Homework 3: Asset Pricing

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1. Consider an economy with a single representative consumer who maximize

$$E\sum_{t=}^{\infty}\beta^t u(c_t) \qquad 0<\beta<1, \qquad u(c_t)=ln(c_t+\alpha)$$

The sole source of single good is an everlasting tree that produce d_t units of the consumption good in period t. At the beginning of time 0, each consumer owns one such tree. The dividend process d_t is Markov, with $Prob\{d_{t+1}\leq d^{'}|d_t=d\}=F(d^{'},d)$.
Assume that the conditional density $f(d^{'},d)$ of F exists. There are competitive markets in titles to trees and in state contingent claims. Let p_t be the price at t of a title to all future dividends from the tree.

   (a) Prove that equilibrium price p_t satisfy:

$$p_t=(d_t+\alpha)\sum_{j=1}^{\infty}\beta^j E_t\left(\frac{d_{t+j}}{d_{t+j}+\alpha}\right)$$

   (b) Find aformula for the risk-free one period interest rate R_{1t}. Prove that, in the special case in which $\{d_t\}$ is independently and identically distributed, R_{1t} is given by $R_{1t}^{-1}=\beta k(d_t+\alpha)$, where k is a constant. Given a formula for k.

   (c) Find a formula for the risk free two peiod interest rate R_{2t}. Prove that, in the special case in which d_t is independently and identically distributed $R_{t2}$is given by $R_{2t}=\beta^2 k(d_t+\alpha)$ where k is the same constant that you found in part (b)

2. Consider a version of a "Lucas tree" economy in which there are two types of trees. Both types of trees are perfectly durable; a type-i tree ($i=1,2$) yields a random amount of dividends equal to dit in period t. Assume that $\{d_{1t}\}_{t=0}^{\infty}$ and $\{d_{2t}\}_{t=0}^{\infty}$ are i.i.d. sequences of random variables and that d_{1t} and d_{2s} are statistically independent for all t and s. In addition, for $i=1,2$, assume that d_{it} equals d_L with probability π_i and equals $d_H>d_L$ with probability $1-\pi_i$.
The economy is populated by a continuum (of measure one) of identical consumers with preferences over consumption streams given by:

$$E_0\sum_{t=0}^{\infty}\beta^t log(c_t)$$

where c_t is consumption in period t. In period 0, each consumer owns one tree of each type. Dividends are non-storable and are the only source of consumption goods. There are competitive markets in which consumers can buy and sell both types of trees.

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(a) Define a sequential competitive equilibrium in which the only assets that consumers trade are the two (types of) trees.

(b) Find an algebraic expression for the equilibrium price of a type-1 tree (measured in terms of today's consumption goods), assuming that the dividends of both types of trees are equal to dL today. Your expression should depend only on primitives (i.e., on the parameters describing preferences and technology).

(c) How many Arrow securities are there in this economy? Express the prices of these securities in terms of primitives.

(d) Use your answer from part (c) to find the price (expressed in terms of today's consumption goods) of an asset that pays one unit of the consumption good in the next period if the dividends of the two trees (in the next period) are not equal to each other and pays zero otherwise.

3. Consider a version of the Lucas "tree" model in which trees not only yield fruit (or dividends) but also enter the utility function directly. In particular, the representative consumer's preferences are given by:

$$E_0 \sum_{t=0}^{\infty} \beta^t [log(c_t) + Alog(s_t)]$$

where c_t is period t consumption, s_t is the number of trees held in period t, and A is positive. Thus owning more trees leads to higher utility: trees are considered "beautiful" and consumers value beauty. The tree yields a stochastic dividend stream $\{d_t\}_t = 0^\infty$. Assume that dt is independent and identically distributed (so that its realization today is statistically independent of its past realizations) and assume that $E(d_t^i) = m_i$ for all nonzero integers i. Dividends are the only source of consumption goods in this economy and they are not storable. Each consumer is endowed initially with one tree. Consumers can buy and sell trees in a competitive market.

(a) Derive the Euler equation of a typical consumer.

(b) Use your answer from part (a) to find the equilibrium price of a tree as a function of the current dividend. (Hint: Guess that the price is equal to a constant B times the current dividend, and then solve for B in terms of parameters.)

4. Consider a two-period exchange economy with identical consumers. Each consumer maximizes $u(c_0) + \beta E[u(c_1)]$, where c_t is consumption in period t. Each consumer is endowed in period 0 with one tree. Each tree yields one unit of the (nonstorable) consumption good in period 0 and a random amount of the consumption good in period 1. In particular, with probability π, each tree yields dH units of the consumption good in period 1; with probability $1-\pi$, each tree yields dL units of the consumption good in period 1, where $d_H > d_L$. (The trees are identical, so they all yield the same number of units of the consumption good-either d_H or d_L- in period 1.) In period 0, consumers trade two assets in competitive markets: trees and riskfree bonds (i.e., sure claims to one unit of the consumption good in period 1).

(a) Are markets complete in this economy? Explain why or why not.

(b) Carefully define a competitive equilibrium for this economy. Find the equilibrium consumption allocation and the equilibrium prices of a tree and of the riskfree bond.

(c) Now suppose that the market for trees is shut down so that consumers can trade only the riskfree bond. (Are markets complete in this case?) Carefully define a competitive equilibrium for this economy and show that the equilibrium consumption allocation and the equilibrium price of a riskfree bond are the same as in part (b). Explain intuitively why this result holds.

5. This problem introduces a "disaster" state (such as a Great Depression) into a Lucas "tree" economy like the one we have developed in lecture. Let the tree's dividend take on one of three values: $\overline{d_1} = 1 + \delta$, $\overline{d_2} = 1 - \delta$, and $\overline{d_3} = 0.5$, where $\delta = 0.02$. Let the transition probability matrix for the dividend be specified as follows:

$$\begin{bmatrix} 0.98 & 0.02 - p & p \\ 0.02 - p & 0.98 & p \\ 0.5 & 0.5 & 0 \end{bmatrix}$$

where p is close 0. Thus the dividend usually fluctuates between $1+\delta$ and $1-\delta$ but occasionally drops by roughly 50% to 0.5. After such a crash, the dividend immediately returns to its "normal" range. The purpose of this problem is to investigate whether the introduction of a rare disaster state can improve the ability of the Lucas tree model to match the equity premium. (This idea was first proposed, in a slightly different form, by T.A. Rietz (1988), "The Equity Risk Premium: A Solution," Journal of Monetary Economics 22, 117"131. See also the critical response by R. Mehra and E.C. Prescott (1988), "The Equity Risk Premium: A Solution?", Journal of Monetary Economics 22, 133-136.)

(a) Set $p = 0.001$. Compute the invariant distribution for the dividend. Use the invariant distribution to verify that the coefficient of variation of the dividend is close to 0.02, matching the standard deviation of the detrended log of aggregate consumption in U.S. data.

(b) Let the consumer's discount factor equal 0.998. Assume that the consumer has constant-relative-risk-aversion (CRRA) preferences, and let the coefficient of relative risk aversion equal 2. Compute the prices of all of the Arrow securities.

(c) Find the prices of a one-period risk-free bond and of a perpetual claim to the tree's dividend in each of the three states of the world.

(d) Compute (the unconditional expected value of) the equity premium. How close is it to 1.5%, its value in quarterly U.S. data? Find the value of the coefficient of relative risk aversion for which the theoretical equity premium matches its observed value.

(e) Using the preference parameters from part (c), compute the prices of a two-period risk-free bond in each of the three states of the world. (A two-period risk-free bond is a promise to pay 1 unit of the consumption good in all states of the world two periods from now.) Use your answer to compute the unconditional expected value of the rate of return on a two-period risk-free bond (call it r_2). How does $(1 + r_2)^{\frac{1}{2}} - 1$ compare to r_1, the unconditional expected value of the rate of return on a one-period risk-free bond? (Optional question: The difference between $(1 + r_2)^{\frac{1}{2}}$ and r_1 is a measure of the slope of the term structure of interest rates. How does this slope compare to the slope in the two-state model that we developed in lecture?)

6. Consider an economy populated by two types of individuals. Each agent of type I initially owns one type I tree. These trees drops dividends according to the stochastic process:

$$y_t^I = s_t y$$

Each type II individual initially owns a type II tree, whose devidends satisfy:

$$y_t^{II} = (1 - s_t) y$$

where the stochastic process $\{s_t\}$ is such that $0 \leq s_t \leq 1$. Assume that, at time 0(befores s_0 is realized) all agents believe that the distribution of So is given by

$$Pr[s_0 \leq s] = F(s)$$

All individuals of type j ($j \in \{I, II\}$) have preferences given by:

$$E\left[\sum_{t=0}^{\infty} \beta^t u_j(c_t)\right]$$

where $0 < \beta < 1$, and the functions $U_j(c)$ are "nice" (i.e. strictly increasing, strictly concave, twice differentiable and whatever else you need short of a functional form) Assume that the economy has complete markets (i.e. a full set of Arrow securities) which open just before the realization of S_0.

(a) Consider the special case in which $F(s)$ is uniform between 0 and 1, and that the distribution of future values of s_t is i.i.d, (not necessarily equal to $F(s)$). Go as far as you can can characterizing the equilibrium allocation of consumption, the one period risk-free interest and the prices of the two types of trees. Show your work.

(b) How would your answers change if the complete set of markets opens after the realization of s_0 but before the realization of s_1

(c) Assume now that the economy is as in 1, except that future (i.e. for $t \geq 1$) values of s_t are given by a Markov process with transition probability density given by $\pi(s_{t+1}, s_t)$. Go as far as you can describing how the equilibrium consumption allocations and the one period risk free interest rate in this economy differ from the equivalent concepts in the economy of part 1. Show your work.

(d) For the economy of section 2 discuss the following claim: "Independently of the coefficient of risk aversion in individual utility functions, the prices of both types of trees are given by the present expected discounted value -using the risk free interest rate- of their future dividends.

(e) Consider the basic economy, except that $F(s)$ is not uniform, and the distribution of s_t , for $t \geq 1$, is arbitrary. Go as far as you can describing properties of individual consumption allocations and asset prices (bonds and stocks in the two types of trees).

7. Time is discrete and starts at $t = 1$. Agents live for 2 periods, and have a labor endowment only when they are young. There is mass one of agents of each age, or if you prefer one agent per cohort. We index agents by the year in which they were born. There is one unit a perishable "tree" in the economy. This tree gives a fruit, in consumption units, of d_t each period. The labor endowment of an agent born at time t is denoted by w_t. The young born at $t = 1, 2$, have no endowment of trees. The old born at $t = 0$ have, at time $t = 1$, an endowment of 1 tree. Labor endowment and fruits are both proportional to the same random variable, Y_t :

$$w_t = (1 - \delta)Y_t \quad and \quad d_t = \delta Y_t$$

where $\delta \in (0, 1)$ is a constant, and where Y_t satisfy $Y_{t+1} = Y_t z_{t+1}$ with $Y_1 = 1$ and $\{z_{t+1}\}$ i.i.d. and $z_{t+1} > 0$. There is one unit of the tree in the economy. Feasibility is then given by

$$c_t^t + c_t^{t-1} = w_t + d_t = Y_t \quad for \quad all \quad t \geq 1.$$

The preferences for the young born at $t \geq 1$ are:

$$(1 - \beta)u(c_t^t) + \beta E_t[u(c_t^t + 1)]$$

where $E_t[.]$ is the conditional expectation. The preferences for the old born at $t = 0$ are c_1^0 The budget constraint of agent born at t when young is:

$$c_t^t + s_t q_t^e + b_t q_t^b = w_t$$

where s_t are his purchases of the tree at price q_t^e , b_t are the purchases of a one period discount bond a bond that pays one unit of consumption at $t+1$, with current price q_t^b . At time $t+1$, when this agent becomes old his consumption is financed by the fruits and sale of the trees, as well as the purchases of the discount bonds:

$$c_{t+1}^t = s_t[q_{t+1}^e + d_{t+1}] + b_t$$

(a) Using the budget constraint of the agent, replace c_{t+1}^t into the preferences and write down the objective function of the young agent born at time t as a function of s_t and b_t, taking current prices q_t^e , q_t^b and the distribution of future prices q_{t+1}^e and fruits d_{t+1} as given.

(b) Write down the first order conditions (foc's) with respect to st and bt of the problem stated above

(c) What has to be the equilibrium holding of trees of for generation $t \geq 1$, i.e. what has to be the equilibrium values of st for $t \geq 1$?

(d) Use the answer for the previous question, and the assumption that initial old (born at $t = 0$) have zero initial endowment of the discount bonds ($b_0 = 0$) to find out what should be the equilibrium value of bt be for all $t \geq 1$?

(e) Specialize the foc's obtained in (b) replacing the equilibrium values of st and bt obtained in the (c) and (d).

(f) Assume from now on that preferences are given by $u(c) = log(c)$. Guess that $q_t^e = Y_t Q^e$ and that $q_t^b = Q^b$ where Q^e and Q^b are two constants. Define $R_{t+1}^e = \frac{q_{t+1}^e + d_{t+1}}{q_t^e}$ and $R_t^b = \frac{1}{q_t^b}$ as the gross returns on the trees and discount bonds. Find

   i. An expression for the price of the three, Q^e as a function of δ and β,

   ii. An expression for the gross expected return of the tree, $E_t[\frac{q_{t+1}^e + d_{t+1}}{q_t^e}]$ as a function of δ, β and $E[z]$,

   iii. An expression for the gross return on the discount bond $\frac{1}{Q^b}$ as a function of β, δ and $E[\frac{1}{z}]$.
   Hints: use the foc's obtained in the previous question, as well as $d_{t+1} = \delta Y_{t+1}$, $w_t = (1-\delta)Y_t$, $z_{t+1} = \frac{Y_{t+1}}{Y_t}$, $q_{t+1}^e = Y_{t+1}Q^e$, $q_t^e = Y_t Q^e$.

(g) To double check your answer to (f.i), consider the following alternative reasoning, which uses that with log utility the share of expenditure is proportional to the share parameter in preferences.

   i. Using that the total wealth of a young agent is given by w_t, express the value of a young agent's consumption c_t^t as a function of β, δ and Y_t.

   ii. Assuming that $b_t = 0$ and $s_t = 1$ use the answer of (i) and the budget constraint of the young agents to derive and expression for Q^e. Compare this expression with your answer to i) in the previous part.

   iii. Give a short explanation of why Q^e is increasing in β and decreasing in δ (maximum two lines for each parameter). Assume from now on that $x = log(z)$ is normally distributed with expected value μ and variance σ^2. Recall that $E[e^x] = e^{\mu + \frac{\sigma^2}{2}}$ Let g be the expected continuously compounded growth rate of Y_t, so $g = \mu + \frac{\sigma^2}{2}$

(h) Under the log normal assumption for z find:

   i. An expression for the expected return of the trees as a function of β, δ, g and σ^2,

   ii. An expression for the level of the gross return on the discount bond as a function of β, δ, g and σ^2, and

iii. an expression for the multiplicative risk premium, i.e. the ratio of the gross expected return on the tree and the gross return on the discount bond as a function of only one of the parameters of the model (β, δ, g and σ^2,).
Hint: To compute $E(\frac{1}{z})$ use the formula displayed above for $(\frac{1}{z}) = e^{-x}$.

(i) Is the equilibrium value of aggregate consumption-using the corresponding Arrow-Debreu prices- finite?
Hint. To answer this, write down the present value of aggregate consumption as a function of the price of a tree. Notice that by paying the price of the tree an agent can obtain the right to the fruits dt at all dates $t \geq 2$. Recall that feasibility can be written as $c_t^t + c_t^{t-1} = Y_t$, and that we assume that $d_t = \delta Y_t$.

(j) Show that for σ^2 large enough, $R^b < e^g$, so that net interest rates are smaller than the expected growth rate of fruits, and hence of GDP.

(k) Given your answer to the previous questions,

i. Is the competitive allocation Pareto Optimal?,

ii. Can the introduction of social security improve the welfare of all generations?,

iii. Does your answer depends on the value of interest rates being smaller, equal or higher than the (expected) growth rate of the economy?,

iv. Recall that in the last 100 years the average for the US economy the real net return on US government bonds (the risk-free rate) is about 3% per year, the average net growth rate of total GDP is about 3%, and the difference between the net real return of a well diversified equity portfolio such as the *SP500* and the net risk-free rate is about 5% per year. Using this simple model as a guide, what you conclude about the existence (or lack of) dynamic inefficiency in the US economy?