Homework 3: Application of Dynamic Programming

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October 17, 2016

1. Consider the following version of the neoclassical growth model with leisure in the utility. More, specifically, Robinson Crusoes preferences are

$$\sum_{t=0}^{T} \beta^t [\mu ln(c_t) + (1-\mu) ln(1-h_t)]$$

where h_t denotes hours worked. Hence, $1-h_t$ is Robinsons leisure. The resource constraint is

$$c_t + k_{t+1} \leq A k_t^{\alpha} h_t^{1-\alpha}$$

Capital, thus, depreciates at 100 percent. Robinson starts out with some capital k_0

   (a) Write down Bellmans Functional Equation

   (b) Guess and verify that the solution takes the form $\gamma_0 + \gamma_1 ln(k)$ (Hint: Guess and verify that $h(k)$ is independent of k.)

2. A consumer seeks to maximize her lifetime utility of consumption and leisure, which is given by: $\sum_{t=0}^{\infty} \beta^t u(c_t, l_t)$, where c_t is consumption in period t and l_t is leisure in period t. The felicity function u is strictly increasing and strictly concave in both of its arguments.
The consumer is endowed with one unit of time in each period, which she allocates between leisure and work. The consumer's labor income in period t equals $wh_t n_t$, where w is the (time-invariant) wage per unit of human capital, h_t is the the consumer's level of human capital in period t, and n_t is the amount of time that the consumer spends working in period t. Human capital accumulates over time according to:

$$h_{t+1} = (1-\delta)h_t + f(h_t, n_t),$$

where f is strictly increasing in both arguments, strictly concave in its first argument, and satisfies: $f(0,n) = f(h,0) = f(0,0) = 0$. In other words, human capital depreciates at rateδ, but the consumer can invest in human capital by working. The consumer has human capital equal to $h_0 > 0$ in period 0. To keep things simple, suppose that the consumer does not participate in asset markets and instead simply consumes her entire labor income in every period.

   (a) Formulate the consumer's dynamic optimization problem as a dynamic programming problem. That is, display the consumer's Bellman equation and identify clearly the control (or choice) variable(s) and the state variable(s).

   (b) Find an equation that determines (implicitly) the steady-state level of human capital.

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(c) Suppose instead that f is linear: $f(h_t, n_t) = g(n_t)h_t$, where g is strictly increasing and satisfies $g(0) = 0$. Do you think that the consumer's optimal path for human capital converges to a steady state in this case? Explain.

3. Consider the following recursive problem:

$$V(a, k) = max_{c,a'}[u(c) + \beta V(a^{'}, k^{'})]$$

$$s.t : a^{'} + c = R(k)a \qquad k^{'} = G(k)$$

Assume that $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$. In the backround of this problem there is a firm (or firms) with a trivial, static maximization problem.
Fully characterize the Recursive Competitive Equilibrium, i.e, observe closed form solutions for the functions $V(a, k), g(a, k), G(k)$. (Hint: Guess that $V(a, k) = B(k)a^{1-\sigma}$, where $B(k)$ is a function of k to be determined)
NOTE: This is a nice example where you can see how a RCE really looks like. However, the algebra is complicated and my advice is not to spend too much time on it if it's not working.