Numerical simulations of localization of electromagnetic waves in two- and three-dimensional disordered media

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Localization of electromagnetic waves in two-dimensional (2D) and three-dimensional (3D) media with random permeittivities is studied by numerical simulations of the Maxwell’s equations. Using the transfer-matrix method, the minimum positive Lyapunov exponent $\gamma_m$ of the model is computed, the inverse of which is the localization length. Finite-size scaling analysis of $\gamma_m$ is carried out in order to check the localization-delocalization transition in 2D and 3D. We show that in 3D disordered media $\gamma_m$ exhibits two distinct types of frequency dependence over two frequency ranges, hence indicating the existence of a localization-delocalization transition at a critical frequency $\omega_c$. The critical exponent $\nu$ of the localization length in 3D is estimated to be, $\nu = 1.57 \pm 0.07$. At the transition point in the 3D media, the distribution function of the level spacings is independent of the system size, and is represented well by the semi-Poisson distribution. The 2D model can be mapped onto the 2D Anderson model and, hence, there is no localization-delocalization transition.

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I. INTRODUCTION

Wave propagation in heterogeneous media is a fundamental phenomenon of great scientific and practical interest, and has been studied for a long time.1 Many years of research have established rigorously that, in one-dimensional (1D) media with diagonal disorder and short-range correlations, even infinitesimally small disorder suffices for localizing the wave function, irrespective of the energy.2,3 Thus, envelope of the wave function $\psi(r)$ for the localized states decays exponentially at large distances $r$ from the domain’s center, $\psi(r) \sim \exp[-r/\xi(E)]$, with $\xi(E)$ being the localization length at energy level $E$. Some rigorous results have also been derived for higher-dimensional disordered media.1

In this paper we study propagation and localization of electromagnetic waves in disordered media. What we study is directly relevant to, among other phenomena, propagation of light through such disordered media as fog and clouds, as well as propagation of electromagnetic waves through stellar atmospheres and interstellar clouds, all of which are subjects of current interest. A study of diffusive wave propagation was first undertaken by astrophysicists,4,5 with the goal of understanding how radiation generated at the center of stars is affected as it traverses through the interstellar clouds. At the same time, the question of whether light can be localized in disordered media,6 in addition to be of much current interest, has still not been addressed completely. Unlike the problem of electron localization in disordered materials, which is of interacting quantum type, electromagnetic waves are noninteracting, even at the quantum level and, therefore, their propagation provides a suitable framework for realizing the properties of the Anderson model of localization.7 In fact, the first observations of strong8–11 and weak localization12–15 motivated much research on the question of light localization,16 as a way of determining the mobility edge.17–19 The standard method of numerical analysis of wave propagation in disordered media and estimating the localization length, in spatial dimensions two and three (where there are not many known exact results), is the transfer-matrix (TM) method.20 In the TM method one fixes the frequency of the wave or the energy level of the incident particles and computes the Lyapunov exponent or its inverse, the localization length. In the limit of weak disorder, however, obtaining numerically convergent and accurate results by the TM method requires intensive computations.

Using the TM method, we study in this paper the localization properties of electromagnetic waves in disordered media over a range of the frequencies. Then, utilizing the finite-size scaling method we study whether there is a localization-delocalization transition in such disordered media. We also study the statistics of the energy levels.21,22 In addition, we compute the number variance, a long-range correlator that represents the variance of the number of the eigenfrequencies in a given interval. The number variance helps us identify which frequencies are localized or delocalized.

Depending on the symmetry of the Hamiltonian that one studies, one may have distinct symmetry classes. For a system of noninteracting electrons moving in a random potential, there are three well-known symmetry classes, namely, the Gaussian orthogonal ensemble (GOE), the Gaussian unitary ensemble, and the Gaussian symplectic ensemble. These represent random-matrix ensembles that are generally called the Wigner-Dyson statistics.23 In this paper we study the level statistic of electromagnetic waves in disordered media, in order to not only identify its universality class, but also check the existence of the localization-delocalization transition as the intensity of the disorder increases. We show below that the behavior of electromagnetic waves in the disordered media that we study belongs to the GOE universality
class (see also24,25 about the new development of wave localization in random media).

The rest of this paper is organized as follows. In the next section the model that we study is introduced, and the governing equations for propagation of electromagnetic waves are outlined. How the TM method is implemented is described in Sec. III, while the TM results are described in Sec. IV. The computations of the density of state and the mobility edge are described in Sec. V. In Sec. VI the statistics of the energy levels are investigated. The results and conclusions are summarized in the last section.

II. MODEL AND GOVERNING EQUATIONS

We consider a medium with a random (white noise) distribution of the dielectric constants. Maxwell’s dynamical equations are utilized for describing the propagation of electromagnetic waves. Discretizing the governing equations enables us to calculate the Hamiltonian matrix, and to carry out the numerical simulations. The dynamical Maxwell’s equations are given by

\[ \nabla \times \mathbf{H} = \partial \mathbf{D}/\partial t, \]

\[ \nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t. \]  

(1)

Here, \( \mathbf{D} = \varepsilon \mathbf{E} \) and \( \mathbf{H} = \mu^{-1} \mathbf{B} \), where \( \varepsilon \) and \( \mu \) are the electric and magnetic fields, respectively. The permittivity field \( \varepsilon(x) \) is considered as a random variable, given by, \( \varepsilon(x) = \bar{\varepsilon} + \eta(x) = k \varepsilon_0 + \eta(x) \), with its mean being, \( \bar{\varepsilon} = \langle \varepsilon(x) \rangle \), where \( \eta(x) \) is a white noise with variance \( \mathcal{W}^2 \), i.e., \( \langle \eta(x_1) \eta(x_2) \rangle = \mathcal{W}^2 \delta(x_1 - x_2) \).

Let us rescale the variables by writing, \( x' = x/x_0 \), \( t' = ct/(x_0 \bar{\varepsilon}) \), \( E' = E/\bar{\varepsilon} \), \( B' = \bar{\varepsilon} B/(E_0/\bar{\varepsilon}) \), and \( \eta'(x') = \eta(x)/\bar{\varepsilon} \), which then yield the following dimensionless equations:

\[ \nabla' \times \mathbf{B'} = \left[ 1 + \eta'(x') \right] \partial \mathbf{E'}/\partial t', \]

\[ \nabla' \times \mathbf{E'} = -\partial \mathbf{B'}/\partial t'. \]  

(2)

The variance of \( \mathcal{W}^2 \) is rescaled as, \( \mathcal{W}'^2 = \mathcal{W}^2/(\bar{\varepsilon}^2 x_0^2) \), so that, \( \langle \eta'(x'_1) \eta'(x'_2) \rangle = (W'/\Delta')^2 \delta_{x'_1 x'_2} = (\bar{\varepsilon}^2 \Delta')^2 \delta_{x_1 x_2} \), where \( \Delta' \) is the lattice spacing of the rescaled system. Thus, the noise \( \eta' \) is characterized by its width, \( \sigma = (W'/\Delta')^{1/2} = W/W_0 \), with, \( W_0 = \bar{\varepsilon}^2 x_0^2 \Delta = \bar{\varepsilon}^2 \Delta \).

To carry out the computations we use Yee’s computational grid, which is commonly used in the numerical simulations of electromagnetic phenomena.26,27 For simplicity, the indices \((i,j,k)\) are used to represent the elementary cubic block of the grid; see Fig. 1. We then derive the discretized forms of the electromagnetic waves equations at frequency \( \omega \), in terms of such indices by using the finite-difference discretization, which enables us to formulate the computations as an eigenvalue problem. If we consider a vector, \( \Psi = (\mathbf{E}, \mathbf{B}) \), the components of which are the (numerical values of the) electric and magnetic fields in the entire computational grid, the eigenvalue problem is expressed by

\[ \sum_\beta H_{\alpha \beta} \Psi_\beta = \omega \Psi_\alpha, \]  

(3)

where we have considered the electromagnetic field in the form, \( \Psi(x,t) = \Psi(x) \exp(i \omega t) \).

In the 2D systems the function \( \Psi \) depends only on \((y,z)\). This leads to dividing Maxwell’s equations into two independent parts. One part contains the governing equations for the fields \((E_x, B_x, B_y)\) (the \(x\)-polarized wave), while the second part contains the governing equations for the fields \((E_y, E_z, E_x)\) (the \(y\)-polarized wave). Therefore, for the 2D media the Hamiltonian \( H \) reduces to two block-diagonal matrices such that each block is investigated separately, with no coupling between the two polarizations.

III. TRANSFER-MATRIX METHOD

Numerical simulations were carried out for 2D strips and 3D bars. Using the TM method, we computed the smallest positive Lyapunov exponent \( \gamma_m \), which represents the inverse of the localization length. The discretized equation for, for example, the field \( B_x(i,j,k+1) \) is

\[ B_x(i,j,k+1) = \frac{1}{\omega} \left[ E_y(i+1,j,k) - E_y(i,j,k) - E_x(i+1,j,k) \right. \]

\[ + E_x(i+1,j-1,k) + \frac{1}{\omega} \left[ E_y(i,j,k) - E_x(i) \right. \]

\[ - 1, j, k) - E_x(i,j,k) + E_x(i,j-1,k) \right] \]

\[ + \Delta \omega \mu \varepsilon(i,j,k) E_x(i,j,k) + B_y(i,j,k). \]  

(4)

For convenience, all the primes have been omitted. The derivatives are accurate to order \( \Delta^2 \). In the numerical simulations we set \( \Delta = 1 \) and distributed uniformly the random variable \( \eta \), representing the disorder, in the interval \([-\sigma, \sigma]\). To formulate the TM computations the equations for the fields \((\mathbf{E}, \mathbf{B})\) are written in the following form:

\[ \begin{pmatrix} \tilde{E}_x \\ \tilde{E}_y \\ \tilde{B}_x \\ \tilde{B}_y \end{pmatrix}_{k+1} = T_k \begin{pmatrix} \tilde{E}_x \\ \tilde{E}_y \\ \tilde{B}_x \\ \tilde{B}_y \end{pmatrix}_k, \]  

(5)

where \( T_k \) is the TM for the slice \( k \), and \((\tilde{E}_x)_k\), the vector that contains the values of \( E_x \) in that slice, and so on. Thus,
through numerical simulations the values of \( B_x \) and \( B_y \) in the slice \( k+1 \) are computed using values of \( \hat{E}_x \), \( \hat{E}_y \), \( B_x \), and \( B_y \) in the slice \( k \), while values of \( E_x \) and \( E_y \) in the slice \( k+1 \) are calculated knowing \( \hat{E}_x \) and \( \hat{E}_y \) in the slice \( k \) and \( B_x \) and \( B_y \) in the slice \( k+1 \), which we compute in the previous step.

In the 2D case, the transfer-matrix \( T \) is, similar to its Hamiltonian \( H \), block diagonal. One block transfers the vector \((\hat{E}_x, \hat{B}_y)_1^t \) to \((\hat{E}_x, \hat{B}_y)_1^t+1 \), where \( t \) denotes the transpose operation, and another vector transfers \((\hat{E}_x, \hat{B}_y)_1^t \) to vector \((\hat{E}_x, \hat{B}_y)_1^t+1 \). The two vectors can be investigated separately.

The disordered medium is represented by \( M \times L \) strips (in 2D) and \( M \times M \times L \) bars (in 3D), with periodic boundary conditions (PBCs), where \( M \) is the width and \( L \) the longitudinal length. The Lyapunov exponents are the logarithm of the eigenvalues of the matrix \((TT^T)^{1/2} \), and, \( T=H_{\text{sort}} \). The Oseledec theorem \(^{20} \) states that, in the limit \( L \to \infty \), all the Lyapunov exponents \( \gamma_n \) of the matrix \( T \) follow the Gaussian distribution with a variance proportional to the mean value. Thus, by specifying the initial values of the fields \( E \) and \( B \) in the first lines of the strips (in 2D) and the first planes of the bars (in 3D), we compute the fields after traversing the length \( L \) by multiplying all the TMs. In the TM method the Lyapunov exponent of the electromagnetic fields indicates that these fields decay exponentially in the transfer length \( L \) by the localization length \( \xi \), which is the inverse of Lyapunov exponent and, thus, the intensity of electromagnetic waves also decays exponentially.

The number \( N \) of the Lyapunov exponents are, \( N=2M \) in 2D and \( N=4M^2 \) in 3D media. The simulations begin by \( N \) orthogonal initial vectors, and are carried out using the Gram-Schmidt (GS) orthogonalization after every two steps of the TM iterations. The reason for the use of the GS orthogonalization is that, after some iterations the direction of all the vectors change to the direction of the vector that corresponds to the largest Lyapunov exponent. The orthogonalization procedure enables us to determine the smallest (positive) Lyapunov exponents. The iterations are continued until we achieve an acceptable relative accuracy \(^{28} \) for the rescaled variable, \( \Lambda^{-1}=\gamma_{\min}/M \).

### IV. RESULTS FOR THE LOCALIZATION LENGTH

Let us consider, first, the 2D disordered media. In our simulation we considered the s-polarized wave. The Maxwell equations reduce to

\[
-\left[1 + \eta(y,z)\right] \omega^2 E_y(y,z) = \nabla^2 E_y(y,z). \tag{6}
\]

Its discretized form is given by

\[
-\left[1 + \eta(j,k)\right] \omega^2 E_y(j,k) = E_y(j+1,k) + E_y(j-1,k) - 2E_y(j,k) + E_y(j,k+1) + E_y(j,k-1) - 2E_y(j,k). \tag{7}
\]

The computed Lyapunov exponents \( \gamma_n \) are presented in Table I for \( \omega=2 \). They occur in pairs \((-\gamma_n, \gamma_n)\), which indicates the symplectic symmetry of the TMs. The inset of Fig. 2 presents the dependence of \( \Lambda^{-1} \) on the frequency for several widths \( M \) of the 2D strips with the PBCs in the transverse direction. There are some peaks that correspond precisely to the frequencies \( \omega_\min \) with \( \omega_\min^2=2-2 \cos(2n\pi/M) \) for the PBCs and, \( \omega_\min^2=2-2 \cos(n\pi/(M+1)) \) for fixed boundary conditions (FBCs). The oscillating behavior is produced by the internal band edges, corresponding to some eigenmodes of the Hamiltonian in one slice of the ordered medium. \(^{29} \) The peaks disappear by increasing the intensity of the disorder. Figure 3 presents the results for a 2D strip of width \( M=8 \), with the FBCs and the disorder intensity \( \sigma=0.5 \), which is smaller than that for Fig. 2. When the FBCs are used, there is a minimum allowed frequency \( \omega_\min \) for the medium without

![FIG. 2.](image-url) Logarithmic plot of the rescaled Lyapunov exponent \( \Lambda^{-1} \) versus frequency of s-polarized wave for several widths \( M \) of the 2D strips. The circle symbols are for Anderson model for comparison. The inset shows the logarithmic-linear plot of \( \Lambda^{-1} \) for different values of strip width \( M \).
disorder, \( \omega_{\text{min}} = \omega_1 \approx 0.35 \) (for \( M=8 \)), for which the Lyapunov exponent is not small for the frequencies \( \omega < \omega_{\text{min}} \). As the width \( M \) increases, the frequency \( \omega_1 \) vanishes.

The main part of Fig. 2 presents the rescaled Lyapunov exponent for larger widths of strip. As the inset indicates, increasing the width of the strip also increases the rescaled Lyapunov exponent \( \Lambda^{-1} \). Such a behavior indicates that all such frequencies are localized for the disorder strength \( \sigma = 0.99 \). For very small \( \sigma \) and \( \omega \), the numerical convergence of \( \gamma_{\text{int}} \) as evaluated by the TM method, is very slow and accurate results are obtained by many more iterations. We carried out similar simulations for the \( p \)-polarized waves and found the behavior of the Lyapunov exponent to be similar to that of the \( s \)-polarized wave.

According to Eq. (7) the model in 2D can be mapped onto the Anderson model by the transformations, \( E = 4 - \omega^2 \), \( V(j,k) = \omega^2 \sigma V(j,k) \), and thus, \( \sigma V = \omega^2 \sigma \), where \( E \) is energy, \( V \) is the random potential energy of Anderson model at node \((j,k)\). The variance \( \sigma V \) is its disorder strength. The positivity of \( \epsilon(x) \) implies imposes some restriction on the variance of the disorder, i.e., \( \sigma < 1 \). For a given frequency and intensity of the disorder in Eq. (6), there is a state in the 2D Anderson model which we know is localized. Therefore, all the states are localized for the \( s \)-polarized wave except for, \( \omega = 0 \), with their equivalent in the Anderson model being, \( E=4 \) and \( \sigma V = 0 \). To make a comparison between the 2D Anderson model and \( s \)-polarized wave, we also calculated the rescaled Lyapunov exponent of the Anderson model. As shown in Fig. 2 (shown with circles) the two model have the same rescaled Lyapunov exponent. This means that the \( s \)-polarized wave in 2D random media does not have a localization-delocalization transition.

Figure 4 depicts the dependence of the rescaled Lyapunov exponent on the frequency for several widths of the 3D bar, with the strength of the disorder being, \( \sigma = 0.99 \). In 3D, one obtains a distinct behavior for \( \Lambda^{-1} \) by increasing the width \( M \) of the bar, as the frequency is varied. The scaling hypothesis\(^{30,31} \) states that, there is a unique scaling function \( F \) such that one has a scaling parameter \( \xi(\omega) \) and a scaling form, \( \Lambda^{-1}(\omega) = F(M/\xi(\omega)) \). The dependence of \( \xi \) on the frequency was computed by rescaling \( \Lambda^{-1} \) to determine the critical frequency \( \omega_c \) at which the behavior changes.

Figure 5 presents a one-parameter scaling representation of \( \Lambda^{-1} \). In the 3D bars one obtains a collapse of the numerical results for the Lyapunov exponent onto two scaling curves, obtained for several widths \( M \). The presence of the two branches is the signature of the existence of the localization-delocalization transition. The lower and upper branches correspond to the delocalized and localized states, respectively. The inset in Fig. 5 presents the frequency dependence of \( \xi \), treated as the scaling parameter. The localization \( \xi \) diverges at the critical frequency \( \omega_c \). We found that, \( \omega_c = 3.7 \pm 0.1 \), for the disorder intensity \( \sigma = 0.99 \). The localization length exponent \( \nu \) defined by, \( \xi \sim (\omega-\omega_c)^{-\nu} \), is determined by fitting the numerical data to the expansion of the one-parameter scaling function of the rescaled Lyapunov exponent.\(^{32,33} \)

\[
\Lambda^{-1} = F(M^{\nu/\nu_p}) = \sum_{(i)} a_i \chi^i M^{\nu/\nu_p}, \tag{8}
\]

where \( \chi = \xi^{-\nu} \) and \( a_0 \) is the critical value of \( \Lambda^{-1} \). Without loss of generality, the coefficient \( a_1 \) is set to be unity in Eq. (8).
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Fig. 6. (Color online) The density of states and mobility edge. The triangular dots show the DOS for several disorder intensities $\sigma$, while the diamond dots indicate the mobility edge for each $\sigma$, which was calculated by the TM method. The long-dashed lines represent the maximum allowed frequency of the DOS for several disorder intensities. The localized and extended areas are also identified in the figure.

The scaling parameter $\chi$ is then expanded as a function of the reduced frequency, $\omega_c=(\omega-\omega_c)/\omega_c$,

$$\chi = \sum_{i=1}^{m} b_i \omega_i^c. \tag{9}$$

For frequencies close to the critical frequency, we can truncate the expansion of Eqs. (8) and (9). $n=3$ and $m=2$ proved to provide accurate results. The estimated parameters and the exponent are

$$\omega_c = 3.67 \pm 0.03, \quad \Lambda_c^{-1} = 3.2 \pm 0.1, \quad \nu = 1.57 \pm 0.07.$$  

We note that the estimated critical exponent $\nu$ is the same as that of the 3D Anderson model.

V. DENSITY OF STATES AND THE MOBILITY EDGE

In 3D, one can determine the phase space of the model, i.e., determine, for a given disorder strength, the frequency interval in which the modes are localized. Let us first determine the density of state (DOS) of the model in 3D. We used the force oscillator method (FOM) to calculate the DOS of the model. The FOM is an algorithm that is particularly suitable for the treating physical systems that are described by very large matrices. The scheme enables us to compute the spectral densities of both Hermitian and non-Hermitian matrices with high speed and accuracy, especially when combined with a fast time-evolution method that is based on the Chebyshev polynomial expansion.

The Hamiltonian matrix introduced in Eq. (3) was used in the computations with the FOM. The Maxwell’s dynamical equations, Eqs. (1), satisfy the conditions, $\mathbf{\nabla} \cdot \mathbf{D} = 0$, and $\mathbf{\nabla} \cdot \mathbf{B} = 0$, for every frequency but $\omega=0$. In Fig. 6 the DOS and the mobility edge of a 3D bar, which were calculated by TM method, are plotted for several disorder intensities. The DOS was calculated so as to determine the allowed frequencies for a given intensity of disorder, and to check whether the critical frequency $\omega_c$ is in the allowed frequency area. The states in a small part of the phase space, between the long-dashed line and the mobility edge line, are all localized. The low frequencies are the delocalized states and belong to the extended regime.

VI. LEVEL STATISTICS

We also studied the level statistic of the models in both 2D and 3D. In what follows we described the method of the computations, after which the results are described.

A. Level-spacing distributions and number variance

To carry out the level-statistics analysis, we computed the eigenvalues of the Hamiltonian by using a numerical diagonalization technique. The level-spacing distribution $P(s)$ was calculated for a given intensity of the disorder and frequency window. Here $P(s)$ is the probability of finding two neighboring eigenvalues at a distance

$$s = \frac{\omega_i+1 - \omega_i}{(\omega_i+1 - \omega_i)}.$$  

The statistics of the level spacings indicate that in the localized regime, where the localization length is small compared to the medium’s linear size, the wave functions have small overlaps, so that the corresponding levels are uncorrelated and follow a Poisson distribution.

The results, shown in Fig. 7, were obtained by exact diagonalization and averaging over 200 realizations of the disorder parameter $\sigma=0.99$, for two frequency windows. The high frequencies results follow the Poisson distribution, whereas the intermediate frequencies are well represented by the WD distribution. The inset shows the level statistics of very low frequencies in the ballistic regime. Here, peak is seen at $s=1$.
is well known, in the ballistic regime the level-spacing distribution \(P(s)\) approaches a delta function centered around the mean, \(s = 1\). Such a \(\delta\) function is evident in the level statistics at very low frequencies, depicted in the inset of Fig. 7.

We note that, as described above, in the 2D disordered media the structure of the Yee’s mesh gives rise to a Hamiltonian that consists of two block-diagonal matrices. Each of the two sets represents the eigenvectors of different blocks of the Hamiltonian. Each block has eigenfrequencies that have no correlations with the frequencies of the other block. However very low frequencies do not still follow the GOE distribution because of finite-size effect they are in the ballistic regime.

Figure 8 presents the frequency dependence of the level statistics for grids of size \(24 \times 24 \times 24\) with \(\sigma = 0.99\), and the same number of realizations. The results at high frequencies are fitted well by the Poisson distribution, whereas the low-frequency statistics are represented accurately by the WD distribution (which is in the universality class of the GOE). These results are similar to those for the 2D systems. In the next section, however, we investigate the finite-size scaling behavior of level-statistics distribution was also studied. If the value of the variance \(\sigma_s^2\) becomes closer to that of the delocalized (localized) states, as the size of the medium increases, the system is in the delocalized (local-

**B. Variance \(\sigma_s^2\)**

To locate the transition point we also studied the variance, \(\sigma_s^2\) of \(P(s)\).

\[
\sigma_s^2 = \langle s^2 \rangle - \langle s \rangle^2 = \int_0^\infty s^2 P(s) ds - 1. \tag{10}
\]

Here, \(\langle \cdot \rangle\) denotes an ensemble average over the realization of the disorder. In the delocalized regime in which the level-spacing distribution is represented by the WD distribution, the variance is, \(\sigma_s^2 = 4/\pi - 1 = 0.27\), while in the localized regime it is described by the Poisson distribution with, \(\sigma_s^2 = 1\).

FIG. 9. (Color online) The number variance \(\Sigma^2\) of the 3D media of size \(24^3\) for two frequency windows, averaged over 200 realizations of the disorder with \(\sigma = 0.99\).

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\]

Here, \(\langle \cdot \rangle\) denotes an ensemble average over the realization of the disorder. In the delocalized regime in which the level-spacing distribution is represented by the WD distribution, the variance is, \(\sigma_s^2 = 4/\pi - 1 = 0.27\), while in the localized regime it is described by the Poisson distribution with, \(\sigma_s^2 = 1\).

Figures 10 and 11 present the frequency dependence of the number variance \(\sigma_s^2\) in 2D and 3D, respectively. In the 2D media at low frequencies the variance \(\sigma_s^2\) is smaller than \(2/\pi - 1 = 0.27\), indicating essentially the ballistic regime [in the completely ballistic regime with \(P(s) = \delta(s-1), \sigma_s^2 = 0\)]. Figure 10 also demonstrates that in the 2D media of finite sizes, there exist crossovers between the three different regimes, namely, the ballistic, diffusive, and localized regimes.

The scaling behavior of level-statistics distribution was also studied. If the value of the variance \(\sigma_s^2\) becomes closer to that of the delocalized (localized) states, as the size of the medium increases, the system is in the delocalized (local-

FIG. 10. (Color online) The number variance \(\Sigma^2\) versus the frequency for sizes \(L = 40, 60, 80\) of the 2D media, averaged over 800, 500, and 200 realizations of the disorder, respectively. The dashed line represents, \(\sigma_s^2 = 0.27\), the variance of the GOE. The low frequencies are in the ballistic regime and with a variance less than 0.27.
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**FIG. 11.** (Color online) The variance $\sigma^2$ versus the frequency for 3D media of sizes $L=8, 16$, and $24$, averaged, respectively, over 200 realizations of the disorder. Here, there is a critical frequency, $\omega_c = 3.8 \pm 0.1$ at which the variance remains unchanged if the system’s size increases. In the inset, the variance is plotted for a wide range of the frequencies. The dashed line represents, $\sigma^2 = 0.27$, the variance of the GOE distribution.

C. Critical distribution

The scaling theory predicts that, at the localization-delocalization transition frequency in 3D, the distribution function of the level spacings is independent of the system size. The distribution is represented well by the semi-Poisson distribution, $P(s) = 4s \exp(-2s)$, and has the combined typical properties of both the localized and delocalized regimes. The level repulsion part of the distribution for small $s$, namely, $P(s) \propto s$, is similar to that of the WD distribution (the GOE universality class with $\beta=1$), while its exponential decay for large $s$ is similar to that of the Poisson distribution.

**FIG. 12.** (Color online) Logarithmic-linear plot of the level statistics in the critical frequency range, $3.7 < \omega < 3.9$, for three sizes of the 3D media. As shown, $P(s)$ is fitted well by the semi-Poisson distribution and is not sensitive to the size $L$. The inset shows that the number variance for the same critical frequency window, which is in good agreement with semi-Poisson distribution.

The localization properties of electromagnetic waves in 2D and 3D disordered medium were studied. Using the TM method and finite-size scaling, we showed that there is a localization-delocalization transition in disordered 3D media, for which we computed the mobility edge. We showed that 2D model for the $s$-polarized wave can be mapped onto the 2D Anderson model. For the 3D electromagnetic waves the critical exponent $\nu$ that characterizes the power-law behavior of the localization length near the transition point is the same as that of the 3D Anderson model.

The statistical properties of the energy levels have distinct distributions over different frequency ranges, hence indicating, in agreement with the TM results for the 3D media, the existence of the localization-delocalization transition. The statistical distribution of the energy levels for electromagnetic model behaves similarly too, and is in the universality class of the Gaussian orthogonal ensemble.

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