Homework 3: New-Keynesian Models

Mohammad Hossein Rahmati *

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1. Time is discrete and denoted $t = 0, 1, \cdots$ There is a representative consumer who faces a random sequence of Markov shocks. The state at t is s_t and a finite history is

$$s^t = (s_t, s_{t-1}, \cdots, s_0)$$

with s_0 known at date zero. As of date zero, the probability of s^t is $f(s^t|verts_0)$.

The economy is comprised of a representative consumer, a representative perfectly competitive final goods firm and a continuum $[0, 1]$ of differentiated monopolistically competitive intermediate firms. The production function of a final goods firm is

$$y_t(s^t) = [\int_0^1 y_t(i, s^t)^\theta di]^{1/\theta}, 0 < \theta \leq 1 \tag{1}$$

where $y_t(s^t)$ is output of the final good and $y_t(i, s^t)$ is output of intermediate i. The profit maximization problem of a final good firm is

$$0 = max_{y_t(i,s^t)}\{\bar{P}_t(s^t)y_t(s^t) - \int_0^1 P_t(i, s^{t-1})y_t(i, s^t)\}$$

subject to the production function (1). The price level in units of account is $\bar{P}_t(s^t)$ while the price of intermediate i in units of account is $P_t(i, s^{t-1})$. The intermediate prices depend only on the history s^{t-1} because the prices set by intermediate firms will be set after the shock s_t has been realized in period t (see below)

(a) Show that this profit maximization problem leads to a demand function for intermediates

$$y_t(i, s^t) = [\frac{\bar{P}_t(s^t)}{P_t(i, s^{t-1})}]^{1/(1-\theta)}y_t(s^t) \tag{2}$$

and an ideal price index

$$\bar{P}_t(s^t) = [\int_0^1 P_t(i, s^{t-1})^{\theta/(\theta-1)}di]^{(\theta-1)/\theta}$$

Intermediate goods are set in a staggered, overlapping, fashion. In particular, each period a uniform fraction $1/N$ of intermediates set their price $P_t(i, s^{t-1})$ before s_t is realized. These prices are then fixed for N periods. The interval $[0, 1]$ is partitioned into N subsets with firms with names $i \in [0, 1/N)$ setting prices on dates $t = 0, N, 2N, \cdots$, and so

*Sharif University of Technology, rahmati@sharif.edu

on. An intermediate that can set its price on date t following s^{t-1} solves the following maximization problem

$$max_{P_t(i,s^{t-1})} \sum_{\tau=t}^{t+N-1} \sum_{s^\tau} Q_{\tau,t-1}(s^\tau|s^{t-1})[P_t(i,s^{t-1}) - \bar{P}_\tau(s^)\nu^\tau(s^\tau)]y_\tau(i,s^\tau) \tag{3}$$

subject to the demand curve (2) for their product. The term $Q_{\tau,t-1}(s^\tau|s^{t-1})$ denotes the price of a unit of account in s^τ discounted back to s^{t-1}. The term $\nu_t(s^\tau)$ denotes the firm's real marginal cost (discussed below).

The production function of an intermediate firm is Cobb-Douglas in capital and labor

$$y_t(i,s^t) = k_t(i,s^t)_t^n(i,s^t)^{1-\alpha}, 0 < \alpha < 1$$

and real marginal cost is given by

$$\nu_t(s^t) = min_{k_t(i,s^t),n_t(i,s^t)}\{r_t(s^t)k_t(i,s^t) + \omega_t(s^t)n_t(i,s^t)|k_t(i,s^t)^\alpha n_t(i,s^t)^{1-\alpha} = 1\}$$

where $r_t(s^t)$ and $\omega_t(s^t)$ denote competitive rental rates for capital and labor.

(b) Show that cost minimization implies

$$\frac{1-\alpha}{\alpha}\frac{k_t(i,s^t)}{n_t(i,s^t)} = \frac{\omega_t(s^t)}{r_t(s^t)}$$

Explain why this implies that all intermediate use the same capital/labor ratio and therefore why all intermediates have the same real marginal cost $\nu_t(s^t)$.

(c) Solve the intermediate price setting problem i.e., that the optimal price to set is

$$P_t(i,s^{t-1}) = \frac{1}{\theta}\frac{P\sum_{\tau=t}^{t+N-1}\sum_{s^\tau} Q_{\tau,t-1}(s^\tau|s^{t-1})\bar{P}_\tau(s^\tau)^{(2-\theta)/(1-\theta)}\nu_\tau(s^\tau)y_\tau(s^\tau)}{\sum_{\tau=t}^{t+N-1}\sum_{s^t au} Q_{\tau,t-1}(s^\tau|s^{t-1})\bar{P}_\tau(s^\tau)^{1/(1-\theta)}y_\tau(s^\tau)}$$

Give intuition for this price. [Hint: what does this reduce to if $N = 1$?].

The consumer has expected utility preferences over consumption, end of period real balances $m_t(s^t) \equiv M_{t+1}(s^t)/P_t(s^t)$, and leisure

$$\sum_{t=0}^{\infty}\sum_{s_t} \beta^t U[c_t(s^t), m_t(s^t), l_t(s^t)]f(s^t|s_0), 0 < \beta < 1$$

The utility function U is assumed to be strictly increasing and strictly concave in all arguments. The representative consumer trades in money, a complete set of nominal, one-period, state contingent bonds, and capital. She has the flow budget constraint

$$\bar{P}_t(s^t)[c_t(s^t) + k_{t+1}(s^t)] + M_{t+1}(s^t) + \sum_{s'} Q_{t+1,t}(s^t,s'|s^t)B_{t+1}(s^t,s')$$

$$\leq P_t\bar{(}s^t)[\omega_t(s^t)n_t(s^t) + (r_t(s^t)+1-\delta)k_t(s^{t-1})] + M_t(s^{t-1}) + B_t(s^{t-1},s_t) +_t(s^t) - T_t(s^t)$$

where $_t(s^t)$ denotes lump-sum profits from intermediate firms and and where $k_0 > 0$,$M_0 > 0$ and $B_0 = 0$ are given initial conditions. She also has the constraint

$$l_t(s^t) + n_t(s^t) \leq 1$$

on her endowment of time for leisure or labor.

(d) Explain in words the representative consumer's flow budget constraints. Explain the dating concepts and any other implicit assumptions.

(e) Derive and interpret the following FONC for the consumer's problem

$$\begin{aligned} \frac{U_{l,t}(s^t)}{U_{c,t}(s^t)} &= \omega_t(s^t) \\ U_{c,t}(s^t) - U_{m,t}(s^t) &= \beta \sum_{s'} U_{c,t+1}(s^t, s') \frac{\bar{P}_t(s^t)}{\bar{P}_{t+1}(s^t, s')} f(s'|s^t) \\ U_{c,t}(s^t) &= \beta \sum_{s'} U_{c,t+1}(s^t, s')[r_{t+1}(s^t, s') + 1 - \delta] f(s'|s^t) \\ Q_{\tau,t}(s^\tau|s^t) &= \beta^{\tau-t} \frac{U_{c,\tau}(s^\tau)}{U_{c,t}(s^t)} \frac{\bar{P}_t(s^t)}{\bar{P}_\tau(s^\tau)}, \tau > t \end{aligned} \tag{4}$$

where the short hand $U_{l,t}(s^t)?U_l[c_t(s^t), m_t(s^t), l_t(s^t)]$, and so on, is used. Money is introduced into the economy by having the exogenous money supply satisfy $M_{t+1}(s^t) = \mu_t(s^t)M_t(s^{t-1})$ where $mu_t(s^t)$ is an exogenous stochastic process, and where the government's budget constraint is $M_{t+1}(s^t) + T_t(s^t) \geq M_t(s^{t-1})$.

(f) Explain the following equilibrium conditions

$$\begin{aligned} k_t(s^{t-1}) &= \int_0^1 k_t(i, s^t) di \\ n_t(s^t) &= \int_0^1 n_t(i, s^t) di \\ c_t(s^t) + k_{t+1}(s^t) &= y_t(s^t) + (1-\delta) k_t(s^{t-1}) \\ M_{t+1}(s^t) &= M_{t+1}(s^t) \\ B_{t+1}(s^t, s') &= 0 \end{aligned} \tag{5}$$

(g) Use the market clearing conditions and the solution to the intermediates' cost minimization problem to show that the relation ship between the output of the final goods firm and aggregate capital and labor is given by

$$\begin{aligned} y_t(s^t) &= A_t(s^t) k_t(s^{t-1})^\alpha n_t(s^t)^{1-\alpha} \\ A_t(s^t) &\equiv \frac{\bar{P}_t(s^t)^{1/(\theta-1)}}{\int_0^1 P_t(i, s^{t-1})^{1/(\theta-1)} di} \end{aligned} \tag{6}$$

How do you interpret the factor $A_t(s^t)$? [Hint:what would $A_t(s^t)$ be if prices were fully flexible?].

(h) Try and reproduce as much as possible of the log-linear analysis of Chari, Kehoe and McGrattan in the case where $N = 2, \alpha = 0$ (only labor is used in production), household utility is

$$U(c, m, n) = \frac{1}{1-\sigma} \{[\phi c^{(\eta-1)/\eta} + (1-\phi) m^{(\eta-1)/\eta}]^{\eta/(\eta-1)} (1-n)^\psi\}^{1-\sigma}$$

and real money demand is assumed to be interest-inelastic

$$m_t = c_t$$

[Note: I am using $m_t = M_{t+1}/\bar{P}_t$ to denote end-of-period real money balances; CKM's notation is slightly different (it is spelled out on p. 1163), so be careful]. It's probably best to begin by solving for the symmetric non-stochastic steady state.

2. The economy is populated by a representative household with preferences represented by

$$U = E_0 \sum_{t=0}^{\infty} \beta^t \left(\frac{C_t^{1-\sigma}}{1-\sigma} + \phi_m \frac{m_t^{1-\xi}}{1-\xi} - \phi_n \frac{N_t^{1+\zeta}}{1+\zeta} \right), \sigma, \xi > 1; \zeta \geq 0; \phi_m; \phi_n > 0$$

where N_t denotes time devoted to work and C_t is a composite consumption index given by the Dixit-Stiglitz aggregator

$$C_t = [\int_0^1 C_t(i)^{\theta-1}\theta di]^{frac\theta\theta-1}, \theta > 1$$

where $C_t(i) \geq 0$ is the quantity of good $i \in [0, 1]$ consumed in period t and θ is the elasticity of substitution among consumption goods. The household's period t budget constraint is

$$\int_0^1 P_t(i)C_t(i)di + B_t + M_t \leq (1 - \tau_t)W_t N_t + (1 + i_{t-1})B_{t-1} + M_{t-1} + T_t + Q_t$$

where B_t are nominal bond holdings, it is the nominal interest rate, W_t is the nominal wage, Q_t are profits from firm ownership, T_t are lump sum transfers from the government and τ_t is the payroll tax rate.

There is a continuum of firms in the economy that are uniformly distributed over the unit interval. Each firm is indexed by $i \in [0, 1]$ and produces a differentiated good using the linear technology

$$Y_t(i) = AN_t(i), A > 0$$

$Y_t(i)$ is output of firm i and $N_t(i)$ is the quantity of labor used by firm i. Labor input is rented in a competitive market. Each firm acknowledges how the demand for its differentiated good i depends on its own price level $P_t(i)$ but regards itself as unable to affect the evolution of aggregate consumption C_t and the general price level P_t and takes these as given.

The government has control over the money supply, must satisfy its budget constraint and the supply of government bonds is constrained to zero in equilibrium. The money growth rate is constant and equal to zero. The government budget constraint in period t is

$$\tau_t W_t N_t = T_t + G$$

where $G > 0$ is a constant level of government spending and

$$\tau_t = (1 - \rho_\tau)\bar{\tau} + \rho_\tau \tau_{t-1} + \epsilon_t^\tau, 0 < \rho < 1, \tau_0 given$$

where ϵ_t^τ is a mean zero i.i.d random variable.

(a) In a symmetric equilibrium with flexible prices, compute the response of $\hat{y}_t, \hat{c}_t$ and $\hat{\imath}_t$ to a positive innovation in payroll taxes. Explain.

(b) In a symmetric equilibrium with one period preset prices, compute the response of $\hat{y}_t$, $\hat{c}_t$ and $\hat{\imath}_t$ to a positive innovation in payroll taxes. Explain.

(c) In a symmetric equilibrium where prices are staggered a la Calvo

i. derive the New Keynesian Phillips curve (note that $\alpha = 1$)

ii. compute the response of $\hat{y}_t$, $\hat{c}_t$ and $\hat{\imath}_t$ to a positive innovation in payroll taxes and explain.