Homework 10: Overlapping Generation Model

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1. Consider an infinite horizon overlapping generations model in which:
   - all agents live two periods
   - t=1,2,....
   - there is no population growth
   - all agents are identical within a generation
   - agents have an endowment of $e > 0$ in youth and 0 in old age
   - agents in generation t have preferences given by $u(c_t^y) + v(c_t^o)$ where c_y^t is consumption in youth and c_t^o is consumption in old age
   - both $u(.)$ and $v(.)$ are strictly increasing and strictly concave
   - agents have access to a private storage the storage technology given by $f(s)$ where $f(.)$ is strictly increasing and strictly concave
   - storage must be non-negative Carefully note the various market structures in the parts which follow.

   (a) **Market Structure: No fiat money, no claims markets.**
   Consider the optimization problem of a representative generation t agent. Write down a necessary condition characterizing the choice of storage. Under what condition is storage strictly positive?

   (b) **Market Structure: No fiat money, claims market in period 0**
   Show that the allocation from part (a) is a competitive equilibrium. Characterize the prices supporting this equilibrium. Is this allocation Pareto optimal? Under what condition are we in the Samuelson case?

   (c) **Market Structure: Fiat money, No claims market**
   Now agents can hold non-negative amounts of fiat money as well as store goods. Consider the optimization problem of a representative generation t agent. Write down the necessary conditions for the optimal decisions on money demand and storage. Under what conditions is there an interior solution in which money is held and storage is strictly positive?

2. Consider an overlapping generations model with a constrant population of two-period lived people. Each has the utility function $u(c_1) + v(c_2)$, where c_i represents consumption in the ith period of life. Assume $u^{'}, v^{'} > 0$ and $u^{''}, v^{''} < 0$. Each is endowed with y goods when young

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and nothing when old. Goods can be stored with a linear technology that delivers x goods in period $t+1$ for each good stored in period t, with $x > 1$.
there is a fixed stock of M units of fiat money at the end of each period t. Each youn person is required to hold real money balanced worth at least γ goods for each good stored, a "reserve requilrment"

(a) Find the conditions defining a monetary equilibrium. Include the Kuhn-Tucker conditions for an equlibrium that is not in interior.

(b) Assume that the reserve requirment binds. Find and graph the equilibrium law of motion for real money balances $q_{t+1} = h(q_t)$. Can there be equilibrium paths with oscillating stock of storage?

(c) Assume a stationary interior solution. Combine the equilibrium conditions into a single equation implicitly defining personal real balances of fiat money, q, as a function of γ. Find an expression defining $q^{'}(\gamma)$.

(d) Now use $q(\gamma)$ and the equilibrium conditions to express stady-state utility as a function $W(\gamma)$. Find the γ that maximizes staedy state utility. The first order condition will suffice. HINT: At some point you will be able to use the agents' first order condition to simplify yout expression for $W^{'}(\gamma)$.

3. Consider the following OG model of production. Each period a new generation of individuals are born and live for 2 periods. There is population growth of size $n \geq 0$ so that the measure of young alive at time $N_t = (1+n)^t N_0$ with $N_0 = 1$(i.e. the measure of agents of born in period $t = 0$ is normalized to 1) and the initial old are taken to be of measure $N_{-1} = \frac{1}{1+n}$. Each agent from a generation born at time t has preferences given by

$$U^t = ln(c_t^t) + \beta ln(c_{t+1}^t)$$

where c_t^g denotes individual consumption (i.e. per capita) of generation g at time t (so that c_t^t represents consumption in youth and c_{t+1}^t represents consumption in old age of a generation t individual) and $\beta \in (0,1)$ such that $\beta(1+n) < 1$. Each individual can only work at most a unit amount of time in youth. There is a constant returns to scale production function where aggregate output Y_t is given by

$$Y_t = K_t^{\alpha} L_t^{1-\alpha}$$

where K_t is aggregate capital, L_t is aggregate labor input, and $\alpha \in (0,1)$. There is complete depreciation of capital (i.e. $\delta = 1$) so that the law of motion for aggregate capital is given by $K_{t+1} = I_t$ with K_0 given. In what follows let per capita and aggregate variables be related as $x_t = \frac{X_t}{N_t}$ so that $k_t = \frac{K_t}{N_t}$.
3.1. Suppose that a social planner chooses $\{c_t^t, c_t^{t-1}, k_{t+1}\}_{t=0}^{\infty}$ to maximize the following welfare function:

$$\frac{1}{1+n} ln(c_0^{-1}) + \sum_{t=0}^{\infty} \beta^t N_t U^t$$

(a) What is the per capita feasibility constraint facing the planner in any given period t?

(b) What are the first order conditions for this problem?

(c) Manipulate the first order conditions to arrive at a condition like an Euler equation.

(d) What do the first order conditions imply about consumption of the young and the old in any given period (i.e. c_t^t and c_t^{t-1})?

(e) Assume that the sequence $\{k_{t+1}\}_{t=0}^{\infty}$ implied by the first order conditions converges. What is the steady state capital stock in the planner's solution? What is the implied steady state interest rate?

(f) Linearize the necessary conditions around the steady state. What conditions need to be satisfied for the system to be locally stable?

(g) Is there balanced growth of aggregate output? At what rate?

(h) How do these results relate to what you would find in a steady state of the RBC model?

3.2. Suppose there are competitive labor, capital and goods markets in the above environment. Specifically, suppose generation t agents can save their labor income in youth for retirement in old age (denoted s_{t+1}^t) and receive interest on those savings R_{t+1}.

(a) Write down the household's optimization problem.

(b) What is an agent's saving function s_{t+1}^t?

(c) What is the capital market clearing condition (i.e. where aggregate investment equals aggregate savings)? What does firm optimization and labor market clearing imply about the wage rate in terms of captial? Solve for the equilibrium difference equation for per capita capital.

(d) Does the difference equation admit a steady state solution? If so, what is the steady state capital stock $k^\star$ in a competitive equilibrium? Solve for the competitiveinterest rate.

3.3. Compare the planner's solution with the competitive solution. If $\alpha = \frac{1}{3}$, the annual population growth is 1% and the annual discount factor is calibrated to be 0.96, then is it possible for the steady state capital stock in a competitive equilibrium to be higher than what the planner using the above weights would choose? Note since a period in this overlapping generations model can be thought of as 30 years, this implies $\beta = (0.96)^{30} = 0.3$, and $1 + n = (1.01)^{30} = 1.35$.

3.4. Suppose there is a government in the competitive equilibrium which runs the following "pay-as-you-go" system. In particular, each agent of generation t must pay a lump sum social security tax τ_t in youth and receive lump sum social security benefit b_{t+1} in old age. The government budget constraint is given by

$$N_{t-1}b_t = N_t\tau_t \Longleftrightarrow b_t = (1+n)\tau_t$$

These are simply transfers from the current young to the current old.

(a) Write down the agent's problem in this environment.

(b) Find a steady state equilibrium.

(c) Is it possible to implement the same allocation in the competitive equilibrium with social security as the planner's solution? Discuss.

4. Consider an overlapping generations economy (call it economy I) where population grows at rate n: The representative consumer in each generations has preferences represented by

$$u(c_t^t; c_t^t + 1) = ln(c_t^t) + ln(c_{t+1}^t)$$

The consumer has endowment $e_t^t = w_1 > 0$ when young and no endowment when old. There is an initial generation of size normalized to 1 that is endowed with $m > 0$ units of Fiat money. Let p_t denote the nominal price level at period t (i.e. fiat money is the numeraire in this economy).

(a) Compute an (Arrow Debreu or Sequential Markets) equilibrium in which fiat money has positive value. Argue that it is unique.

(b) Now consider economy II. It is identical to economy I, but it has a payas-you-go social security system of size $\tau > 0$; where τ is the payroll tax paid by the young generation and $b = (1+n)\tau$ are the social security benefits when old. Note that economy II still has the initial old generation endowed with fiat currency $m > 0$: Does economy II have an (AD or SM) equilibrium in which money has positive value? Justify your answer. Describe the restrictions on the parameters $(w_1; n; \tau)$; if any, that are needed to assure the existence of such an equilibrium.

(c) If an equilibrium with valued fiat money exists, is it unique? Justify your answer.

(d) Consider a stationary equilibrium with valued fiat currency. Does it exist? Is it unique? Justify your answers. Describe how the value of money over time, as measured by the sequences $\{\frac{1}{p_t}\}_{t=1}^{\infty}$ varies across economies with different sizes of the social security system, as measured by τ.