Homework 16: Recursive Commitment

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1. Consider a two period economy in which agents face uncertainty regarding their preferences over the timing of their consumption. Number the periods $t = 0, 1$. Each egent has preferences over consumption given by $\theta log(c_0) + \beta log(c_1)$. Where θ is the random taste shoch realized at the beginig the of period $t = 0$. The shock θ is drawn from the set $\Theta = \{\theta^1, \cdots, \theta^N\}$ with probability $\pi(\theta^n)$. We assume that $\pi(\theta^n)$ also represents the fraction of agents who recieve shock θ^n. Let $\{c_0(\theta^n), c_1(\theta^n)\}_{n=1}^N$ denote the consumption allocation in this economy. Each period each agent is endowed with y unit of consumption good. This good can not be stored. The resource constraint for this economy are $\sum_{n=1}^N c_t(\theta^n)\pi(\theta^n) = y$ for $t = 0, 1$.

   (a) **Bond Economy** Assume that at date 0 agents only trade an uncontingent bond that pays off 1 unit of consumption for sure at date 1. Let q denote the price of this bond and $b(\theta^n)$ the quantity purchased by an agent with shock θ^n. Given this notation, agents have budget constraint:

   $$c_0(\theta^n) + qb(\theta^n) = y \quad and \quad c_1(\theta^n) = y + b(\theta^n)$$

   Solve for the equilibrium bond price q and the consition allocation $\{c_0(\theta^n), c_1(\theta^n)\}_{n=1}^N$.

   (b) **Full Information Social Optimum** Now solve for the allocation that maximizes the utilitarian social welfare function

   $$\sum_{n=1}^{N} [\theta log(c_0(\theta^n)) + \beta log(c_1(\theta^n))]\pi(\theta^n)$$

   subject to the resource constraint. Define agents's marginal rate of substitution q^n by:

   $$q^n = \frac{\frac{\beta}{c_1(\theta^n)}}{\frac{\theta^n}{c_0(\theta^n)}}$$

   Is q^n equated across agents in the full information social optimum allocation? Is do, call this q^*. Does q^* equal the q that you found in the bond economy in bart (a)? Does the social optimum allocation satisfy the budget constraints from the bond economy in part (a)?

   (c) **Private Information** Now consider the problem of finding an optimal allocation $\{c_0(\theta^n), c_1(\theta^n)\}_{n=1}^N$ that is also incentive compatible in an economy in which agents' taste shocks θ^n are private information. Specially, we say that an allocation is incentive compatobe if :

   $$\theta^n log(c_0(\theta^n)) + \beta log(c_1(\theta^n)) \geq \theta^n log(c_0(\theta^i)) + \beta log(c_1(\theta^i))$$

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for all $i = 1, \cdots N$. Is the allocation that you solved for in the bond economy in part (a) incentive compatible? If you say yes, prove it. In you say no, give a specific example that violates the incentive compatibility constraints.

(d) **Optimla Private information** Define an optimal allocation under private information as one that maximizes the ulitarian social welfare function in part (b) subject to the resource constraints and the incentive compatibility constraints. Explain why the equilibrium allocation fromt the bond economy is not an optimal allocation inder the private information.

2. Consider a simplified version of the two period hidden information problem studied in R. Townsend (1982). It generates the following programming problem (called PI.2) where the principal minimizes the cost of providing the agent with a "utility" allocation $\{u^\theta, \omega^\theta\}_{\theta\in\{H,L\}}$ that respects incentive feasibility:

$$min_{\{u^\theta,\omega^\theta\}_{\theta\in\{H,L\}}} \sum_{\theta\in\{H,L\}} \pi^\theta[C(u^\theta) + \beta v(\omega^\theta)] \tag{1}$$

$$s.t. \quad \sum_{\theta\in\{H,L\}} \pi^\theta[u^\theta + \beta\omega^\theta] = \omega \tag{2}$$

$$u^H + \beta\omega^H \geq u(C(u^L) + \Delta) + \beta\omega^L \tag{3}$$

$$u^L + \beta\omega^L \geq u(C(u^H) - \Delta) + \beta\omega^H \tag{4}$$

where $\Delta = y^H - y^L > 0$. Equation (2) is known as the promise keeping constraint and inequalities (3) and (4) are incentive compatibility constraints in the H and L states, respectively. You are to prove that while this problem does not yield the full risk sharing allocation, it improves upon the allocation that would occur in a repetition of the static problem (i.e. it improves upon autarky). In particular, show that $y^L < c^L < c^H < y^H$, $\omega^L < \omega^H$, (3) binds and (4) is slack. The following 4 parts will help you establish these results.

(a) Show (3) must bind. Use a proof by contradiction; that is, start by assuming (3) is slack. But this implies $\omega^L \geq \omega^H$ by convexity of V. Furthermore it implies that $u^L \geq u^H$. But these latter two results lead to a contradiction with (3) slack.

(b) Show $u^H > u^L$. In two parts. First, assume (3) and (4) bind. Construct a new equation by adding (3) and (4), call it (5). Defining $f(x) = u(c^L + x) + u(^cH - x)$,then (5) can be written $f(0) = f(\Delta)$ and we can use the properties of $u()$ to show $u^H > u^L$. Second, assume that only (3) binds to show $u^H > u^L$. These results show $c^H > c^L$

(c) Show $\omega^H > \omega^L$.Solve problem PI.2. You will have 4 first order conditions corresponding to $\{u^H, u^L, \omega^H, \omega^L\}$ with multipliers λ on (2) and μ^H and μ^L on (3) and (4) respectively. Manipulate the first order conditions and use the properties of $u()$ to show that $\mu^H > \mu^L$ which with convexity of V yields $\omega^H > \omega^L$.

(d) Show that (4) is slack. Use a proof by contradiction. As in part 2, construct a new equation by adding (3) and (4) and use the properties of $u()$ to yield a mathematical inconsistency.