Homework 5: Money

Mohammad Hossein Rahmati *

January 16, 2019

1. Consider an infinite horizon overlapping generations model in which:

   - all agents live two periods
   - t=1,2,....
   - there is no population growth
   - all agents are identical within a generation
   - agents have an endowment of $e > 0$ in youth and 0 in old age
   - agents in generation t have preferences given by $u(c_t^y) + v(c_t^o)$ where c_y^t is consumption in youth and c_t^o is consumption in old age
   - both $u(.)$ and $v(.)$ are strictly increasing and strictly concave
   - agents have access to a private storage the storage technology given by $f(s)$ where $f(.)$ is strictly increasing and strictly concave
   - storage must be non-negative Carefully note the various market structures in the parts which follow.

   (a) **Market Structure: No fiat money, no claims markets.**
   Consider the optimization problem of a representative generation t agent. Write down a necessary condition characterizing the choice of storage. Under what condition is storage strictly positive?

   (b) **Market Structure: No fiat money, claims market in period 0**
   Show that the allocation from part (a) is a competitive equilibrium. Characterize the prices supporting this equilibrium. Is this allocation Pareto optimal? Under what condition are we in the Samuelson case?

   (c) **Market Structure: Fiat money, No claims market**
   Now agents can hold non-negative amounts of fiat money as well as store goods. Consider the optimization problem of a representative generation t agent. Write down the necessary conditions for the optimal decisions on money demand and storage. Under what conditions is there an interior solution in which money is held and storage is strictly positive?

2. Consider an overlapping generations model with a constrant population of two-period lived people. Each has the utility function $u(c_1) + v(c_2)$, where c_i represents consumption in the ith period of life. Assume $u^{'}, v^{'} > 0$ and $u^{''}, v^{''} < 0$. Each is endowed with y goods when young

*Sharif University of Technology, rahmati@sharif.edu

and nothing when old. Goods can be stored with a linear technology that delivers x goods in period $t+1$ for each good stored in period t, with $x > 1$.
there is a fixed stock of M units of fiat money at the end of each period t. Each youn person is required to hold real money balanced worth at least γ goods for each good stored, a "reserve requilrment"

(a) Find the conditions defining a monetary equilibrium. Include the Kuhn-Tucker conditions for an equlibrium that is not in interior.

(b) Assume that the reserve requirment binds. Find and graph the equilibrium law of motion for real money balances $q_{t+1} = h(q_t)$. Can there be equilibrium paths with oscillating stock of storage?

(c) Assume a stationary interior solution. Combine the equilibrium conditions into a single equation implicitly defining personal real balances of fiat money, q, as a function of γ. Find an expression defining $q^{'}(\gamma)$.

(d) Now use $q(\gamma)$ and the equilibrium conditions to express stady-state utility as a function $W(\gamma)$. Find the γ that maximizes staedy state utility. The first order condition will suffice. HINT: At some point you will be able to use the agents' first order condition to simplify yout expression for $W^{'}(\gamma)$.

3. Consider an overlapping generations economy (call it economy I) where population grows at rate n: The representative consumer in each generations has preferences represented by

$$u(c_t^t; c_t^t + 1) = ln(c_t^t) + ln(c_{t+1}^t)$$

The consumer has endowment $e_t^t = w_1 > 0$ when young and no endowment when old. There is an initial generation of size normalized to 1 that is endowed with $m > 0$ units of Fiat money. Let p_t denote the nominal price level at period t (i.e. fiat money is the numeraire in this economy).

(a) Compute an (Arrow Debreu or Sequential Markets) equilibrium in which fiat money has positive value. Argue that it is unique.

(b) Now consider economy II. It is identical to economy I, but it has a payas-you-go social security system of size $\tau > 0$; where τ is the payroll tax paid by the young generation and $b = (1+n)\tau$ are the social security benefits when old. Note that economy II still has the initial old generation endowed with fiat currency $m > 0$: Does economy II have an (AD or SM) equilibrium in which money has positive value? Justify your answer. Describe the restrictions on the parameters $(w_1; n; \tau)$; if any, that are needed to assure the existence of such an equilibrium.

(c) If an equilibrium with valued fiat money exists, is it unique? Justify your answer.

(d) Consider a stationary equilibrium with valued fiat currency. Does it exist? Is it unique? Justify your answers. Describe how the value of money over time, as measured by the sequences $\{\frac{1}{p_t}\}_{t=1}^{\infty}$ varies across economies with different sizes of the social security system, as measured by τ.

4. We will consider a cash-in-advance environment to study "Unpleasant Monetarist Arithmetic". The technology is given by $y_t = n_t$ where $n_t \in [0, 1]$ is fraction of hours worked. Preferences are given by $\sum_{t=0}^{\infty} \beta^t \{U(c_t) - \gamma n_t\}$ where $U(c_t) = \frac{c_t^{1-\theta}}{1-\theta}$. Households can hold money and/or government bonds. Let M_{t+1} and B_{t+1} be money (dollars) and nominal bonds (claims to dollars next period) held by households between t and $t+1$. The government expands or contracts the money supply at a constant rate μ according to $M_{t+1} = (1+\mu)M_t$. Let p_t be the

dollar price of consumption goods and q_t be the consumption good price of a bond (so that $p_t q_t$ is the dollar price of a bond). Let τ_t be real lump sum taxes/transfers the government levies to help pay for its constant real expenditure g on goods. Assume that only money M_t accumulated last period can be used to purchase consumption goods c_t this period at price p_t.

(a) Write down the household's budget constraint in nominal terms. Transform it into real terms (e.g. let $m_t = \frac{M_t}{P_t}$).

(b) Write down the household's cash-in-advance constraint in nominal terms. Transform it into real terms.

(c) Write down the government's budget constraint in nominal terms. Transform it into real terms.

(d) Define a competitive equilibrium.

(e) Characterize a stationary competitive equilibrium where agents take $\tau_t = \tau$ as given. Under what conditions does the cash-in-advance constraint bind? Under the assumption that the c-i-a constraint binds, characterize real seignorage revenue in a steady state (i.e. defined as $\frac{\bar{M}_{t+1} - \bar{M}_t}{P_t} = \mu m_t = \mu m$ in a steady state). Are there values of θ such that an increase in μ leads to a decrease in real seignorage? That is, can the monetary authority be on the wrong side of the seignorage "Laffer" curve?

(f) Suppose there is an equilibrium where $B_t = B$ and M_t satisfies the government budget constraint for a constant money growth rate μ and all other government policy variables (i.e. τ_t) constant, taking as given M_0 and B_0. Now analyse the effect of an open market sale of bonds, defined as a decrease in the money supply at $t = 1$ to say $\widehat{M_1}$ accompanied by an increase in $B_t = \widehat{B}$ for $t \geq 1$ with the other fiscal government policy variables (i.e. τ) the same as before. This issue is known as "Unpleasant Monetarist Arithmetic". Hint: Increase B in the $t = 0$ and $t \geq 1$ government budget constraints separately.

5. Consider the following search model of money. Time is discrete and there is a continuum of agents with population normalized to 1. Any particular agent specializes in the production of one service (a nonstorable good) but likes other services in an interval of size $x \in (0, 1)$. She derives utility $u(q) = q^{1/2}$ from consuming $q \in R_+$ units of the service provided it falls in her desired interval. An agent discounts the future at rate $(1 + r)^{-1}$. There is a constant disutility $-q$ to producing q units of a service. Production and consumption occur at the end of the period (and hence should be appropriately discounted). At the beginning of time, a fraction of agents $M \in (0, 1)$ are randomly given one unit of currency. Currency is indivisible and can be stored only one unit at a time. Agents are exogenously matched in the following way. Agents with money (we will term them buyers) are randomly matched in pairs with agents without money. Thus, the probability that a buyer is matched with a seller whose good she desires is $x(1 - M)$. Also, the probability that a seller is matched with a buyer who wants her good is xM. Every agent's trading history is private information. Finally, assume that buyers submit take-it-or-leave-it offers (which amount to a trade of 1 unit of money for Q units of the seller's service).

(a) Taking the quantity of services bargained for Q as given, write down the value functions for a buyer $V_b(Q)$ and a seller $V_s(Q)$ respectively .

(b) Taking the value functions V_b and V_s as given, what is the value of Q from the buyer's take-it-or-leave-it offer? To answer this question, proceed as follows. What condition assures that a seller accepts money in exchange for the production of his services? In particular, what is the seller's utility if he accepts the offer (produces the service and obtains the unit of currency)? What is the seller's utility if he rejects the offer (and goes back into the search pool)? Under what conditions on Q then will the seller accept the offer? Hence, if the buyer is trying to get as much services as possible, what value of Q will she demand from the seller?

(c) Define a monetary equilibrium.

(d) Does a monetary equilibrium exist? If so, under what conditions on r, x, and M? Do any other equilibria exist?

(e) Does the price level vary with increases in M?

(f) If we define ex-ante welfare as $W = MV_b + (1 - M)V_s$, how is welfare affected by changes in the money supply?

6. **Based on Kiyotaki, Wright [1993]** Consider the following environment where the goods and money are indivisible. The exogenous parameter $0 < x < 1$ equals the proportion of commodities that can be consumed by any given agents and x also equals the proportion of agents that can consumes any given commodities. One unit of consumption yields $U > 0$, while consuming other commodities or money yields zero utilities. A fraction of M of the total agents at each period own money while $1 - M$ are producing goods or own commodities .Money and commodities are costlessly storable. There is a production sector. That is, once an agent consumes enter in production sector and during one time could produce one unit of output with probabilities of $\alpha > 0$. In exchange sector, agent who has just produced looks for other agent to trade. Traders in the exchange sector meet pairwise and with probability $\beta > 0$ find other traders. The exchange take place if and only if it is mutually agreeable, that is, if and only if both agents are at least as well off after the trade. Also there is a transaction cost $0 < \epsilon < U$, that must be paid by the receiver whenever any real commodity is accepted in trade.In the exchange sector two types of agents, commodities trader and money traders, exist. Let μ denote the fraction of trader in the exchange sector who are money trader, so that a trader located at random has money with probability μ and a real commodity with probability $1 - \mu$. Let Π denote the probability that a commodities trader accepts money and let π be the best response of the representative individual. Let V_j denote the value function for the individual in state $j = 0, 1, m$ indicates that he is a producer, a commodity trader or the money trader, respectively.

   (a) Assume that we do not have double coincide problem. write the Bellman's Equations. In the rest of problem assume that double coincide matching is possible.

   (b) For this case write the Bellman's Equations.

   (c) Assumes that N_0 and N_1 and N_m denote the number of producer, commodity trader or the money trader, respectively, in the steady state.Find the implicit function for the μ as a function of M and Π. Is the μ is increasing with respect to M? what about Π? or show that it is indeterministic?

   (d) What is the value of Π which there exist mixed strategy? The equilibrium is called mixed-monetary equilibrium. (hint: what happen if $\pi < x$, what is the best response of commodity traders.)

   (e) For simplicity assume that $\alpha \to \infty$ thus the production is instantaneous and $N_m = M$, $N_1 = 1 - M$ and $\mu = M$. Assume $x < \frac{1}{2}$, find the value μ^0 which maximize the welfare function.

7. **Based on Lotz, Shevchenko, Waller [2007]** Assume the following environment where the money and goods are indivisible. Agents discount future at rate r. Agents meet each other with probability α and the probability an agent can produce one's desired consumption good is

x. There is no double coincide of wants. The agents are (ex ante) two types, $i = H, L$, whose measure are given by μ_H and μ_L, respectively. Type i agents get utility $u_i(q)$ from consuming q and incur disutility $c_i(q)$ from producing q units. We assume that $\dot{u_i} > 0$, $\ddot{u_i} < 0$, $u_i(0) = 0$, $\dot{c_i} > 0$, $\ddot{c_i} > 0$, $\frac{\dot{u_i}(0)}{\dot{c_i}(0)} = \infty$ for all i, j. When goods are indivisible we assume that $u_i(1) = U_i$ and $c_i(1) = C_i$ where $U_i > C_H > C_L$ for $i = H, L$ and $\frac{C_h}{C_L} > \frac{U_H}{U_L}$.
Let M be the fraction of agents with money and they are constraint to hold no more than one unit of indivisible money. Let m_H denote the fraction of high types holding a unit of money and m_L denotes the same for the low type. Thus the fraction of money holder in each types differs from each other. Now we introduce the lottery in the economy. [1] We introduce two probability. If the match happen between buyer i and seller j, the buyer enjoy from utility U_i with probability λ_{ij} while pay the unit indivisible money with probability τ_{ij}. Similarly, assume that the seller j accepts to sell the unit of good to buyer i, then he will incur cost C_j with probability λ_{ij} and give money with probability τ_{ij}. Assume that V_i^1 denotes the stationary value function of the agent holding one unit of money and V_i^0 denotes the agents without money

(a) Consider the case where buyer has full bargaining power and take all the surplus. Thus when a type $i (i = H, L)$ buyer meets a type $j (j = H, L)$ seller, he makes a take-it-or-leave-it offer to the seller. The offer consist of the pair $(\lambda_{ij}, \tau_{ij})$. Write the bargaining problem and find the $(\lambda_{ij}, \tau_{ij})$ as a function of value functions.

(b) Prove that if $\tau_{LH} > \tau_{HL}$ then $m_H > M > m_L$.

(c) Define the stationary Bellman Equation and find the stationary value function. You can assume that $\rho = \frac{r}{x\alpha}$ For simplicity in the rest of problem assume that $\tau_{ij} = \tau_j > 0$ and $\lambda_{ij} = \lambda_j > 0$. Thus the lotteries prices only depend on the seller's type.

(d) It is important question that is it possible that money and lottery exist in equilibrium simultaneously. Show that for $\rho \in (0, \rho_1)$ and $M \in (0, \bar{M}_1]$, a unique monetary equilibrium exists with $\tau_H, \tau_L < 1$ and $\lambda_H, \lambda_L = 1$.

8. Consider the following cash-in-advance economy where firms are monopolistically competitive and must set prices for their goods one period in advance. The cash-in-advance constraints mean that expected inflation cause agents to inefficiently economize on money holdings. The monopolistic competition means that equilibrium output falls short of the efficient level. Sticky prices mean that unanticipated money changes have real effects. Thus the government faces a tradeoff between the costs of expected inflation and the benefits of unexpected inflation. Specifically, the economy consists of a representative family (composed of a worker/shopper household and a continuum of firms indexed by $i \in [0, 1]$) and a government. Each firm produces a distinct, nonstorable good i with the technology $y_t(i) = n_t(i)$ where $n_t(i)$ is the labor input at firm i. The representative household (HH) consumes a basket of goods (call it c_t) and supplies labor to each of the firms (call the total labor supply n_t). HH preferences are given by

$$\sum_{t=0}^{\infty} \beta^t \{ \frac{c_t^\alpha}{\alpha} -_n t \}$$

where $0 < \beta < 1, 0 < \alpha < 1$ and the composite goods are defined by

$$c_t = \left[\int_0^1 c_t(i)^{(\theta-1)/\theta} di \right]^{\theta/(\theta-1)}$$

[1] Remember the indivisible lobar Hansen (1985) that to resolve the noncovexity of labor introduce the lottery to number of agent participate in the labor market.

where $1 < \theta$ and $n_t = \int_0^1 n_t(i)di$.

The HH trades bonds as well as money. Bonds costing B_{t+1}/R_t dollars at time t return B_{t+1} dollars at time $t+1$ where R_t is the gross nominal interest rate between t and $t+1$ (i.e. the nominal bond price $Q_t = 1/R_t$). Bonds are available in zero net supply so $B_{t+1} = 0$ must hold in equilibrium. Let M_{t+1} be a HH's choice of money balances in period t and M^s_{t+1} is the government supply of money. The money supply evolves according to $M^s_{t+1} = x_t M^s_t$ via government lump sum taxes/transfers $\tau_t = (x_t - 1)M^s_t$ (note that this is also the government budget constraint). We assume the growth rate of the money supply lies in the set $x_t \in [\beta, x]$ where $x > 1$ so that the government might actually shrink the money supply. Timing in any period is given by:

1. The HH enters t with M_t and B_t chosen at $t-1$. Each firm enters t taking as given nominal price for its goods $P_t(i)$ chosen at $t-1$
2. The HH receives/pays nominal transfer/tax τ_t and its bonds mature (generating nominal balances of $M_t + \tau_t + B_t$).
3. The HH purchases B_{t+1} bonds and pays B_{t+1}/R_t out of its nominal balances
4. The shopper must use its nominal balances to purchase consumption goods from each of the firms
$$\int P_t(i)c_t(i)di \leq M_t + \tau_t + B_t - B_{t+1}/R_t$$
5. The firm hires $n_t(i)$ worker hours, pays nominal wages $W_t n_t(i)$ to the worker, and chooses next period's price $P_{t+1}(i)$ to maximize future profits.
6. The firm brings home any current nominal profits $D_t(i) = P_t(i)y_t(i) - W_t(i)n_t(i)$
7. The family reassembles and brings into period $t+1$ any unspent money holdings and income
$$M_{t+1} = W_t n_t + \int D_t(i)di + M_t + \tau_t + B_t - B_{t+1}/R_t - \int P_t(i)c_t(i)di$$

This ends the description of the environment. Before characterizing an equilibrium, in order to induce stationarity, normalize the household's cash-in-advance and budget constraints by setting $m_t = \frac{M_t}{M^s_t}, b_t = \frac{B_t}{M^s_t}, x_{t-1} = \frac{\tau_t}{M^s_t}, d_t(i) = D_t(i)M^s_t, p_t(i) = \frac{P_t(i)}{M^s_t}, \omega_t = \frac{W_t}{M^s_t}$. In this case, the c-i-a equations can be written

$$\int p_t(i)c_t(i)di \leq m_t + x_{t-1} + b_t - \frac{b_{t+1}x_t}{R_t}$$

and budget constraint equation as

$$m_{t+1}x_t = \omega_t n_t + \int d_t(i)di + m_t + x_{t-1} + b_t - \frac{b_{t+1}x_t}{R_t} - \int p_t(i)c_t(i)di$$

On Ramsey Equilibrium (RE):

(a) Let x denote the government's policy sequence (i.e. $x = (x0, x1, \cdots)$). Firm $i's$ allocation rule $\pi^i(x)$ is a specification of price settings $p_t(i), t = 0, 1, \cdots$ for each possible x. Let $\pi(x) = \{\pi_i(x), \forall i\}$. The representative household's allocation rule $f(x, \pi(x))$ is a specification of $(c_t, c_t(i), n_t, m_{t+1}, b_{t+1}), t = 0, 1, \cdots$ for each possible x and $\pi(x)$. Define a Ramsey Equilibrium

(b) The first step is to choose $\{c_t(i)\}$ to maximize c_t subject to $\int p_t(i)c_t(i)di \leq A$ given $\{p_t(i)\}$. Defining the aggregate price level as $p_t = \left[\int_0^1 p_t(i)^{1-\theta}di\right]^{1/(1-\theta)}$, show that $c_t(i) = \left(\frac{p_t(i)}{p_t}\right)^{-\theta}$ Also, show that $\int p_t(i)c_t(i) = p_t c_t$. Finally, rewrite the cash-in-advance constraint and budget constraint in terms of p_t and c_t.

(c) In the second step, use the results in part (b) when stating the necessary conditions for the household's choice of $\{c_t, n_t, m_{t+1}, b_{t+1}\}$.

(d) Assuming the c-i-a constraint binds, show that in a competitive equilibrium:

$$c_t = \frac{x_t}{p_t}$$

$$\omega_t = \frac{1}{\beta}x_t x_{t+1}\left(\frac{p_{t+1}}{x_{t+1}}\right)^{\alpha}$$

(e) State the firm's problem. In equilibrium, show that each firm charges the same fixed markup price over marginal cost so that $p_t(i) = p_t$ where

$$p_t = \frac{\theta}{\theta - 1}\omega_t$$

(f) Using the above results, what are equilibrium interest rates on bonds?

(g) Combining above equations yields the difference equation for c_t that must hold in any equilibrium under commitment

$$c_t = \beta\left(\frac{\theta - 1}{\theta}\right)\left(\frac{c_{t+1}^{\alpha}}{x_{t+1}}\right)$$

Suppose the government solves the social problem by maximizing subject to this equation and $c_t = n_t$. Interpret this problem in terms of the RE. Show the RE has $x_t = \beta$ for all t. In a stationary equilibrium, show $R_t = 1$. What does this say about optimal monetary policy with commitment? Interpret.

(h) Show that the worst equilibrium under commitment has $x_t = \bar{x}$

On Sustainable Equilibria without Commitment

(i) Without commitment, families and the government can revise their plans after any history of money growth rates. Let $h_t = (x_0, x_1, \cdots, x_t)$ where h_{-1} is a null history. Let the sequence $(\sigma_t(h_{t-1})), t = 0, 1, \cdots$ denote a government plan which specifies money growth at any time t conditional on the realization of history h_{t-1}.Firm i's allocation rule is now a sequence of price setting functions $p_t(i, h_{t-1})$ for each possible h_{t-1}. Again call it π. The representative household's allocation rule $f(h_t, \pi)$ is a specification of $(c_t, c_t(i), n_t, m_{t+1}, b_{t+1})$ for each possible h_t and any possible π consistent with h_t. Define a Sustainable Equilibrium

(j) Show that the government policy $\sigma_t^{worst}(h_{t-1}) = \bar{x}, \forall h_{t-1}$,constitutes a sustainable equilibrium. Hint: show the government has no incentive to deviate from $\bar{x}$ in any period, given that agents believe the government will play $\bar{x}$ in all periods.

(k) Suppose $1 < \bar{x} < \beta\left(\frac{\theta}{\theta-1}\right)^{1/(1-\alpha)}$. Furthermore, suppose HHs and firms play "revert to worst" strategies: as long as the government chooses $x_t = \beta$, they play according to the outcomes in the RE while if the government ever plays $x_t \neq \beta$, then they revert to the allocations in the worst equilibrium. Under what conditions is it possible to support the RE? Is β important for this result? Interpret.

(l)

9. Consider the following Diamond-Dybvig model with aggregate uncertainty about household type. In particular, while all agents are ex-ante (i.e. $t = 0$) identical, there are two equally likely events $\omega \in \{H, L\}$ at $t = 1$ such that when $\omega = H$ there is a high probability that any given agent is an early consumer and when $\omega = L$ there is a low probability she is an early consumer. Specifically, each agent faces an iid preference shock $\theta \in \{1, 2\}$ that is realized at $t = 1$ which depends on the aggregate state such that the probability of being an early consumer is given by $prob(\theta = 1|\omega = H) = \pi_H = \frac{3}{4}$ and $prob(\theta = 1|\omega = L) = \pi_L = \frac{1}{4}$. Otherwise, all other aspects of the environment are identical as that presented in class. In particular, early consumers have preferences $u(c_1) = \sqrt{c_1}$ and late consumers have preferences $u(c_2) = \sqrt{c_2}$. There are two storage technologies: a long term productive technology

$t = 0$	$t = 1$	$t = 2$
-1	1	R

and a pillow technology

$t = 1$	$t = 2$
-1	1

in which storage is privately observable.

(a) State and solve the planner's problem when the aggregate state and type are observable (First Best)

(b) State and solve the planner's problem when the aggregate state but not type is observable.

(c) Explain why a deposit contract cannot implement the planner's problem. (Note: a deposit contract is simply time contingent, but not state contingent).

(d) Devise a resource feasible government deposit insurance scheme that improves upon the autarkic solution.