Homework 13: Asset Pricing

Mohammad Hossein Rahmati *

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1. Consider an economy with a single representative consumer who maximize

$$E\sum_{t=}^{\infty}\beta^t u(c_t) \qquad 0<\beta<1, \qquad u(c_t)=ln(c_t+\alpha)$$

The sole source of single good is an everlasting tree that produce d_t units of the consumption good in period t. At the beginning of time 0, each consumer owns one such tree. The dividend process d_t is Markov, with $Prob\{d_{t+1}\leq d^{'}|d_t=d\}=F(d^{'},d)$.
Assume that the conditional density $f(d^{'},d)$ of F exists. There are competitive markets in titles to trees and in state contingent claims. Let p_t be the price at t of a title to all future dividends from the tree.

   (a) Prove that equilibrium price p_t satisfy:

$$p_t=(d_t+\alpha)\sum_{j=1}^{\infty}\beta^j E_t\left(\frac{d_{t+j}}{d_{t+j}+\alpha}\right)$$

   (b) Find aformula for the risk-free one period interest rate R_{1t}. Prove that, in the special case in which $\{d_t\}$ is independently and identically distributed, R_{1t} is given by $R_{1t}^{-1}=\beta k(d_t+\alpha)$, where k is a constant. Given a formula for k.

   (c) Find a formula for the risk free two peiod interest rate R_{2t}. Prove that, in the special case in which d_t is independently and identically distributed $R_{t2}$is given by $R_{2t}=\beta^2 k(d_t+\alpha)$ where k is the same constant that you found in part (b)

2. Consider a version of a Lucas tree economy in which there are two types of trees. Both types of trees are perfectly durable; a type-i tree ($i=1,2$) yields a random amount of dividends equal to dit in period t. Assume that $\{d_{1t}\}_{t=0}^{\infty}$ and $\{d_{2t}\}_{t=0}^{\infty}$ are i.i.d. sequences of random variables and that d_{1t} and d_{2s} are statistically independent for all t and s. In addition, for $i=1,2$, assume that d_{it} equals d_L with probability π_i and equals $d_H>d_L$ with probability $1-\pi_i$.
The economy is populated by a continuum (of measure one) of identical consumers with preferences over consumption streams given by:

$$E_0\sum_{t=0}^{\infty}\beta^t log(c_t)$$

where c_t is consumption in period t. In period 0, each consumer owns one tree of each type. Dividends are non-storable and are the only source of consumption goods. There are competitive markets in which consumers can buy and sell both types of trees.

*Sharif University of Technology, rahmati@sharif.edu

(a) Define a sequential competitive equilibrium in which the only assets that consumers trade are the two (types of) trees.

(b) Find an algebraic expression for the equilibrium price of a type-1 tree (measured in terms of todays consumption goods), assuming that the dividends of both types of trees are equal to dL today. Your expression should depend only on primitives (i.e., on the parameters describing preferences and technology).

(c) How many Arrow securities are there in this economy? Express the prices of these securities in terms of primitives.

(d) Use your answer from part (c) to find the price (expressed in terms of todays consumption goods) of an asset that pays one unit of the consumption good in the next period if the dividends of the two trees (in the next period) are not equal to each other and pays zero otherwise.

3. Consider a version of the Lucas tree model in which trees not only yield fruit (or dividends) but also enter the utility function directly. In particular, the representative consumer's preferences are given by:

$$E_0 \sum_{t=0}^{\infty} \beta^t [log(c_t) + Alog(s_t)]$$

where c_t is period t consumption, s_t is the number of trees held in period t, and A is positive. Thus owning more trees leads to higher utility: trees are considered beautiful and consumers value beauty. The tree yields a stochastic dividend stream $\{d_t\}_t = 0^\infty$. Assume that dt is independent and identically distributed (so that its realization today is statistically independent of its past realizations) and assume that $E(d_t^i) = m_i$ for all nonzero integers i. Dividends are the only source of consumption goods in this economy and they are not storable. Each consumer is endowed initially with one tree. Consumers can buy and sell trees in a competitive market.

(a) Derive the Euler equation of a typical consumer.

(b) Use your answer from part (a) to find the equilibrium price of a tree as a function of the current dividend. (Hint: Guess that the price is equal to a constant B times the current dividend, and then solve for B in terms of parameters.)

4. Consider a two-period exchange economy with identical consumers. Each consumer maximizes $u(c_0) + \beta E[u(c_1)]$, where c_t is consumption in period t. Each consumer is endowed in period 0 with one tree. Each tree yields one unit of the (nonstorable) consumption good in period 0 and a random amount of the consumption good in period 1. In particular, with probability π, each tree yields dH units of the consumption good in period 1; with probability $1-\pi$, each tree yields dL units of the consumption good in period 1, where $d_H > d_L$. (The trees are identical, so they all yield the same number of units of the consumption good-either d_H or d_L- in period 1.) In period 0, consumers trade two assets in competitive markets: trees and riskfree bonds (i.e., sure claims to one unit of the consumption good in period 1).

(a) Are markets complete in this economy? Explain why or why not.

(b) Carefully define a competitive equilibrium for this economy. Find the equilibrium consumption allocation and the equilibrium prices of a tree and of the riskfree bond.

(c) Now suppose that the market for trees is shut down so that consumers can trade only the riskfree bond. (Are markets complete in this case?) Carefully define a competitive equilibrium for this economy and show that the equilibrium consumption allocation and the equilibrium price of a riskfree bond are the same as in part (b). Explain intuitively why this result holds.