Homework 5: Dynamic Programming & Speed of Convergence

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1. **(Adjustment cost model)** Consider the following discrete-time dynamic programming problem in sequence formulation;

$$\max_{\{x_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t [h(x_t) - a(x_{t+1} - x_t)]$$

where x_0 given. Assume that $h^{''}(x) \leq 0$ all $x \in R$, that there is a unique x^* such that $h^{'}(x^*) = 0$, and that $a^{''}(z) \geq 0$ for all $z \in R$ and that $a^{'}(z) = a(z) = 0$ only for $z = 0$.

   (a) Write down the Bellman equation for this problem. Write down explicitly the period return function in terms of h and a. Use x for the current state and y for next period state. Use v for the value function.

   (b) Let $F(x, y)$ the period return function used in (a). Answer true or false, and give a short proof or counter-example.

      i. Is $F(x, y)$ increasing in x for all the values of y?

      ii. Is $F(x, y)$ concave in (x, y)?

   (c) Let v the value function. Just answer true or false, and give a short proof or counterexample.

      i. Is $v(x)$ concave?

      ii. Is $v(x)$ increasing?

   (d) Assuming differentiability of v, write down the first order conditions for the problem. Use $y = g(x)$ for the optimal decision rule.

   (e) Use the envelope to write down an expression for the derivative of the value function v. Use $y = g(x)$ for the optimal decision rule.

   (f) Let $\bar{x} = g(\bar{x})$ denote a steady state. Use your answer to (e) and (f) to show that $\bar{x} = x^*$ is the unique steady state (2 lines maximum).

   (g) First order conditions and shape of the optimal decision rule g.

      i. Let $F(x, y)$ be the period return function for this problem. Plot the function $?F_y(x, y)$ (in terms of derivatives of h and a) for a fixed value of x with $x < x^*$, and $\beta v^*(y)$ with y in the horizontal axis. Indicate in the horizontal axis the value of x and the value of y that corresponds to $g(x)$.

      ii. In the same plot used for (i) draw the function $?F_y(x^{'}, y)$ for a higher value of x, i.e. for $x < x^{'} < x^*$. Make sure to identify the new value of $g(x^{'})$ in your plot.

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iii. Given your answer to (i) and (ii) is $g(x)$ increasing or decreasing in x? [one word].

(h) Roots of Euler Equation
Let $F_y(x, g(x)) + \beta F_x[g(x), g(g(x))] = 0$ be the Euler equation for an arbitrary dynamic programing problem with period return $F(x, y)$ concave and differentiable and optimal policy $y = g(x)$. Assume that the state x is one dimensional and that g is differentiable.

i. Differentiate the expression for the Euler equation w.r.t. x. Make sure that each of the derivatives is evaluated at the correct arguments [one line].

ii. Evaluate the expression in (i) at the steady state values $\bar{x} = g(\bar{x})$ [one line].

iii. Assuming that $F_{yx}(\bar{x}, \bar{x}) \neq 0$, divide the resulting expression in (ii) by this quantity and write down a quadratic function on the variable λ whose zeros are solved by $g^{'}(\bar{x})$. Denote this quadratic expression by $Q(\lambda)$ where

$$Q(\lambda) = 1 + b\lambda + \beta\lambda^2$$

and

$$b \equiv \frac{F_{yy}(\bar{x}, \bar{x}) + \beta F_{xx}(\bar{x}, \bar{x})}{F_{yx}(\bar{x}, \bar{x})}$$

(i) Speed of Convergence:

i. Write down an expression for each of the following derivatives: $F_{xx}(\bar{x}, \bar{x})$, $F_{xy}(\bar{x}, \bar{x})$,and $F_{yy}(\bar{x}, \bar{x})$ for the problem with adjustment cost using the functions h and a as well as $x = x^*$. Write down an expression for b in terms of the derivatives of h and a evaluated at the steady state values.

ii. Write down the quadratic equation $Q(\lambda)$ derived in (h-iii) for the problem of adjustment cost [this should be a function of λ and parameters $a^{'}(0)$, β and $h^{''}(x^*)$ only]

iii. Compute the values of $Q(\lambda)$ for $\lambda = 0$, $\lambda = 1$, $\lambda = \frac{1}{\beta}$ and λ^*, where λ^* is such that $Q^{'}(\lambda^*) = 0$. Is $\lambda^* > 1$? What is the sign of $Q^{'}(\lambda)$ in $\lambda \in (0, 1)$? What is the limit $\lim_{\lambda\to\infty} Q(\lambda)$?

iv. Plot $Q(\lambda)$ with λ in the horizontal axis. Make sure to identify $\lambda = 0$, $\lambda = 1$, $\lambda = \lambda^*$ and $\lambda = \frac{1}{\beta}$ as well as the corresponding values of $Q(\lambda)$. Make sure you label the smallest root of Q, and denote it by λ_1. How does λ_1 depend on b?

v. Draw a second quadratic function, denoted as $\hat{Q}(\lambda)$, that corresponds to a problem with a larger value of $-\frac{h^{''}(x^*)}{a^{''}(0)} = |\frac{h^{''}(x^*)}{a^{''}(0)}|$. How does $|\frac{h^{''}(x^*)}{a^{''}(0)}|$ relate to b? Denote the smallest root of this equation by $\hat{\lambda_1}$. How is $\hat{\lambda_1}$ compared with λ_1?

vi. Recall that if $|\lambda_1| < 1$, then $g^{'}(\bar{x}) = \lambda_1$. How does the speed of convergence of $\{x_t\}$ depend on $-\frac{h^{''}(x^*)}{a^{''}(0)} = |\frac{h^{''}(x^*)}{a^{''}(0)}|$? What is the economic intuition for this dependence? (Explain the intuition for each of the parameters: $|h^{''}(x^*)|$ and $|a^{''}(0)|$).

2. Here is a problem facing a monopolist. He faces a demand curve each period given by $q = (1 - p)$, That is, if the price is p he can sell the quantity q. Production is costless but at each period in time the monopolist faces a capacity constraint, $q = c$, where c is the upper bound on his production. Capacity can be increased with a one period time delay according to the cost function $(c' - c)^2$, where $c' \geq c$ is the level of capacity that the monopolist chooses for next period. The monopolist faces the time-invariant gross interest rate r.

(a) Formulate the monopolist dynamic programming problem.

(b) Is the value function strictly increasing? If so, outline an argument.

(c) Is the value function strictly concave? If so, outline an argument.

(d) Is the value function differentiable? If so, outline an argument.

(e) Using the guess and verify technique, provide a closed-form solution for the Value and Policy functions.

3. Consider a representative household in an infinite horizon endowment economy. Time is discrete; $y_t = y$ is the endowment at date t, which can be divided into consumption of a perishable good, c_t, and investment in a durable good, d_t^x. The durable depreciates at the rate $\delta \in (0, 1)$, but it is not directly productive. The stock of durables at any date, d_t, produces a flow of services that enters the utility function. Thus, the problem faced by the representative household with initial stock d_0 is:

$$\max_{\{c_t, d_t, d_t^x\}} \sum_{t=0}^{\infty} \beta^t \{u(c_t) + v(d_t)\}$$

subject to

$$\begin{aligned} c_t + d_t^x &\leq y \\ d_{t+1} &\leq d_t^x + (1-\delta)d_t \\ c_t, d_t &\geq 0, \quad d_0 \quad given \end{aligned}$$

where both u and v are strictly increasing and continuous.

(a) Write an equivalent problem in the following form:

$$\max_{\{d_{t+1}\}} \sum_{t=0}^{\infty} \beta^t \{F(d_t, d_{t+1})\}$$

subject to

$$\begin{aligned} d_{t+1} &\in \Gamma(d_t) \\ & d_0 \quad given \end{aligned}$$

where $\Gamma(d_t) \subset \mathbb{R}^+$.

What is F? What is the correspondence Γ? Justify your answer.

(b) Write the Bellman equation for this problem and argue that there exists a unique continuous value function, $h(d)$.

(c) State additional conditions on u and v such that the value function is both strictly increasing and strictly concave. Prove these two properties.

For the remaining questions, assume that both u and v satisfy the Inada conditions and are continuously differentiable.

(d) State the Benveniste-Scheinkman condition and the FOC for the functional equation problem in (b).

(e) Let $c(d)$ and $d' = g(d)$ be the corresponding policy functions for consumption and durables, respectively. Show that there is a unique steady state value of the stock, d^*, such that $d^* = g(d^*) > 0$.

(f) Show that the policy functions are increasing in d.