Homework 9: Stochastic Optimization

Mohammad Hossein Rahmati
Optional to hand in*

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1. Do Exercise 8.10 *Early resolution of uncertainty*, from Ljungquist, Sarget, 2nd ed.

2. Do Exercise 13.4 *The term structure and consumption*, from Ljungquist, Sarget, 2nd ed.

3. Consider a two-period exchange economy with two (types of) consumers labelled A and B. The two types of consumers have identical preferences given by $u(c_0) + \beta E[u(c_1)]$, where u is strictly increasing and strictly concave. Each consumer is endowed with one consumption good in period 0. In period 1, each type A consumer is endowed with θy consumption goods and each type B consumer is endowed with $(1 - \theta)y$ consumption goods. The random variable θ can be interpreted as the consumer's share of the aggregate endowment y. Let θ equal $\frac{1}{2} + z$ with probability p and equal $\frac{1}{2} - z$ with probability $1 - p$, where $0 < z < \frac{1}{2}$. The aggregate endowment y is also random: it equals $1 + x$ with probability one-half and equals $1 - x$ with probability one-half. The random variables y and θ are statistically independent.

   (a) Assume that markets are complete: in period 0, consumers can trade a full set of Arrow securities. Express the competitive equilibrium allocations and prices in terms of primitives as explicitly as you can. You might want to start with the case where $p = \frac{1}{2}$ (so that the consumers are identical in all respects) and then consider the more general case $p \neq \frac{1}{2}$.

   (b) Use the prices of the Arrow securities from part (a) to find the equilibrium period-0 price of a risk-free bond (i.e., an asset that pays one unit of the consumption in all states of the world in period 1). If you are unable to solve for the Arrow prices explicitly, then show how you would use these prices to compute the price of a risk-free bond.

   (c) Now suppose that markets are incomplete: in period 0, the only asset that consumers are allowed to trade is a risk-free bond. The net supply of bonds is zero (since the economy is closed). Find the competitive equilibrium allocations and the equilibrium price of the bond as explicitly as you can. Compare your answers to those in parts (a) and (b). Show that eliminating complete markets makes consumers worse off. Does eliminating complete markets increase or decrease the risk-free rate of return (i.e., the inverse of the bond price)? Why? (Again, you might want to start with the case $p = \frac{1}{2}$ before considering the case $p \neq \frac{1}{2}$.)

   (d) Introduce a second asset into the economy you studied in part (c). Specifically, in addition to the endowments described above (which can be viewed as labor income), let each consumer be endowed with one Lucas tree in period 0. Each tree yields a dividend of d consumption goods in period 1, where d equals d_H if y equals $1 + x$ and equals $d_L < d_H$

*Sharif University of Technology, rahmati@sharif.edu

if y equals 1?x. Trees, as well as risk-free bonds, can be bought and sold in competitive markets in period 0. Without doing any explicit calculations, show how you would go about solving for the equilibrium prices of the two assets in this economy.

(e) Determine (as completely as you can) the prices of a risk-free bond and of a Lucas tree under the assumption that consumers can trade a full set of Arrow securities in the economy that you studied in part (d). Compare (if possible) these prices to the corresponding prices in part (d).

4. **Aggregation** .Suppose that there are N consumption goods every period. Agent i's preferences are represented by expected, discounted time-separable utility for which the instantaneous utility function is

$$u(c_{i,1}, ..., c_{i,n}, ..., c_{i,N}) = \left(\sum_{n=1}^{N} \phi_n \frac{(c_{i,n} - \gamma_{i,n})^{1-\sigma}}{1-\sigma} \right)$$

where $\gamma_{i,n}$ is the minimum consumption required by agent i of commodity $n = 1, ..., N$.

Consider an economy where a full set of Arrow securities can be traded every period (very importantly, the corresponding competitive equilibrium allocation is a PO parametrized by some welfare weight α). Assume that discount factors are identical (i.e., $\beta_i = \beta$ for all $i = 1, ..., I$). Show that there exists a representative aggregate consumer with preferences represented by

$$\left\{ \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \; \pi(s^t) \; \sum_{n=1}^{N} \phi_n \frac{\left(C_n(s^t) - \gamma_n\right)^{1-\sigma}}{1-\sigma} \right\}$$

where $C_n = \sum_{i=1}^{I} c_{i,n}$ stands for aggregate consumption of the $n - th$ commodity and $\gamma = \sum_{i=1}^{I} \gamma_i$ denotes the aggregate minimum consumption requirement of the $n - th$ commodity. In particular, show that the intertemporal marginal rates of substitution of the aggregate representative agent coincides with the IMRS's of any agent $i = 1, ..., I$.

5. Consider a household that lives for a finite number of periods, T. Wealth, W_t, at the beginning of the period is allocated towards current consumption, c_t, purchases of a risky asset x_t, and a risk-free asset, y_t, according to

$$c_t + x_t + y_t \leq W_t.$$

Wealth evolves according to

$$W_{t+1} = R_{x_t} + R_f y_t$$

where R denotes the (gross) return to the risky asset and R_f denotes the gross return to the risk-free asset. Assume that R is i.i.d over time and is drawn from a distribution $F(R)$. The household's preferences are given by $\sum_{t=0}^{T} \beta^t u(c_t)$ where β is the discount function and $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$. The household maximizes expected utility.

(a) Set up the household's problem as a dynamic program.

(b) Financial planners recommend that households' portfolios should become less risky as they age. Evaluate this recommendation.
Hint: Guess at the functional form of the value function. Derive the households optimal asset allocation given this guess.