Homework 2: Limited Information and Recursive Commitment

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1. Consider an insurance problem for a static economy with a large number of agents who are subject to idiosyncratic production shocks. We will consider several different assumptions about the information structure of those shocks.

   Environment

   - Population: A unit measure of agents.
   - Technology:
     - Each agent has one unit of time to allocate. She can choose between working ($n = 1$) and enjoying leisure ($n = 0$).
     - If the agent expends labor effort (i.e. $n = 1$), the output from that effort can be in one of two states $s \in \{h, l\}$. With probability $1 - p$ state h occurs where output from the effort is given by $y^h = A > 0$. With probability p state l occurs where output from the effort is $y^l = 0$.
     - If the agent expends no effort (i.e. $n = 0$), output is zero.
   - Preferences: The utility from consuming c units of the good is given by $u(c)$ where $u(.)$ is positive, strictly increasing and strictly concave when $c > 0, u(0) = 0$, and $u(c) = -\infty$ when $c < 0$. The utility from enjoying leisure (i.e. from choosing $n = 0$) is $b > 0$ and is additive to the utility from consuming the produced good. That is, if the agent chooses not to work and consumes c units of good, then total utility is $u(c) + b$.
   - Timing: Agents first exert effort $n \in \{0, 1\}$. If they chose $n = 1$, the productivity shocks are then realized.
   - Parametric Assumptions: $A > u^{-1}\left(\frac{b}{1-p}\right) > \frac{u^{-1}(b)}{1-p}$ where the second inequality can be shown given $u(.)$ is strictly concave.

   Case 1 - Full Information

   Assume the planner can observe both effort n and output y of each agent. State and solve the planner's problem given that she gives equal weight to every agent in the economy in the following steps:

   (a) If the planner chooses $n = 0$ for everyone, what is aggregate welfare?

   (b) State and solve the planner's problem if she chooses $n = 1$ for everyone and allocates c^h to agents with output $y^h = A$ and c^l to the agents with output $y^l = 0$

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(c) Verify that parametric assumption is sufficient to guarantee the social planner would choose $n = 1$ instead of $n = 0$ under full information.

Case 2 - Unobservable effort, observable output

Now suppose that effort $n \in \{0, 1\}$ is private information so that the planner cannot see the effort taken by each individual agent. Output of each agent, however, still remains observable to the planner so she can make allocations contingent on it.

(d) In what state does the planner face an inference problem? That is, under what choices of n and realizations of y are two agents indistinguishable? Is the allocation from question (1-b) incentive feasible? That is, if the social planner promises c^h and c^l that you solved for in the full information environment when $n = 1$, is this an incentive and resource feasible allocation?

(e) Suppose the social planner wants to induce every agent to exert effort $n = 1$ by allocating state contingent consumption $\hat{c}^h$ and $\hat{c}^l$. Under the case where output is observable, what constraint(s) must the social planner satisfy such that all agents indeed choose $n = 1$ instead of choosing $n = 0$ and falsely claiming to have worked? Hint: the incentive problem must respect the timing that the decision to work precedes the realization of the productivity shock (i.e. the incentive constraint must be feasible ex-ante).

(f) State and solve the problem of the social planner who wishes to induce $n = 1$. You do not need to provide a closed-form solution for the allocation but you need to provide the system of equations that characterizes the allocation.

(g) Is it possible to support some insurance in this case? Specifically, prove that when the social planner strictly prefers $n = 1$ to $n = 0$, the optimal allocation has $c^l > 0$. In other words, show that when it is optimal to incentivize agents to work, the social planner needs to provide partial insurance.

Case 3 - Unobservable effort, unobservable output

Now suppose neither effort nor output is observed by the planner. To be consistent with the social planner not being able to observe output, if an agent worked and received y^h he can keep his output concealed. However, it is not individually feasible for him to report something that he cannot resource feasibly transfer to the planner. The social planner now chooses a report contingent consumption allocation implemented through report contingent tax/transfers.

(h) List the constraints the social planner must now obey in order to induce $n = 1$ and also successfully implement insurance after the output realization (i.e. expost as opposed to ex-ante in case 2). Prove that the social planner cannot implement any insurance in this case and must resort to autarky.

Welfare Comparison

(i) Compare ex-ante welfare under the three cases with the parametric assumption on A.

2. Consider a two period economy in which agents face uncertainty regarding their preferences over the timing of their consumption. Number the periods $t = 0, 1$. Each egent has preferences over consumption given by $\theta log(c_0) + \beta log(c_1)$. Where θ is the random taste shoch realized at the beginig the of period $t = 0$. The shock θ is drawn from the set $\Theta = \{\theta^1, \cdots, \theta^N\}$ with probability $\pi(\theta^n)$. We assume that $\pi(\theta^n)$ also represents the fraction of agents who recieve shock θ^n. Let $\{c_0(\theta^n), c_1(\theta^n)\}_{n=1}^N$ denote the consumption allocation in this economy. Each period each agent is endowed with y unit of consumption good. This good can not be stored. The resource constraint for this economy are $\sum_{n=1}^N c_t(\theta^n)\pi(\theta^n) = y$ for $t = 0, 1$.

(a) **Bond Economy** Assume that at date 0 agents only trade an uncontingent bond that pays off 1 unit of consumption for sure at date 1. Let q denote the price of this bond and $b(\theta^n)$ the quantity purchased by an agent with shock θ^n. Given this notation, agents have budget constraint:

$$c_0(\theta^n) + qb(\theta^n) = y \quad and \quad c_1(\theta^n) = y + b(\theta^n)$$

Solve for the equilibrium bond price q and the consition allocation $\{c_0(\theta^n), c_1(\theta^n)\}_{n=1}^N$.

(b) **Full Information Social Optimum** Now solve for the allocation that maximizes the utilitarian social welfare function

$$\sum_{n=1}^{N}[\theta log(c_0(\theta^n)) + \beta log(c_1(\theta^n))]\pi(\theta^n)$$

subject to the resource constraint. Define agents's marginal rate of substitution q^n by:

$$q^n = \frac{\frac{\beta}{c_1(\theta^n)}}{\frac{\theta^n}{c_0(\theta^n)}}$$

Is q^n equated across agents in the full information social optimum allocation? Is do, call this q^*. Does q^* equal the q that you found in the bond economy in bart (a)? Does the social optimum allocation satisfy the budget constraints from the bond economy in part (a)?

(c) **Private Information** Now consider the problem of finding an optimal allocation $\{c_0(\theta^n), c_1(\theta^n)\}_{n=1}^N$ that is also incentive compatible in an economy in which agents' taste shocks θ^n are private information. Specially, we say that an allocation is incentive compatobe if :

$$\theta^n log(c_0(\theta^n)) + \beta log(c_1(\theta^n)) \geq \theta^n log(c_0(\theta^i)) + \beta log(c_1(\theta^i))$$

for all $i = 1, \cdots N$. Is the allocation that you solved for in the bond economy in part (a) incentive compatible? If you say yes, prove it. In you say no, give a specific example that violates the incentive compatibility constraints.

(d) **Optimla Private information** Define an optimal allocation under private information as one that maximizes the ulitarian social welfare function in part (b) subject to the resource constraints and the incentive compatibility constraints. Explain why the equilibrium allocation fromt the bond economy is not an optimal allocation inder the private information.

3. Consider a simplified version of the two period hidden information problem studied in R. Townsend (1982). It generates the following programming problem (called PI.2') where the principal minimizes the cost of providing the agent with a "utility" allocation $\{u^\theta, \omega^\theta\}_{\theta \in \{H,L\}}$ that respects incentive feasibility:

$$min_{\{u^\theta, \omega^\theta\}_{\theta\in\{H,L\}}} \sum_{\theta \in \{H,L\}} \pi^\theta [C(u^\theta) + \beta v(\omega^\theta)] \tag{1}$$

$$s.t. \quad \sum_{\theta \in \{H,L\}} \pi^\theta [u^\theta + \beta \omega^\theta] = \omega \tag{2}$$

$$u^H + \beta\omega^H \geq u(C(u^L) + \Delta) + \beta\omega^L \tag{3}$$

$$u^L + \beta\omega^L \geq u(C(u^H) - \Delta) + \beta\omega^H \tag{4}$$

where $\Delta = y^H - y^L > 0$. Equation (2) is known as the promise keeping constraint and inequalities (3) and (4) are incentive compatibility constraints in the H and L states, respectively. You are to prove that while this problem does not yield the full risk sharing allocation, it improves upon the allocation that would occur in a repetition of the static problem (i.e. it improves upon autarky). In particular, show that $y^L < c^L < c^H < y^H$, $\omega^L < \omega^H$, (3) binds and (4) is slack. The following 4 parts will help you establish these results.

(a) Show (3) must bind. Use a proof by contradiction; that is, start by assuming (3) is slack. But this implies $\omega^L \geq \omega^H$ by convexity of V. Furthermore it implies that $u^L \geq u^H$. But these latter two results lead to a contradiction with (3) slack.

(b) Show $u^H > u^L$. In two parts. First, assume (3) and (4) bind. Construct a new equation by adding (3) and (4), call it (5). Defining $f(x) = u(c^L + x) + u(^cH - x)$,then (5) can be written $f(0) = f(\Delta)$ and we can use the properties of $u(\Delta)$ to show $u^H > u^L$. Second, assume that only (3) binds to show $u^H > u^L$. These results show $c^H > c^L$

(c) Show $\omega^H > \omega^L$.Solve problem PI.2. You will have 4 first order conditions corresponding to $\{u^H, u^L, \omega^H, \omega^L\}$ with multipliers λ on (2) and μ^H and μ^L on (3) and (4) respectively. Manipulate the first order conditions and use the properties of $u(\Delta)$ to show that $\mu^H > \mu^L$ which with convexity of V yields $\omega^H > \omega^L$.

(d) Show that (4) is slack. Use a proof by contradiction. As in part 2, construct a new equation by adding (3) and (4) and use the properties of $u(\Delta)$ to yield a mathematical inconsistency.