Time Series PhD, Fall 2021
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1. Consider the following model:

$$y_t . \gamma_0 = u_t$$

Where y_t and $\gamma_0$each have $k+1$ entries. y_t is a vector of observables variables with a finite second moment and γ_0 is a parameter vector to be estimated.

a. Suppose that u_t satisfies the conditional moment restriction:

$$E(u_t|F_{t-1}) = 0$$

Let z_t be an r-dimensional vector in the information set associated with F_{t-1}. This vector has a finite second moment and $E[z_t z_t']$ is nonsingular. Moreover $r > k$. In addition suppose that

$$\gamma_0 = \begin{pmatrix} 1 \\ \beta_0 \end{pmatrix}$$

i. Show that $E(z_t u_t) = 0$

ii. Suppose that $\{(y_t, z_t)\}$ is a stationary and ergodic process. In addition suppose that $\gamma_0 = \begin{pmatrix} 1 \\ \beta_0 \end{pmatrix}$. Suggest a way to estimate β_0

iii. Suppose that $y_t$depends only on information in F_t. Show that:

$$M_N = \sum_{t=1}^{N} z_t u_t$$

Is a martingale. Provided that $z_t u_t$ has a finite second moment, what can you say about

$$\frac{1}{\sqrt{N}} M_N$$

Be as precise as possible.

iv. A two-stage least squares estimator of β_0 solves:

$$a_N \frac{1}{N} \sum_{t=1}^{N} z_t y_t' \begin{bmatrix} 1 \\ b_N \end{bmatrix} = 0$$

Where

$$a_N = \frac{1}{N} \sum_{t=1}^{N} y_{2,t} (z_t)' \left[\frac{1}{N} \sum_{t=1}^{N} z_t z_t' \right]^{-1}$$

Where $y_{2,t}$ contains all of the entries of y_t except for the first entry. Under what circumstances will this estimator be efficient among the class of GMM estimators? In particular, consider two cases. First, suppose the u is conditionally homoscedastic. That is, suppose that $E(u^2|z) = \sigma^2$ independent of z. Second suppose that u conditionally heteroskedastic. That is, $E(u^2|z)$ varies with z.

b. Suppose that $u_t$satisfies the conditional moment restriction:

c. $E(u_t|F_{t-2}) = 0$

Let $z_t$be an r- dimensional vector that is in the information set associated with F_{t-2}.

i. Provided that $z_t u_t$has a finite second moment, what can you say about

$$\frac{1}{\sqrt{N}}\sum_{t=1}^{N} z_t u_t$$

Be precise as possible?

ii. Show how to construct an efficient GMM estimator. Be as precise as possible.

iii. Provide a confidence set for β. What asymptotic results are you using in your result? Hint: This should involve a χ^2distribution. Recall that in class we showed that

$$\sqrt{N}(b_N - \beta_0) \approx -(ad)^{-1} a \frac{1}{\sqrt{N}}\sum_{t=1}^{N} f(x_t, \beta_0)$$

For a limiting selection matrix a and $d = E\left[\frac{\partial f(x_t, \beta 0)}{\partial \beta}\right]$

2. Suppose we wish to estimate an MA(2) process $y_t = \mu + e_t + \theta_1 e_{t-1} + \theta_2 e_{t-2}$ where the e_t are iid $N(0, \sigma^2)$ random variables. Although estimation is possible using ML, explain how you could estimate the parameters of the model using indirect inference. Also indicate how the models specification can be tested using indirect inference.

3. Suppose x_t and y_t are each $AR(1)$ processes:

$$x_t = \rho x_{t-1} + \varepsilon_t, y_t = v_t + \gamma v_{t-1}$$

Show how to construct an ARMA representation for $x_t + y_t$

4. Suppose that a time series, $\{X_t\}$, has the following representation:

$$X_t = x_t + u_t$$

where x_t and u_t are purely indeterministic and ergodic, covariance stationary processes and $x_t \perp u_s$ for all s, t. Suppose that u_t is the source of seasonality in X_t. That is, the spectrum of u_t, $S_u\,(e^{-i\omega})$, has much power concentrated in the seasonal frequencies (i.e., those near $\omega = 2\pi/4$ in quarterly data). The spectrum of x_t, $S_x\,(e^{-i\omega})$, is smooth and does not display a peak in the seasonal frequencies. The econometrician seeks to estimate the "seasonally adjusted data", x_t, by projecting x_t onto a complete realization of X_t (i.e., $\{\ldots, X_{-1}, X_0, X_1, \ldots\}$):

$$x_t = \sum_{k=-\infty}^{\infty} h_k X_{t-k} + v_t \ ,$$

where v_t is uncorrelated with X_{t-k} for all k. Let $\hat{x}_t$ denote the 'season ally adjusted' data:

$$\hat{x}_t = \sum_{k=-\infty}^{\infty} h_k X_{t-k}$$

a. Derive the formula for $h\,(e^{-i\omega})$ in terms of the known objects, $S_u = (e^{-i\omega})$ and $S_x\,(e^{-i\omega})$.

b. Show that $S_{\hat{x}}\,(e^{-i\omega}) < S_x\,(e^{-i\omega})$ for all ω.

c. Show that if $S_x\,(e^{-i\omega})$ is smooth across all frequencies, while $S_u\,(e^{-i\omega})$ has sharp peaks at the seasonal frequencies, then $S_{\hat{x}}\,(e^{-i\omega})$ will have substantial $dips$ at the seasonal frequencies. (It may seem ironic that optimal seasonal adjustment produces a series, $\hat{x}_t$, that itself displays seasonality.)