN*
E-Mail: razvan@karun.ipm.ac.ir

Abstract

Let φ^t be a continuous flow on a metric space X and I be an isolated invariant set with an index pair (N, L) and a Morse decomposition $\{M_i\}_{i=1}^n$. For every category ν on N/L, we prove that $\nu(N/L) \leq \nu([L]) + \sum_{i=1}^n \nu(M_i)$. As a result if $\varphi^t|_I$ is gradient-like and X is semi-locally contractible, then φ^t has at least $\nu_H(h(I)) - 1$ rest points in I where $h(I)$ is the Conley index of I and ν_H is the Homotopy Lusternik-Schnirelmann category.

Keywords: Conley Index, gradient-like flow, Lusternik-Schnirelmann category.

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1 Introduction

Conley's homotopy index theory was first introduced as a generalization of Morse theory [2] and it was indeed a landmark in this field and related subjects. From the view point of critical points, Morse theory concerns the relations between the topology of a manifold and the number of analytically distinct critical points of smooth functions on it. However the natural problem is the number of geometrically distinct critical points and this is investigated by Lusternik-Schnirelmann theory. An important result in this theory is that if there is a smooth function $f : M \longrightarrow \mathbb{R}$ on a compact boundaryless manifold M with n critical points, then M is covered by n contractible open subsets [1]. The above result can be generalized for compact metric spaces as follows: If a compact locally contractible metric space X admits a gradient-like flow with n rest points, then X is covered by n contractible open subsets of X.

The compactness assumption is crucial in the above results. Indeed every noncompact manifold admits a smooth function without critical points. But in some applications of critical point theory, we deal with a compact subset of a noncompact space [3]. In [2] and [4], the results of Morse theory were generalized for every isolated invariant set. In this paper we use Conley index theory to obtain Lusternik-Schnirelamnn results for isolated invariant sets.

We need some basic results from Conley index theory which are presented in the next section. Then in section 3, we define the concept of category and prove a Ljusternik-Schnirelmann result in Conley index theory. In section 4, we consider a well-known example of Lusternik-Schnirelmann category and we obtain the results of [7].

2 Conley Index Theory

Let φ^t be a continuous flow on a metric space X. An isolated invariant set is a subset I of X which is the maximal invariant set in a **compact** neighborhood of itself. Such a neighborhood is called an isolating neighborhood.

Definition. A Morse decomposition of I is a collection $\{M_i\}_{i=1}^n$ where each M_i is an isolated invariant subset of I and for all $x \in I - \bigcup_{i=1}^n M_i$ there exist $i, j \in \{1, \cdots, n\}$ such that $i < j$, $\alpha(x) \in M_i$ and $\omega(x) \in M_j$.

Let V be any isolating neighborhood for I. We define

$$I^+ = I^+(V) = \{x \in V \mid \varphi^{[0,\infty)}(x) \subset N\},$$
$$I_j^+ = I^+(N) = \{x \in N \mid \varphi^{[0,\infty)}(x) \subset N \text{ and } \omega(x) \subset M_1 \cup \cdots M_j\}$$

for $j = 1, \cdots, n$. In [4] it is proved that I_j is compact and for every $x \in N$ with $\varphi^{[0,\infty)}(x) \subset N$, there is a $j \in \{1, \cdots, n\}$ such that $\omega(x) \subset M_j$ (See [10], Lemma 3.8).

In order to define the Conley index of an isolated invariant set, we follow [8]. Given a compact pair (N, L) we define the induced semi-flow on N/L by:

$$\varphi_{\#}^t : N/L \longrightarrow N/L, \quad \varphi_{\#}^t(x) = \begin{cases} \varphi^t(x) & \text{if } \varphi^{[0,t]}(x) \subset N - L \\ [L] & \text{otherwise.} \end{cases}$$

In [8] it is proved that $\varphi^t_{\#}$ is continuous if and only if:

i) L is positively invariant relative to N, i.e.

$$x \in L, \varphi^{[0,t]}(x) \subset N \Rightarrow \varphi^{[0,t]}(x) \subset L.$$

ii) Every orbit which exits N goes through L first:

$$x \in N, \varphi^{[0,\infty)}(x) \not\subset N \Rightarrow \exists_{t\geq 0} \text{ with } \varphi^{[0,t]}(x) \subset N, \varphi^t(x) \in L,$$

or equivalently if $x \in N - L$ then there is a $t > 0$ such that $\varphi^{[0,t]}(x) \subset N$.

An index pair for an isolated invariant set $I \subseteq X$ is a compact pair (N, L) in X such that $\overline{N - L}$ is an isolating neighborhood for I and the semi-flow $\varphi^t_{\#}$ induced by φ^t is continuous.In [2], [10] and [8] it is shown that every isolated invariant set I admits an index pair (N, L) and the homotopy type of the pointed space N/L is independent of the choice of the index pair. The Conley index of I is the homotopy type of N/L.

Note. We shall not distinguish between $N - L$ and $N/L - \{[L]\}$.

Lemma 2.1. If A is a compact subset of N/L with $A \cap I^+ = \varnothing$, then there exist a $T \in \mathbb{R}^+$ such that $\varphi^T_{\#}(A) = [L]$. In particular $[L]$ admits a contractible neighborhood in N/L.

Proof. Since I is the maximal invariant set in $\overline{N - L}$, for every $x \in A$, there is a $t \in \mathbb{R}^+$ such that $\varphi^t(x) \notin \overline{N - L}$. Thus $\varphi^t(y) \notin \overline{N - L}$ for every y in a neighborhood of x. Now by compactness, we can find a $T \in \mathbb{R}^+$ with $\varphi^{[0,T]}(x) \not\subset \overline{N - L}$ for every $x \in A$ and the proof is complete. It also shows that $(\varphi^t_{\#})^{-1}([L])$ is a compact contractible neighborhood of $[L]$ for sufficiently large amounts of t. $\square$

Lemma 2.2. Let I be an isolated invariant set with an isolating neighborhood N and a Morse decomposition $\{M_i\}_{i=1}^n$. If A is a closed subset of $I^+_j - I^+_{j-1}$ and U is a neighborhood of M_j, then there exists a $T \in \mathbb{R}^+$ such that $\varphi^t(A) \subset U$, for every $t \geq T$.

Proof. We may assume that U is a compact neighborhood of M_i such that $U \subset N$ and $U \cap I^+_{j-1} = \varnothing$. Now for every $x \in U \cap (I^+_j - \overset{\circ}{U})$ there is a $t \in \mathbb{R}^+$ such that $\varphi^{-t}(x) \notin U$. Since $U \cap (I^+_j - \overset{\circ}{U})$ is compact, there exists a $T \in \mathbb{R}^+$ such that $\varphi^{[-T,0]}(x) \not\subset U$ for every

$x \in U \cap (I_j^+ - \overset{\circ}{U})$. Now choose a neighborhood V of M_j such that $\varphi^{[0,T]}(V) \subset \overset{\circ}{U}$. We claim that $\varphi^{[0,\infty)}(V \cap I_j^+) \subset \overset{\circ}{U}$. Suppose the contrary, then $\varphi^t(x) \notin \overset{\circ}{U}$ for some $x \in V \cap I_j^+$ and $t \in \mathbb{R}^+$. Since $\varphi^0(x) = x \in \overset{\circ}{U} \cap I_j^+$, there is a $t_0 < t$ such that $\varphi^{t_0}(x) \in U \cap (I_j^+ - \overset{\circ}{U})$ for the first time, i.e. $\varphi^s(x) \in \overset{\circ}{U}$ for $s \in [0, t_0)$. Since $\varphi^{[0,T]}(V) \subset \overset{\circ}{U}$, we have $T < t_0 < t$. Thus $\varphi^{t_0 - T}(x) \notin U$ which is a contradiction with $\varphi^s(x) \in \overset{\circ}{U}$ for $0 < s < t_0$. Now for every $x \in A$, there is a $t \in \mathbb{R}^+$ such that $\varphi^t(x) \in V$ and since A is compact, there is a $T \in \mathbb{R}^+$ such that $\varphi^s(x) \in V$ for some $s \in [0, T]$. Since $\varphi^s(x) \in V \cap I_j^+$ and $\varphi^{[0,\infty)}(V \cap I_j^+) \subset U$, we conclude that $\varphi^t(A) \subset U$, for every $t \geq T$. $\square$

3 Lusternik-Schnirelmann Theory

Let M be a topological space. A category on M is a map $\nu : 2^M \longrightarrow \mathbb{Z} \cup \{+\infty\}$ which satisfies the following axioms:
i) If $A \subset B$, then $\nu(A) \leq \nu(B)$.
ii) $\nu(A \cup B) \leq \nu(A) + \nu(B)$.
iii) For every $A \subset M$, there is an open set $U \subset M$ with $A \subset U$ and $\nu(A) = \nu(U)$.
iv) If $f : M \longrightarrow M$ is homotopic to the identity id_M, then $\nu(A) \leq \nu(f(A))$ for every $A \subset M$.

It has been shown that if ν is a category on a compact metric space X satisfying the following axiom:
v) If A consists of a single point, then $\nu(A) = 1$,
then every gradient-like flow on X possesses at least $\nu(X)$ lest points (cf.[6]). The following theorem gives a generalization of this result and also the resuls of [7] and [9].

Theorem 3.1. Let φ^t be a continuous flow on a metric space X and I be an isolated invariant set for φ^t with a Morse decomposition $\{M_i\}_{i=1}^n$. If (N, L) is an index pair for I and ν is a category on N/L, then $\nu(N/L) \leq \nu([L]) + \sum_{i=1}^n \nu(M_i)$. Moreover, if $L = \varnothing$, then $\nu(N) \leq \sum_{i=1}^n \nu(M_i)$.
Proof. Recall that for every $1 \leq j \leq n$, I_j^+ is compact where $I_j^+ = \{x \in N | \varphi^{[0,\infty)}(x) \subset N$ and $\omega(x) \subset M_1 \cup \cdots \cup M_j\}$, $I_j^+ \subseteq I_{j+1}^+$ and $I_n^+ = I^+ = \{x \in N | \varphi^{[0,\infty)}(x) \subset N\}$. We construct $V_j \supset M_j$ open in N/L such that $\nu(V_j) = \nu(M_j)$ and $I_j \subset \bigcup_{i=1}^j V_i$ for $1 \leq j \leq n$

by induction. For $j = 1$, by axioms (i) and (iii), there is an open set $U_1 \subset N - L$ such that $M_1 \subset U_1$ and $\nu(M_1) = \nu(U_1)$. Now by Lemma 2.2, there is a $t_1 \in \mathbb{R}^+$ such that $\varphi^{t_1}(I_1^+) \subset U_1$ and hence $I_1^+ \subset V_1 := \{x \in N | \varphi^{[0,t_1]}(x) \subset N \text{ and } \varphi^{t_1}(x) \in U_1\}$. It is easy to see that V_1 is an open set in N, $M_1 \subset V_1 \subset N - L$ and $M_1 \subset \varphi_{\#}^{t_1}(V_1) \subset U_1$. Since $\varphi_{\#}^t$ is homotopic to identity, by axioms (i) and (iv) $\nu(M_1) \leq \nu(V_1) \leq \nu(\varphi_{\#}^t(V_1)) \leq \nu(V_1) = \nu(M_1)$ and hence $\nu(V_1) = \nu(M_1)$. V_1 is the desired open set in N/L.

Now suppose that we have constructed $V_1, \cdots, V_{j-1}$ open in N/L with $M_i \subset V_i$ and $I_{j-1}^+ \subset \bigcup_{i=1}^{j-1} V_i$. Similar to the case $j = 1$, we choose an open set $U_j \subset N - L$ with $\nu(U_j) = \nu(M_j)$. Now $I_j^+ - \bigcup_{i=1}^{j-1} V_i$ is a compact subset of $I_j^+ - I_{j-1}^+$ and by Lemma 2.1 there exists a $t_j \in \mathbb{R}^+$ such that $\varphi^{t_j}(I_j^+ - \bigcup_{i=1}^{j-1} V_i) \subset U_j$ and hence

$$I_j^+ - \bigcup_{i=1}^{j-1} V_i \subseteq V_j := \{x \in N | \varphi^{[0,t_j]}(x) \subset N \text{ and } \varphi^{t_j}(x) \in U_j\}$$

It is not hard to see that V_j is open in N, $M_j \subset V_j \subset N - L$ and $M_j \subset \varphi_{\#}^{t_j}(V_j) \subset U_j$. Thus $\nu(M_j) = \nu(V_j)$ and $I_j^+ \subseteq \bigcup_{i=1}^{j} V_i$.

Now if $L = \varnothing$, then $N = I^+ = I_n^+ = \bigcup_{i=1}^n V_i$ and by axiom (ii), $\nu(N) \leq \sum_{i=1}^n \nu(V_i) = \sum_{i=1}^n \nu(M_i)$. If $L \neq \varnothing$, then by Lemma 2.1. there is a $T \in \mathbb{R}^+$ such that $\varphi_{\#}^T(N/L - \bigcup_{i=1}^n V_i) = [L]$. This shows that $\nu(N/L - \bigcup_{i=1}^n V_i) = \nu([L])$ by axiom (iv) and now by axiom (ii)

$$\nu(N/L) \leq \nu([L]) + \sum_{i=1}^n \nu(V_i) = \nu([L]) + \sum_{i=1}^n \nu(M_i). \ \square$$

Corollary 3.2. Let I be a compact invariant subset of X and $\{M_i\}_{i=1}^n$ be a Morse decomposition for I. Then for every category ν on X, $\nu(I) \leq \sum_{i=1}^n \nu(M_i)$

Proof. Consider the flow $\varphi^t|_I$ on compact metric space I. Then $(I, \varnothing)$ is an index pair for I with respect to $\varphi^t|_I$. Now the above argument shows that there exist $V_1, \cdots, V_n$ open in I and $t_1, \cdots, t_n \in \mathbb{R}^+$ such that $\bigcup_{i=1}^n V_i = I$ and $\varphi^{t_i}(V_i) \subset U_i$ where U_i is an open subset of I with $\nu(M_i) = \nu(U_i)$. Since φ^{t_i} is defined on X and homotopic to id_X, we have $\nu(V_i) \leq \nu(U_i) = \nu(M_i)$ by axiom (iv) and hence $\nu(I) \leq \sum_{i=1}^n \nu(V_i) \leq \sum_{i=1}^n \nu(M_i)$. $\square$

Now suppose that X is a compact manifold and φ^t is the gradient flow of a smooth function with n critical points. Then in the above corollary, each U_i can be chosen a disk and it follows that X can be covered by n open disks. Moreover if $f : X \longrightarrow \mathbb{R}$ is a smooth function with n critical value $c_1 < \cdots < c_n$ and M_i is the set of critical points in $f^{-1}(c_i)$, then we can use the above corollary to obtain the results of [9]. Similarly if X is a locally contractible metric space and φ^t is a gradient-like flow with n rest points, then X can be covered by n contractible open sets.

Theorem 3.3. Let φ^t be a continuous flow and I be an isolated invariant set for φ^t such that $\varphi^t|_I$ is gradient-like. Then for every index pair (N, L) for I and every category ν on N/L which satisfies axiom (v) in I, φ^t has at least $\nu(N/L) - \nu([L])$ rest points in I.
Proof. We may assume that $\varphi^t|_I$ has a finite number of rest points $x_1, \cdots, x_n$. Since $\varphi^t|_I$ is gradient-like, these rest point give a Morse decomposition of I. Now by Theorem 3.1., $\nu(N/L) \leq \nu([L]) + \sum_{i=1}^{n} \nu(x_i)$. Since ν satisfies axiom (v) in I, $\nu(N/L) \leq \nu([L]) + n$. Thus the number of rest points of $\varphi^t|_I$ is not less than $\nu(N/L) - \nu([L])$. $\square$

4 HLS Category

Let M be a topological space. A subset $A \subset M$ is called contractible in M if the inclusion map $A \longrightarrow M$ is homotopic to a constant. We define Homotopy Lusternik-Schnirelmann category as follows:

Definition. The HLS-category $\nu_H(A) = \nu_H(A, M)$ of a subset $A \subset M$ is defined to be the minimum number of open sets contractible in M required to cover M. If such a cover does not exist, we set $\nu_H(A) = \infty$ and if it exists, A is called H-categorizable (in M).

A subset $A \subset M$ is H-categorizable if and only if $\nu_H(\{x\}) = 1$ for every $x \in A$. Thus ν_H satisfies axiom (v) if and only if M is H-categorizable. It is easy to see that ν_H satisfies axioms (i)-(iii). The following result [5] gives a generalization of axiom (iv).

Lemma 4.1. If Y dominates X, i.e. there are continuous maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ with $g \circ f \sim id_X$), then for every H-categorizable subset $A \subset Y$, $f^{-1}(A)$ is H-categorizable and $\nu_H(f^{-1}(A)) \leq \nu_H(A)$. In particular if Y is H-categorizable, then so is X and $\nu_H(X) \leq \nu_H(Y)$.
Proof: It is enough to prove that for every open set $U \subset Y$ contractible in Y, $f^{-1}(U)$

is contractible in X. Consider the following commutative diagram in which i and j are inclusion maps:

$$\begin{tikzcd} X \arrow[r, shift left, "f"] & Y \arrow[l, shift left, "g"] \\ V = f^{-1}(U) \arrow[u, "i"] \arrow[r, "f|_V"'] & U \arrow[u, "j"'] \end{tikzcd}$$

$f \circ i = j \circ f|_V \Rightarrow g \circ f \circ i = g \circ j \circ f|_V \Rightarrow i \sim g \circ j \circ f|_V \sim$ constant. $\square$

Corollary 4.2. Let X and Y be topological spaces of the same homotopy type. If X is H-categorizable then so is Y and $\nu_H(X) = \nu_H(Y)$.

Proof: We have two maps $f : X \longrightarrow Y$ and $g : Y \longrightarrow X$ such that $f \circ g \sim id_Y$ and $g \circ f \sim id_X$. The above lemma shows that Y is H-categorizable and $\nu_H(Y) \leq \nu_H(X)$. Now since Y is H-categorizable, then so is X and $\nu_H(X) \leq \nu_H(Y)$. $\square$

Remark 4.3. Let I be an isolated set with a Morse decomposition $\{M_i\}_{i=1}^n$ and two index pairs (N, L) and (N', L'). Then there is a flow-defined homotopy equivalence between N/L and N'/L' which leaves each M_i invariant [10]. Thus by Lemma 4.1., $\nu_H(M_i, h(I))$, $\nu_H(I, h(I))$ and $\nu_H(h(I), h(I))$ make sense. Moreover by Lemma 2.1, $[L]$ admits a contractible neighborhood in N/L and hence $\nu_H([L], N/L) = 1$.

Theorem 4.4. Let I be an isolated invariant set with a Morse decomposition $\{M_i\}_{i=1}^n$. If $\nu_H(M_i, h(I)) < \infty$ for $1 \leq i \leq n$, then $h(I)$ is H-categorizable and $\nu_H(h(I), h(I)) \leq 1 + \sum_{i=1}^n \nu_H(M_i, h(I))$.

The above theorem is an immediate consequence of Theorem 3.1. In order to apply theorem 3.3, we need $h(I)$ to be H-categorizable. We introduce a class of metric spaces in which the Conley index of every isolated invariant set is H-categorizable.

Definition. A topological space X is called semi-locally contractible if for every $x \in X$ and open set $U \subseteq X$ with $x \in U$, there exists a neighborhood V of x such that $x \in V \subset U$ and V is contractible in U.

Lemma 4.5. Let I be an isolated invariant set in a semi-locally contractible metric space X. Then $\nu_H(I, h(I)) < \infty$ and hence $h(I)$ is H-categorizable.

Proof. Suppose that (N, L) is an index pair for I. Since X is semi-locally contractible, every $x \in I$ admits an open set contractible in $N-L$. Since I is compact, $\nu_H(I, h(I)) < \infty$ and by Theorem 4.4, case $n = 1$, $h(I)$ is H-categorizable. $\square$

Theorem 4.6. Let I be an isolated invariant set for a continuous flow φ^t on a semi-locally contractible metric space X. If $\varphi^t|_I$ is gradient-like, then φ^t has at least $\nu_H(h(I))-1$ rest points in I.

Proof. ν_H satisfies axiom (v) on $h(I)$ by lemma 4.6. and we can use Theorem 3.4. $\square$

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References

[1] Bott, R., Lectures on Morse theory, old and new. *Bull. Amer. Math. Soc*, **7** (1982), 331–358.

[2] Conley, C. , Isolated invariant set and the Morse index. CBMS Notes, **38** (1978), AMS Providence, RI.

[3] Conley, C. and Zehnder, E., The Birkhoff-Lewis fixed point theorem and a conjecture of V.I. Arnold, *Invent. Math.* **73** (1983), 33–49.

[4] Conley, C. and Zehnder, E., Morse type index theory for flows and periodic solutions for Hamiltonian systems. *Comm. Pure Appl. Math.*, **37** (1984), 207–253.

[5] James, I.M., On category, in the sense of Lusternik-Schnirelmann. *Topology*, **17** (1978), 331–349.

[6] McDuff, D. and Salamon, D., *Introduction to Symplectic Topology*, Oxford Scientific publications, 1994.

[7] Pozniak, M., Lusternik-Schnirelmann category of an isolated invariant set. *Univ. Iagel. Acta. Math.*, **31** (1994), 129–139.

[8] Robbin, J.W. and Salamon D.A., Dynamical system, shape theory and the Conley index. *Ergodic Theory Dynamical system*, **8*** (1988), 375–393.

[9] Rudyak, Y., A remark on fixed point sets of gradient-like flows, Preprint math.DG/9908017.

[10] Salamon, D., Connected simple systems and the Conley index for isolated invariant sets. *Trans. Amer. Math. Soc.*, **291** (1985), 1–41.