Homework 8: Stochastic Optimization

Mohammad Hossein Rahmati *

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1. **Transition Functions**: Let

$$\Pi = \begin{bmatrix} 1-\gamma & \alpha\beta & \beta\gamma \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

   be a transition matrix associated with a 3-state Markov chain. Further, let $\alpha, \beta, \gamma, \in (0,1)$ and $\alpha + \beta = 1$

   (a) What is the stationary probability distribution associated with the above transition matrix?

   (b) How many ergodic sets are there? What are they?

2. Consider a household which can be located on one of three islands in a given period. Label the islands $i = 0; 1; 2$. The household enjoys utility u^i if it is located on island i in a given period. Assume that $u^0 = 0$ and that $u^i > 0$ for $i = 1; 2$. The household lives forever and discounts at a rate of β with $0 < \beta < 1$.
   Households are restricted in terms of their movement between islands. If the household is located on island i in the current period, the probability of being on island j in the next period is given by π_{ij} . Assume that the household is unable to go directly from island $i = 1; 2$ to island $j = 1; 2$, for $i \neq j$ without going through island 0. Thus a household on island $i = 1; 2$ in period t remains on that island or switches to island 0 for period t + 1.

   (a) Write down a transition matrix describing the evolution of the household from one period to the next. What is the probability that a household on island 1 in the current period is on island 2 two periods from now?

   (b) Write down the system of equations (the Bellman equations) which describe the value of being on each of the three islands. To do so, let V^i be the value of being on island $i = 0;$ $1; 2$.

   (c) Explain in words why an increase in u^1 will increase V^0, V^1 and V^2.

   (d) Suppose that, in addition to the assumptions made above, $\pi_{10} = \pi_{20} = 0$. Further suppose that $u^2 > u^1$.Prove that $V^2 > V^1$. Provide conditions such that V^0 is greater than V^1. Explain these conditions.

3. Consider an exchange economy with two infinitely-lived consumers with identical preferences given by:

$$E\left(\sum_{t=0}^{\infty} \beta^t log(c_t)\right)$$

*Sharif University of Technology, rahmati@sharif.edu

Both of the consumers have random endowments that depend on an (exogenous) sequence of state variables $\{s_t\}_{t=0}^{\infty}$. The s_t's are statistically independent random variables with identical probability distributions. Specifically, for each t, $s_t = H$ with probability π and $s_t = L$ with probability $1 - \pi$, where π does not depend on time or on the previous realization of states. If $s_t = H$, then the first consumer's endowment is 2 and the second consumer's endowment is 1; if $s_t = L$, then the first consumer's endowment is 1 and the second consumer's endowment is 0. Markets are complete.

(a) Carefully define a competitive equilibrium with date-0 trading for this economy. (Assume that consumers make decisions before observing the realization of the state in period 0.)

(b) Determine the competitive equilibrium allocation in terms of primitives.

(c) Determine the prices of the Arrow securities in terms of primitives.

(d) Use your answer from part (c) to determine the average rate of return on a (oneperiod) riskless bond in this economy.

4. Consider the following investment problem faced by a firm. The firm's revenues are given by $R(K_t, Z_t)$ and there are costs of adjusting its capital holdings $C(K_{t+1}, K_t)$ where K_t is the capital stock at the beginning of period t, $K_{t+1} \geq 0$ is capital chosen at the beginning of period t to come on line in $t + 1$, Z_t is a technology innovation, and $K_0 > 0$ is given. Gross investment is given by $I_t = K_{t+1} - (1 - \delta)K_t$ and the relative price of investment goods in terms of consumption goods is p_t. Firms discount the future at rate $\beta \in (0, 1)$. The $firm's$ problem is:

$$\max_{\{K_{t+1}\}_{t=0}^{\infty}} E\left[\sum_{t=0}^{\infty} \beta^t \{R(K_t, Z_t) - C(K_{t+1}, K_t) - p_t[K_{t+1} - (1 - \delta)K_t]\}\right].$$

We will assume the following:

$$R(K_t, Z_t) = Z_t K_t, \quad C(K_{t+1}, K_t) = \frac{\gamma}{2}\left(\frac{I_t}{K_t}\right)^2 K_t, \quad and p_t = p, \forall t.$$

There are 4 parts to this question.

(a) What are the first order conditions? Interpret them in terms of the marginal cost of adding capital versus the marginal benefit

(b) Assume that $Z_t = 1$ for all t. Solve for the decision rule $K_{t+1} = G(K_t)$. Hint: Conjecture $G(K_t)$ is a linear function and verify it satisfies the f.o.c.

(c) Assume that $Z_t \in \{Z_1, Z_2\}$ follows a two-state Markov process and that $\delta = 1$.

a) Is the decision rule still a linear function of K_t (i.e. is $K_{t+1} = H(Z_t)K_t$ consistent with firm optimization)? State the system of equations that characterize the decision rules as a time invariant function of each shock.

b) In addition, denote the value of the firm (i.e. the value function) by

$$V(K_t, Z_t) = E_t\left[\sum_{j=0}^{\infty} \beta^j \{R(K_{t+j}, Z_{t+j}) - C(K_{t+j+1}, K_{t+j}) - pK_{t+j+1}\}\right].$$

If $K_{t+1} = H(Z_t)K_t$, is the value function linear in K_t? In that case, is the marginal value of capital (sometimes called "marginal q" and given by $\frac{\partial V(K_t,Z_t)}{\partial K_t}$ equal to the average value of capital (sometimes called "average Q" and given by $\frac{V(K_t,Z_t)}{K_t}$?

(d) Assume that Z_t follows $Z_t = (1-\rho) + \rho Z_{t?1} + \epsilon_t$ where ϵ_t is an i.i.d. shock drawn from a continuous distribution with mean zero and constant variance. Again, let $\delta = 1$. Find the linearized decision rule $\hat{K_{t+1}} = \gamma_{kk}\hat{K}_t + \gamma_{kz}\hat{Z}_t$ where $\hat{X}_t$ denotes the deviation of X_t from a steady state.