4 OPTIMAL LINEAR RECONSTRUCTION OF THE STATE

4.1 INTRODUCTION

All the versions of the regulator and tracking problems solved in Chapter 3 have the following basic assumption in common: *the complete state vector can be measured accurately*. This assumption is often unrealistic. The most frequent situation is that for a given system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \qquad x(t_0) = x_0, \tag{4-1}$$

only certain linear combinations of the state, denoted by y, can be measured:

$$y(t) = C(t)x(t). \tag{4-2}$$

The quantity y, which is assumed to be an l-dimensional vector, with l usually less than the dimension n of the state x, will be referred to as the *observed variable*.

The purpose of this chapter is to present methods of reconstructing the state vector, or finding approximations to the state vector, from the observed variable. In particular, we wish to find a functional F,

$$x'(t) = F[y(\tau),\ t_0 \le \tau \le t], \qquad t_0 \le t, \tag{4-3}$$

such that $x'(t) \simeq x(t)$, where $x'(t)$ represents the *reconstructed* state. Here t_0 is the initial time of the observations. Note that $F[y(\tau),\ t_0 \le \tau \le t]$, the reconstructed $x(t)$, is a function of the *past* observations $y(\tau)$, $t_0 \le \tau \le t$, and does not depend upon future observations, $y(\tau)$, $\tau \ge t$. Once the state vector has been reconstructed, we shall be able to use the control laws of Chapter 3, which assume knowledge of the complete state vector, by replacing the *actual* state with the *reconstructed* state.

In Section 4.2 we introduce the *observer*, which is a dynamic system whose output approaches, as time increases, the state that must be reconstructed. Although this approach does not explicitly take into account the difficulties that arise because of the presence of noise, it seeks methods of reconstructing the state that implicitly involve a certain degree of filtering of the noise.

In Section 4.3 we introduce all the stochastic phenomena associated with the problem explicitly and quantitatively and find the *optimal observer*, also referred to as the *Kalman–Bucy filter*. The derivation of the optimal observer is based upon the fact that the optimal observer problem is "dual" to the optimal regulator problem presented in Chapter 3.

Finally, in Section 4.4 the steady-state and asymptotic properties of the Kalman–Bucy filter are studied. These results are easily obtained from optimal regulator theory using the duality of the optimal regulator and observer problems.

4.2 OBSERVERS

4.2.1 Full-Order Observers

In order to reconstruct the state x of the system **4-1** from the observed variable y as given by **4-2**, we propose a linear differential system the output of which is to be an approximation to the state x in a suitable sense. It will be investigated what structure this system should have and how it should behave. We first introduce the following terminology (Luenberger, 1966).

Definition 4.1. *The system*

$$\begin{aligned}\dot{q}(t) &= F(t)q(t) + G(t)y(t) + H(t)u(t),\\ z(t) &= K(t)q(t) + L(t)y(t) + M(t)u(t),\end{aligned} \qquad \textbf{4-4}$$

is an observer for the system

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t),\\ y(t) &= C(t)x(t),\end{aligned} \qquad \textbf{4-5}$$

if for every initial state $x(t_0)$ of the system **4-5** *there exists an initial state q_0 for the system* **4-4** *such that*

$$q(t_0) = q_0 \qquad \textbf{4-6}$$

implies

$$z(t) = x(t), \qquad t \geq t_0, \qquad \textbf{4-7}$$

for all $u(t)$, $t \geq t_0$.

We note that the observer **4-4** has the system input u and the system observed variable y as inputs, and as output the variable z. We are mainly interested in observers of a special type where the state $q(t)$ of the observer itself is to be an approximation to the system state $x(t)$:

Definition 4.2. *The n-dimensional system*

$$\dot{\hat{x}}(t) = F(t)\hat{x}(t) + G(t)y(t) + H(t)u(t) \qquad \textbf{4-8}$$

*is a **full-order observer** for the n-dimensional system*

$$\dot{x}(t) = A(t)x(t) + B(t)u(t), \tag{4-9a}$$

$$y(t) = C(t)x(t), \tag{4-9b}$$

if

$$\hat{x}(t_0) = x(t_0) \tag{4-10}$$

implies

$$\hat{x}(t) = x(t), \qquad t \geq t_0, \tag{4-11}$$

for all $u(t)$, $t \geq t_0$.

The observer **4-8** is called a full-order observer since its state $\hat{x}$ has the same dimension as the state x of the system **4-9**. In Section 4.2.3 we consider observers of the type **4-4** whose dimension is less than that of the state x. Such observers will be called *reduced-order observers.*

We now investigate what conditions the matrices F, G, and H must satisfy so that **4-8** qualifies as an observer. We first state the result.

Theorem 4.1. *The system* **4-8** *is an observer for the system* **4-9** *if, and only if,*

$$\begin{aligned} F(t) &= A(t) - K(t)C(t), \\ G(t) &= K(t), \\ H(t) &= B(t), \end{aligned} \tag{4-12}$$

where $K(t)$ *is an arbitrary time-varying matrix. As a result, full-order observers have the following structure:*

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t) + B(t)u(t) + K(t)[y(t) - C(t)\hat{x}(t)]. \tag{4-13}$$

This theorem can be proved as follows. By subtracting **4-8** from **4-9a** and using **4-9b**, the following differential equation for $x(t) - \hat{x}(t)$ is obtained:

$$\dot{x}(t) - \dot{\hat{x}}(t) = [A(t) - G(t)C(t)]x(t) - F(t)\hat{x}(t) + [B(t) - H(t)]u(t). \tag{4-14}$$

This immediately shows that $x(t) = \hat{x}(t)$ for $t \geq t_0$, for all $u(t)$, $t \geq t_0$, implies **4-12**. Conversely, if **4-12** is satisfied, it follows that

$$\dot{x}(t) - \dot{\hat{x}}(t) = [A(t) - K(t)C(t)][x(t) - \hat{x}(t)], \tag{4-15}$$

which shows that if $x(t_0) = \hat{x}(t_0)$ then $x(t) = \hat{x}(t)$ for all $t \geq t_0$, for all $u(t)$, $t \geq t_0$. This concludes the proof of the theorem.

The structure **4-13** follows by substituting **4-12** into **4-8**. Therefore, a full-order observer (see Fig. 4.1) consists simply of a model of the system with as extra driving variable a term that is proportional to the difference

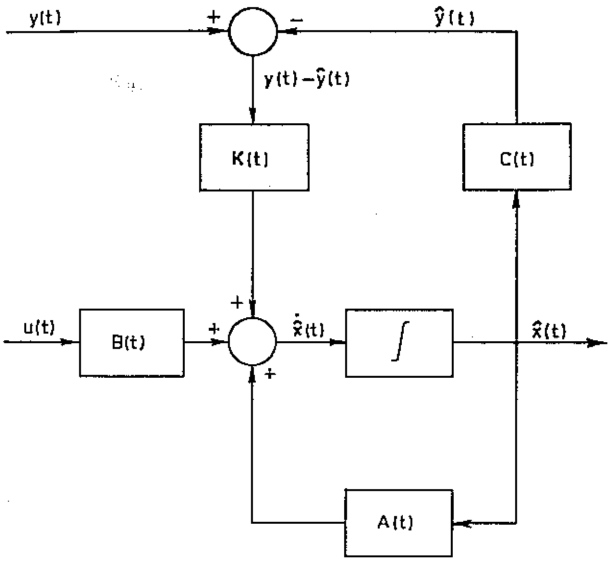

Fig. 4.1. Block diagram of a full-order observer.

$y(t) - \hat{y}(t)$, where

$$\hat{y}(t) = C(t)\hat{x}(t) \quad \textbf{4-16}$$

is the observed variable as reconstructed by the observer. We call the matrix $K(t)$ the *gain matrix* of the observer. Up to this point the choice of $K(t)$ for $t \geq t_0$ is still arbitrary.

From **4-13** we see that the observer can also be represented as

$$\dot{\hat{x}}(t) = [A(t) - K(t)C(t)]\hat{x}(t) + B(t)u(t) + K(t)y(t). \quad \textbf{4-17}$$

This shows that the *stability* of the observer is determined by the behavior of $A(t) - K(t)C(t)$. Of course stability of the observer is a desirable property in itself, but the following result shows that stability of the observer has further implications.

Theorem 4.2. *Consider the observer*

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t) + B(t)u(t) + K(t)[y(t) - C(t)\hat{x}(t)] \quad \textbf{4-18}$$

for the system

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t), \\ y(t) &= C(t)x(t). \end{aligned} \quad \textbf{4-19}$$

Then the ***reconstruction error***

$$e(t) = x(t) - \hat{x}(t) \quad \textbf{4-20}$$

satisfies the differential equation

$$\dot{e}(t) = [A(t) - K(t)C(t)]e(t). \qquad \textbf{4-21}$$

The reconstruction error has the property that

$$e(t) \to 0 \qquad \text{as } t \to \infty, \qquad \textbf{4-22}$$

for all $e(t_0)$, *if, and only if, the observer is asymptotically stable.*

That the reconstruction error, as defined by **4-20**, satisfies the differential equation **4-21** immediately follows from **4-15**. Comparing **4-21** and **4-17**, we see that the stability of the observer and the asymptotic behavior of the reconstruction error are both determined by the behavior of the matrix $A(t) - K(t)C(t)$. This clearly shows that the reconstruction error $e(t)$ approaches zero, irrespective of its initial value, if and only if the observer is asymptotically stable. This is a very desirable result.

Observer design thus revolves about determining the gain matrix $K(t)$ for $t \geq t_0$ such that the reconstruction error differential equation **4-21** is asymptotically stable. In the time-invariant case, where all matrices occurring in the problem formulation are constant, including the gain K, the stability of the observer follows from the locations of the characteristic values of the matrix $A - KC$. We refer to the characteristic values of $A - KC$ as the *observer poles.* In the next section we prove that, under a mildly restrictive condition (complete reconstructibility of the system), all observer poles can be arbitrarily located in the complex plane by choosing K suitably (within the restriction that complex poles occur in complex conjugate pairs).

At this point we can only offer some intuitive guidelines for a choice of K to obtain satisfactory performance of the observer. To obtain fast convergence of the reconstruction error to zero, K should be chosen so that the observer poles are quite deep in the left-half complex plane. This, however, generally must be achieved by making the gain matrix K large, which in turn makes the observer very sensitive to any observation noise that may be present, added to the observed variable $y(t)$. A compromise must be found. Section 4.3 is devoted to the problem of finding an *optimal* compromise, taking into account all the statistical aspects of the problem.

Example 4.1. *Positioning system*

In Example 2.4 (Section 2.3), we considered a positioning system described by the state differential equation

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \kappa \end{pmatrix} \mu(t). \qquad \textbf{4-23}$$

Here $x(t) = \text{col}\,[\xi_1(t), \xi_2(t)]$, where $\xi_1(t)$ denotes the angular displacement and $\xi_2(t)$ the angular velocity. Let us assume that the observed variable $\eta(t)$ is the angular displacement, that is,

$$\eta(t) = (1, \quad 0)x(t).$$

A time-invariant observer for this system is given by

$$\dot{\hat{x}}(t) = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix}\hat{x}(t) + \begin{pmatrix} 0 \\ \kappa \end{pmatrix}\mu(t) + \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}[\eta(t) - (1, \quad 0)\hat{x}(t)], \quad \textbf{4-24}$$

where the constant gains k_1 and k_2 are to be selected. The characteristic polynomial of the observer is given by

$$\det\left[\begin{pmatrix} s & 0 \\ 0 & s \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix} + \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}(1, \quad 0)\right] = \det\left[\begin{pmatrix} s + k_1 & -1 \\ k_2 & s + \alpha \end{pmatrix}\right]$$
$$= s^2 + (\alpha + k_1)s + k_2. \quad \textbf{4-25}$$

With the numerical values of Example 2.4, the characteristic values of the system **4-23** are located at 0 and $-\alpha = -4.6\ \mathrm{s}^{-1}$. In order to make the observer fast as compared to the system itself, let us select the gains k_1 and k_2 such that the observer poles are located at $-50 \pm j50\ \mathrm{s}^{-1}$. This yields for the gains:

$$k_1 = 95.40\ \mathrm{s}^{-1}, \qquad k_2 = 4561\ \mathrm{s}^{-2}. \quad \textbf{4-26}$$

In Fig. 4.2 we compare the output of the observer to the actual response of the system. The initial conditions of the positioning system are

$$\xi_1(0) = 0.1\ \mathrm{rad}, \qquad \xi_2(0) = 0.5\ \mathrm{rad/s}, \quad \textbf{4-27}$$

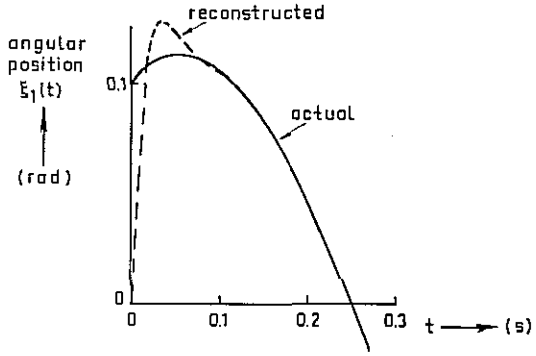

Fig. 4.2. Actual response of a positioning system and the response as reconstructed by a full-order observer.

while the input voltage is given by

$$\mu(t) = -10 \text{ V}, \qquad t \geq 0. \tag{4-28}$$

The observer has zero initial conditions. Figure 4.2 clearly shows the excellent convergence of the reconstructed angular position to its actual behavior.

4.2.2* Conditions for Pole Assignment and Stabilization of Observers

In this section we state necessary and sufficient conditions for pole assignment and stabilization of time-invariant full-order observers. We first have the following result, which is dual to Theorem 3.1 (Section 3.2.2).

Theorem 4.3. *Consider the time-invariant full-order observer*

$$\dot{\hat{x}}(t) = A\hat{x}(t) + K[y(t) - C\hat{x}(t)] + Bu(t) \tag{4-29}$$

for the time-invariant system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t). \end{aligned} \tag{4-30}$$

Then the observer poles, that is, the characteristic values of $A - KC$, can be arbitrarily located in the complex plane (within the restriction that complex characteristic values occur in complex conjugate pairs), by choosing the constant matrix K suitably, if and only if the system **4-30** *is completely reconstructible.*

To prove this theorem we note that

$$\det[\lambda I - (A - KC)] = \det[\lambda I - (A^T - C^TK^T)], \tag{4-31}$$

so that the characteristic values of $A - KC$ are identical to those of $A^T - C^TK^T$. However, by Theorem 3.1 the characteristic values of $A^T - C^TK^T$ can be arbitrarily located by choosing K appropriately if and only if the pair $\{A^T, C^T\}$ is completely controllable. From Theorem 1.41 (Section 1.8), we know that $\{A^T, C^T\}$ is completely controllable if and only if $\{A, C\}$ is completely reconstructible. This completes the proof.

If $\{A, C\}$ is not completely reconstructible, the following theorem, which is dual to Theorem 3.2 (Section 3.2.2) gives conditions for the stability of the observer.

Theorem 4.4. *Consider the time-invariant observer*

$$\dot{\hat{x}}(t) = A\hat{x}(t) + K[y(t) - C\hat{x}(t)] + Bu(t) \tag{4-32}$$

for the time-invariant system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t). \end{aligned} \tag{4-33}$$

Then a matrix K can be found such that the observer is asymptotically stable if and only if the system **4-33** *is detectable.*

Detectability was defined in Section 1.7.4. The proof of this theorem follows by duality from Theorem 3.2.

4.2.3* Reduced-Order Observers

In this section we show that it is possible to find observers of dimension *less* than the dimension of the system to be observed. Such observers are called *reduced-order observers*. For simplicity we discuss only the time-invariant case. Let the system to be observed be described by

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t), \end{aligned} \qquad \textbf{4-34}$$

where the dimension of the state $x(t)$ is n and the dimension of the observed variable $y(t)$ is given by l. Since the observation equation $y(t) = Cx(t)$ provides us with l linear equations in the unknown state $x(t)$, it is necessary to reconstruct only $n - l$ linear combinations of the components of the state. This approach was first considered by Luenberger (1964, 1966). We follow the derivation of Cumming (1969).

Assuming that C has full rank, we introduce an $(n - l)$-dimensional vector $p(t)$,

$$p(t) = C'x(t), \qquad \textbf{4-35}$$

such that

$$\begin{pmatrix} C \\ C' \end{pmatrix} \qquad \textbf{4-36}$$

is nonsingular. By the relations

$$\begin{aligned} y(t) &= Cx(t), \\ p(t) &= C'x(t), \end{aligned} \qquad \textbf{4-37}$$

it follows that

$$x(t) = \begin{pmatrix} C \\ C' \end{pmatrix}^{-1} \begin{pmatrix} y(t) \\ p(t) \end{pmatrix}. \qquad \textbf{4-38}$$

It is convenient to write

$$\begin{pmatrix} C \\ C' \end{pmatrix}^{-1} = (L_1, L_2), \qquad \textbf{4-39}$$

so that

$$x(t) = L_1 y(t) + L_2 p(t). \qquad \textbf{4-40}$$

Thus if we reconstruct $p(t)$ and denote the reconstructed value by $\hat{p}(t)$, we can write the reconstructed state as

$$\hat{x}(t) = L_1 y(t) + L_2 \hat{p}(t). \qquad \textbf{4-41}$$

An observer for $p(t)$ can be found by noting that $p(t)$ obeys the following differential equation

$$\dot{p}(t) = C'Ax(t) + C'Bu(t), \qquad \textbf{4-42}$$

or

$$\dot{p}(t) = C'AL_2 p(t) + C'AL_1 y(t) + C'Bu(t). \qquad \textbf{4-43}$$

Note that in this differential equation $y(t)$ serves as a forcing variable. If we now try to determine an observer for p by replacing p with $\hat{p}$ in **4-43** and adding a term of the form $K(t)[y(t) - C\hat{x}(t)]$, where K is a gain matrix, this is unsuccessful since from **4-41** we have $y - C\hat{x} = y - CL_1 y - CL_2\hat{p} = y - y = 0$; apparently, y does not carry any information about p. New information must be laid bare by differentiating $y(t)$:

$$\begin{aligned}\dot{y}(t) &= CAx(t) + CBu(t) \\ &= CAL_2 p(t) + CAL_1 y(t) + CBu(t).\end{aligned} \qquad \textbf{4-44}$$

Equations **4-43** and **4-44** suggest the observer

$$\begin{aligned}\dot{\hat{p}}(t) = C'AL_2\hat{p}(t) + C'AL_1 y(t) + C'Bu(t) \\ + K[\dot{y}(t) - CAL_1 y(t) - CBu(t) - CAL_2\hat{p}(t)].\end{aligned} \qquad \textbf{4-45}$$

We leave it as an exercise to show that, if the pair $\{A, C\}$ is completely reconstructible, also the pair $\{C'AL_2, CAL_2\}$ is completely reconstructible, so that by a suitable choice of K all the poles of **4-45** can be placed at arbitrary positions (Wonham, 1970a).

In the realization of the observer, there is no need to take the derivative of $y(t)$. To show this, define

$$q(t) = \hat{p}(t) - Ky(t). \qquad \textbf{4-46}$$

It is easily seen that

$$\begin{aligned}\dot{q}(t) = &[C'AL_2 - KCAL_2]q(t) \\ &+ [C'AL_2K + C'AL_1 - KCAL_1 - KCAL_2K]y(t) \\ &+ [C'B - KCB]u(t).\end{aligned} \qquad \textbf{4-47}$$

This equation does not contain $\dot{y}(t)$. The reconstructed state follows from

$$\hat{x}(t) = L_2 q(t) + (L_1 + L_2 K)y(t). \qquad \textbf{4-48}$$

Together, **4-47** and **4-48** constitute an observer of the form **4-4**.

Since the reduced-order observer has a direct link from the observed variable $y(t)$ to the reconstructed state $\hat{x}(t)$, the estimate $\hat{x}(t)$ will be more sensitive to measurement errors in $y(t)$ than the estimate generated by a full-order observer. The question of the effects of measurement errors and system disturbances upon the observer is discussed in Section 4.3.

Example 4.2 *Positioning system*

In this example we derive a one-dimensional observer for the positioning system we considered in Example 4.1. For this system the observed variable is given by

$$\eta(t) = (1, \quad 0)x(t). \tag{4-49}$$

Understandably, we choose the variable $p(t)$, which now is a scalar, as

$$p(t) = (0, \quad 1)x(t), \tag{4-50}$$

so that $p(t)$ is precisely the angular velocity. It is immediately seen that $p(t)$ satisfies the differential equation

$$\dot{p}(t) = -\alpha p(t) + \kappa\mu(t). \tag{4-51}$$

Our observation equation we obtain by differentiation of $\eta(t)$:

$$\dot{\eta}(t) = \dot{\xi}_1(t) = \xi_2(t) = p(t). \tag{4-52}$$

An observer for $p(t)$ is therefore given by

$$\dot{\hat{p}}(t) = -\alpha\hat{p}(t) + \kappa\mu(t) + \lambda[\dot{\eta}(t) - \hat{p}(t)], \tag{4-53}$$

where the scalar observer gain λ is to be selected. The characteristic value of the observer is $-(\alpha + \lambda)$. To make the present design comparable to the full-order observer of Example 4.1, we choose the observer pole at the same distance from the origin as the pair of poles of Example 4.1. Thus we let $\alpha + \lambda = 50\sqrt{2} = 70.71\ \text{s}^{-1}$. With $\alpha = 4.6\ \text{s}^{-1}$ this yields for the gain

$$\lambda = 66.11\ \text{s}^{-1}. \tag{4-54}$$

The reconstructed state of the original system is given by

$$\hat{x}(t) = \begin{pmatrix} \eta(t) \\ \hat{p}(t) \end{pmatrix}, \qquad t \geq 0. \tag{4-55}$$

To obtain a reduced-order observer without derivatives, we set

$$q(t) = \hat{p}(t) - \lambda\eta(t). \tag{4-56}$$

By using **4-53** it follows that $q(t)$ satisfies the differential equation

$$\dot{q}(t) = -(\alpha + \lambda)q(t) + \kappa\mu(t) - (\alpha + \lambda)\lambda\eta(t). \tag{4-57}$$

In terms of $q(t)$ the reconstructed state of the original system is given by

$$\hat{x}(t) = \begin{pmatrix} \eta(t) \\ q(t) + \lambda\eta(t) \end{pmatrix}. \tag{4-58}$$

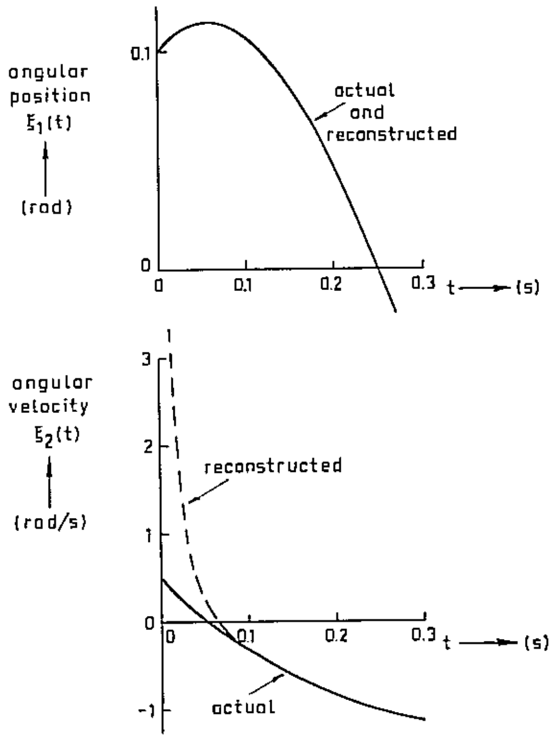

Fig. 4.3. Actual response of a positioning system and the response as reconstructed by a reduced-order observer.

In Fig. 4.3 we compare the output of the reduced-order observer described by **4-57** and **4-58** to the actual behavior of the system. The initial conditions of the system are, as in Example 4.1:

$$\xi_1(0) = 0.1 \text{ rad}, \qquad \xi_2(0) = 0.5 \text{ rad/s}, \tag{4-59}$$

while the input is given by

$$\mu(t) = -10 \text{ V}, \qquad t \geq 0. \tag{4-60}$$

The observer initial condition is

$$q(0) = 0 \text{ rad/s}. \tag{4-61}$$

Figure 4.3 shows that the angular position is of course faithfully reproduced and that the estimated angular velocity quickly converges to the correct value, although the initial estimate is not very good.

4.3 THE OPTIMAL OBSERVER

4.3.1 A Stochastic Approach to the Observer Problem

In Section 4.2 we introduced observers. It has been seen, however, that in the selection of an observer for a given system a certain arbitrariness remains in the choice of the gain matrix K. In this section we present methods of finding the *optimal* gain matrix. To this end we must make specific assumptions concerning the disturbances and observation errors that occur in the system that is to be observed. We shall then be able to define the sense in which the observer is to be optimal.

It is assumed that the actual system equations are

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + w_1(t), \qquad \textbf{4-62a}$$

$$y(t) = C(t)x(t) + w_2(t). \qquad \textbf{4-62b}$$

Here $w_1(t)$ is termed the *state excitation noise*, while $w_2(t)$ is the *observation* or *measurement noise*. It is assumed that the joint process col $[w_1(t), w_2(t)]$ can be described as white noise with intensity

$$V(t) = \begin{pmatrix} V_1(t) & V_{12}(t) \\ V_{12}^T(t) & V_2(t) \end{pmatrix}, \qquad \textbf{4-63}$$

that is,

$$E\left\{\begin{pmatrix} w_1(t_1) \\ w_2(t_1) \end{pmatrix}[w_1^T(t_2), w_2^T(t_2)]\right\} = V(t_1)\,\delta(t_1 - t_2). \qquad \textbf{4-64}$$

If $V_{12}(t) = 0$, the state excitation noise and the observation noise are *uncorrelated*. Later (in Section 4.3.5) we consider the possibility that $w_1(t)$ and $w_2(t)$ can not be represented as white noise processes. A case of special interest occurs when

$$V_2(t) > 0, \qquad t \geq t_0. \qquad \textbf{4-65}$$

This assumption means in essence that all components of the observed variable are corrupted by white noise and that it is impossible to extract from $y(t)$ information that does not contain white noise. If this condition is satisfied, we call the problem of reconstructing the state of the system **4-62** *nonsingular*.

Finally, we denote

$$E\{x(t_0)\} = \bar{x}_0, \qquad E\{[x(t_0) - \bar{x}_0][x(t_0) - \bar{x}_0]^T\} = Q_0. \qquad \textbf{4-66}$$

Suppose now that a full-order observer of the form

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t) + B(t)u(t) + K(t)[y(t) - C(t)\hat{x}(t)] \qquad \textbf{4-67}$$

is connected to the system **4-62**. Then the *reconstruction error* is given by

$$e(t) = x(t) - \hat{x}(t). \tag{4-68}$$

The *mean square reconstruction error*

$$E\{e^T(t)W(t)e(t)\}, \tag{4-69}$$

with $W(t)$ a given positive-definite symmetric weighting matrix, is a measure of how well the observer reconstructs the state of the system at time t. The mean square reconstruction error is determined by the choice of $\hat{x}(t_0)$ and of $K(\tau)$, $t_0 \le \tau \le t$. The problem of how to choose these quantities optimally is termed the *optimal observer problem*.

Definition 4.3. *Consider the system*

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) + w_1(t), \\ y(t) &= C(t)x(t) + w_2(t), \end{aligned} \qquad t \ge t_0. \tag{4-70}$$

Here col $[w_1(t), w_2(t)]$ *is a white noise process with intensity*

$$\begin{pmatrix} V_1(t) & V_{12}(t) \\ V_{12}^T(t) & V_2(t) \end{pmatrix}, \qquad t \ge t_0. \tag{4-71}$$

Furthermore, the initial state $x(t_0)$ is uncorrelated with w_1 and w_2,

$$E\{x(t_0)\} = \bar{x}_0, \qquad E\{[x(t_0) - \bar{x}_0][x(t_0) - \bar{x}_0]^T\} = Q_0, \tag{4-72}$$

and $u(t)$, $t \ge t_0$, is a given input to the system. Consider the observer

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t) + B(t)u(t) + K(t)[y(t) - C(t)\hat{x}(t)]. \tag{4-73}$$

Then the problem of finding the matrix function $K(\tau)$, $t_0 \le \tau \le t$, and the initial condition $\hat{x}(t_0)$, so as to minimize

$$E\{e^T(t)W(t)e(t)\}, \tag{4-74}$$

where

$$e(t) = x(t) - \hat{x}(t), \tag{4-75}$$

and where $W(t)$ is a positive-definite symmetric weighting matrix, is termed the ***optimal observer problem****. If*

$$V_2(t) > 0, \qquad t \ge t_0, \tag{4-76}$$

the optimal observer problem is called ***nonsingular****.*

In Section 4.3.2 we study the nonsingular optimal observer problem where the state excitation noise and the observation noise are assumed moreover to be uncorrelated. In Section 4.3.3 we relax the condition of uncorrelatedness, while in Section 4.3.4 the singular problem is considered.

4.3.2 The Nonsingular Optimal Observer Problem with Uncorrelated State Excitation and Observation Noises

In this section we consider the nonsingular optimal observer problem where it is assumed that the state excitation noise and the observation noise are uncorrelated. This very important problem was first solved by Kalman and Bucy (Kalman and Bucy, 1961), and its solution has had a tremendous impact on optimal filtering theory. A historical account of the derivation of the so-called *Kalman–Bucy filter* is given by Sorenson (1970).

Somewhat surprisingly the derivation of the optimal observer can be based on Lemma 3.1 (Section 3.3.3). Before proceeding to this derivation, however, we introduce the following lemma, which shows how time can be reversed in any differential equation.

Lemma 4.1. *Consider the differential equations*

$$\frac{dx(t)}{dt} = f[t, x(t)], \qquad t \geq t_0, \qquad x(t_0) = x_0, \tag{4-77}$$

and

$$-\frac{dy(t)}{dt} = f[t^* - t, y(t)], \qquad t \leq t_1, \qquad y(t_1) = y_1, \tag{4-78}$$

where $t_0 < t_1$, *and*

$$t^* = t_0 + t_1. \tag{4-79}$$

Then if

$$x_0 = y_1, \tag{4-80}$$

the solutions of **4-77** *and* **4-78** *are related as follows:*

$$\begin{aligned} x(t) &= y(t^* - t), \qquad t \geq t_0, \\ y(t) &= x(t^* - t), \qquad t \leq t_1. \end{aligned} \tag{4-81}$$

This lemma is easily proved by a change in variable from t to $t^* - t$.

We now proceed with our derivation of the optimal observer. Subtracting **4-67** from **4-62a** and using **4-62b**, we obtain the following differential equation for the reconstruction error $e(t) = x(t) - \hat{x}(t)$:

$$\begin{aligned} \dot{e}(t) &= [A(t) - K(t)C(t)]e(t) + (I, -K(t))\begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix}, \\ e(t_0) &= e_0, \end{aligned} \tag{4-82}$$

where

$$e_0 = x(t_0) - \hat{x}(t_0), \tag{4-83}$$

and where, as yet, $K(t)$, $t \geq t_0$, is an arbitrary matrix function. Let us denote by $\tilde{Q}(t)$ the variance matrix of $e(t)$, and by $\bar{e}(t)$ the mean of $e(t)$:

$$E\{e(t)\} = \bar{e}(t),$$
$$E\{[e(t) - \bar{e}(t)][e(t) - \bar{e}(t)]^T\} = \tilde{Q}(t). \qquad \textbf{4-84}$$

Then we write

$$E\{e(t)e^T(t)\} = \bar{e}(t)\bar{e}^T(t) + \tilde{Q}(t). \qquad \textbf{4-85}$$

With this, using **1-469**, the mean square reconstruction error can be expressed as

$$E\{e^T(t)W(t)e(t)\} = \bar{e}^T(t)W(t)\bar{e}(t) + \text{tr}\,[\tilde{Q}(t)W(t)]. \qquad \textbf{4-86}$$

The first term of this expression is obviously minimal when $\bar{e}(t) = 0$. This can be achieved by letting $\bar{e}(t_0) = 0$, since by Theorem 1.52 (Section 1.11.2) $\bar{e}(t)$ obeys the homogeneous differential equation

$$\dot{\bar{e}}(t) = [A(t) - K(t)C(t)]\bar{e}(t), \qquad t \geq t_0. \qquad \textbf{4-87}$$

We can make $\bar{e}(t_0) = 0$ by choosing the initial condition of the observer as

$$\hat{x}(t_0) = \bar{x}_0. \qquad \textbf{4-88}$$

Since the second term of **4-86** does not depend upon $\bar{e}(t)$, it can be minimized independently. From Theorem 1.52 (Section 1.11.2), we obtain the following differential equation for $\tilde{Q}(t)$:

$$\dot{\tilde{Q}}(t) = [A(t) - K(t)C(t)]\tilde{Q}(t) + \tilde{Q}(t)[A(t) - K(t)C(t)]^T$$
$$+ V_1(t) + K(t)V_2(t)K^T(t). \qquad \textbf{4-89}$$

The corresponding initial condition is

$$\tilde{Q}(t_0) = Q_0. \qquad \textbf{4-90}$$

Let us now introduce a differential equation in a matrix function $\tilde{P}(t)$, which is derived from **4-89** by reversing time (Lemma 4.1):

$$-\dot{\tilde{P}}(t) = [A^T(t^* - t) - C^T(t^* - t)K^T(t^* - t)]^T\tilde{P}(t)$$
$$+ \tilde{P}(t)[A^T(t^* - t) - C^T(t^* - t)K^T(t^* - t)]$$
$$+ V_1(t^* - t) + K(t^* - t)V_2(t^* - t)K^T(t^* - t), \qquad t \leq t_1. \qquad \textbf{4-91}$$

Here

$$t^* = t_0 + t_1, \qquad \textbf{4-92}$$

with $t_1 > t_0$. We associate with **4-91** the terminal condition

$$\tilde{P}(t_1) = Q_0. \qquad \textbf{4-93}$$

It immediately follows from Lemma 4.1 that

$$\tilde{Q}(t) = \tilde{P}(t^* - t), \qquad t \leq t_1. \qquad \textbf{4-94}$$

Let us now apply Lemma 3.1 (Section 3.3.3) to **4-91**. This lemma shows that the matrix $\tilde{P}(t)$ is minimized if $K(t^* - \tau)$, $t \le \tau \le t_1$, is chosen as $K^0(t^* - \tau)$, $t \le \tau \le t_1$, where

$$K^0(t^* - \tau) = V_2^{-1}(t^* - \tau)C(t^* - \tau)P(\tau). \qquad \textbf{4-95}$$

In this expression $P(t)$ is the solution of **4-91** with K replaced by K^0, that is,

$$-\dot{P}(t) = V_1(t^* - t) - P(t)C^T(t^* - t)V_2^{-1}(t^* - t)C(t^* - t)P(t) + P(t)A^T(t^* - t) + A(t^* - t)P(t), \qquad t \le t_1, \qquad \textbf{4-96}$$

with the terminal condition

$$P(t_1) = Q_0. \qquad \textbf{4-97}$$

The minimal value of $\tilde{P}(t)$ is $P(t)$, where the minimization is in the sense that

$$P(t) \le \tilde{P}(t), \qquad t \le t_1. \qquad \textbf{4-98}$$

By reversing time back again in **4-96**, we see that the variance matrix $\tilde{Q}(t)$ of $e(t)$ is minimized in the sense that

$$\tilde{Q}(t) \ge Q(t), \qquad t \ge t_0, \qquad \textbf{4-99}$$

by choosing $K(\tau) = K^0(\tau)$, $t_0 \le \tau \le t$, where

$$K^0(\tau) = Q(\tau)C^T(\tau)V_2^{-1}(\tau), \qquad \tau \ge t_0, \qquad \textbf{4-100}$$

and where the matrix $Q(t)$ satisfies the matrix Riccati equation

$$\dot{Q}(t) = V_1(t) - Q(t)C^T(t)V_2^{-1}(t)C(t)Q(t) + Q(t)A^T(t) + A(t)Q(t), \qquad t \ge t_0, \qquad \textbf{4-101}$$

with the initial condition

$$Q(t_0) = Q_0. \qquad \textbf{4-102}$$

Since **4-99** implies that

$$\text{tr}\,[Q(t)W(t)] \le \text{tr}\,[\tilde{Q}(t)W(t)] \qquad \textbf{4-103}$$

for any positive-definite symmetric matrix $W(t)$, we conclude that the gain matrix **4-100** optimizes the observer. We moreover see from **4-86** that for the optimal observer the mean square reconstruction error is given by

$$E\{e^T(t)W(t)e(t)\} = \text{tr}\,[Q(t)W(t)], \qquad \textbf{4-104}$$

while the variance matrix of $e(t)$ is $Q(t)$.

We finally remark that the result we have obtained is independent of the particular time t at which we have chosen to minimize the mean square reconstruction error. Thus if the gain is determined according to **4-100**, the mean square reconstruction error is simultaneously minimized *for all* $t \ge t_0$.

Our findings can be summarized as follows.

Theorem 4.5. *Consider the optimal observer problem of Definition* 4.3. *Suppose that the problem is nonsingular and that the state excitation and observation noise are uncorrelated. Then the solution of the optimal observer problem is obtained by choosing for the gain matrix*

$$K^0(t) = Q(t)C^T(t)V_2^{-1}(t), \qquad t \geq t_0, \tag{4-105}$$

where $Q(t)$ *is the solution of the matrix Riccati equation*

$$\dot{Q}(t) = A(t)Q(t) + Q(t)A^T(t) + V_1(t) - Q(t)C^T(t)V_2^{-1}(t)C(t)Q(t), \qquad t \geq t_0, \tag{4-106}$$

with the initial condition

$$Q(t_0) = Q_0. \tag{4-107}$$

The initial condition of the observer should be chosen as

$$\hat{x}(t_0) = \bar{x}_0. \tag{4-108}$$

If **4-105** *and* **4-108** *are satisfied,*

$$E\{[x(t) - \hat{x}(t)]^T W(t)[x(t) - \hat{x}(t)]\} \tag{4-109}$$

is minimized for all $t \geq t_0$. *The variance matrix of the reconstruction error is given by*

$$E\{[x(t) - \hat{x}(t)][x(t) - \hat{x}(t)]^T\} = Q(t), \tag{4-110}$$

while the mean square reconstruction error is

$$E\{[x(t) - \hat{x}(t)]^T W(t)[x(t) - \hat{x}(t)]\} = \text{tr}\,[Q(t)W(t)]. \tag{4-111}$$

It is noted that the solution of the optimal observer problem is, surprisingly, independent of the weighting matrix $W(t)$.

The optimal observer of Theorem 4.5 is known as the *Kalman–Bucy filter.* In this section we have derived this filter by first assuming that it has the form of an observer. In the original derivation of Kalman and Bucy (1961), however, it is proved that this filter is the *minimum mean square linear estimator*, that is, we cannot find another linear functional of the observations $y(\tau)$ and the input $u(\tau)$, $t_0 \leq \tau \leq t$, that produces an estimate of the state $x(t)$ with a smaller mean square reconstruction error. It can also be proved (see, e.g., Jazwinski, 1970) that if the initial state $x(t_0)$ is Gaussian, and the state excitation noise w_1 and the observation noise w_2 are Gaussian white noise processes, the Kalman–Bucy filter produces an estimate $\hat{x}(t)$ of $x(t)$ that has minimal mean square reconstruction error among *all* estimates that can be obtained by processing the data $y(\tau)$ and $u(\tau)$, $t_0 \leq \tau \leq t$.

The close relationship between the optimal *regulator* problem and the optimal *observer* problem is evident from the fact that the matrix Riccati equation for the observer variance matrix is just the time-reversed Riccati equation that holds for the regulator problem. In later sections we make further use of this relationship, which will be referred to as the *duality property*, in deriving facts about observers from facts about regulators.

The gain matrix $K^0(t)$ can be obtained by solving the matrix Riccati equation **4-106** in real time and using **4-105**. Alternatively, $K^0(t)$ can be computed in advance, stored, and played back during the state reconstruction process. It is noted that in contrast to the optimal regulator described in Chapter 3 the optimal observer can easily be implemented in real time, since **4-106** is a differential equation with given *initial* conditions, whereas the optimal regulator requires solution of a Riccati equation with given *terminal* conditions that must be solved backward in time.

In Theorem 3.3 (Section 3.3.2), we saw that the regulator Riccati equation can be obtained by solving a set of $2n \times 2n$ differential equations (where n is the dimension of the state). The same can be done with the observer Riccati equation, as is outlined in Problem 4.3.

We now briefly discuss the steady-state properties of the optimal observer. What we state here is proved in Section 4.4.3. It can be shown that under mildly restrictive conditions the solution $Q(t)$ of the observer Riccati equation **4-106** converges to a *steady-state solution* $\bar{Q}(t)$ which is independent of Q_0 as the initial time t_0 approaches $-\infty$. In the time-invariant case, where all the matrices occurring in Definition 4.3 are constant, the steady-state solution $\bar{Q}$ is, in addition, a constant matrix and is, in general, the unique nonnegative-definite solution of the *algebraic observer Riccati equation*

$$0 = AQ + QA^T + V_1 - QC^TV_2^{-1}CQ. \qquad \textbf{4-112}$$

This equation is obtained from **4-106** by setting the time derivative equal to zero.

Corresponding to the steady-state solution $\bar{Q}$ of the observer Riccati equation, we obtain the *steady-state optimal observer gain matrix*

$$\bar{K}(t) = \bar{Q}(t)C^T(t)V_2^{-1}(t). \qquad \textbf{4-113}$$

It is proved in Section 4.4.3, again under mildly restrictive conditions, that the observer with $\bar{K}$ as gain matrix is, in general, asymptotically stable. We refer to this observer as the *steady-state optimal observer*. Since in the time-invariant case the steady-state observer is also time-invariant, it is very attractive to use the steady-state optimal observer since it is much easier to implement. In the time-invariant case, the steady-state optimal observer is optimal in the sense that

$$\lim_{t_0 \to -\infty} E\{e^T(t)We(t)\} = \lim_{t \to \infty} E\{e^T(t)We(t)\} \qquad \textbf{4-114}$$

is minimal with respect to all other time-invariant observers.

We conclude this section with the following discussion which is restricted to the time-invariant case. The optimal observer provides a compromise between the *speed of state reconstruction* and the *immunity to observation noise*. The balance between these two properties is determined by the magnitudes of the white noise intensities V_1 and V_2. This balance can be varied by keeping V_1 constant and setting

$$V_2 = \rho M, \qquad \textbf{4-115}$$

where M is a constant positive-definite symmetric matrix and ρ is a positive scalar that is varied. It is intuitively clear that decreasing ρ improves the speed of state reconstruction, since less attention can be paid to filtering the observation noise. This increase in reconstruction speed is accompanied by a shift of the observer poles further into the left-half complex plane. In cases where one is not sure of the exact values of V_1 or V_2, a good design procedure may be to assume that V_2 has the form **4-115** and vary ρ until a satisfactory observer is obtained. The limiting properties of the optimal observer as $\rho \downarrow 0$ or $\rho \to \infty$ are reviewed in Section 4.4.4.

Example 4.3. *The estimation of a "constant"*

In many practical situations variables are encountered that stay constant over relatively long periods of time and only occasionally change value. One possible approach to model such a constant is to represent it as the state of an undisturbed integrator with a stochastic initial condition. Thus let $\xi(t)$ represent the constant. Then we suppose that

$$\begin{aligned} \dot{\xi}(t) &= 0, \\ \xi(0) &= \xi_0, \end{aligned} \qquad \textbf{4-116}$$

where ξ_0 is a scalar stochastic variable with mean $\bar{\xi}_0$ and variance Q_0. We assume that we measure this constant with observation noise $\nu_2(t)$, that is, we observe

$$\eta(t) = \xi(t) + \nu_2(t), \qquad \textbf{4-117}$$

where $\nu_2(t)$ is assumed to be white noise with constant scalar intensity V_2.

The optimal observer for $\xi(t)$ is given by

$$\begin{aligned} \dot{\hat{\xi}}(t) &= k(t)[\eta(t) - \hat{\xi}(t)] \\ \hat{\xi}(0) &= \bar{\xi}_0, \end{aligned} \qquad \textbf{4-118}$$

where the scalar gain $k(t)$ is, from **4-105**, given by

$$k(t) = \frac{Q(t)}{V_2}. \qquad \textbf{4-119}$$

The error variance $Q(t)$ is the solution of the Riccati equation

$$\dot{Q}(t) = -\frac{Q^2(t)}{V_2}, \qquad Q(0) = Q_0. \tag{4-120}$$

Equation **4-120** can be solved explicitly:

$$Q(t) = \frac{Q_0 V_2}{V_2 + Q_0 t}, \qquad t \geq 0, \tag{4-121}$$

so that

$$k(t) = \frac{Q_0}{V_2 + Q_0 t}, \qquad t \geq 0. \tag{4-122}$$

We note that as $t \rightarrow \infty$ the error variance $Q(t)$ approaches zero, which means that eventually a completely accurate estimate of $\xi(t)$ becomes available. As a result, also $k(t) \rightarrow 0$, signifying that there is no point in processing any new data.

This observer is not satisfactory when the constant occasionally changes value, or in reality varies slowly. In such a case we can model the constant as the output of an integrator driven by white noise. The justification for modeling the process in this way is that integrated white noise has a very large low-frequency content. Thus we write

$$\begin{aligned} \dot{\xi}(t) &= \nu_1(t), \qquad \xi(0) = \xi_0 \\ \eta(t) &= \xi(t) + \nu_2(t), \end{aligned} \tag{4-123}$$

where ν_1 is white noise with constant intensity V_1 and ν_2 is white noise as before, independent of ν_1. The steady-state optimal observer is now easily found to be given by

$$\dot{\hat{\xi}}(t) = \bar{k}[\eta(t) - \hat{\xi}(t)], \tag{4-124}$$

where

$$\bar{k} = \sqrt{V_1/V_2}. \tag{4-125}$$

In transfer function form we have

$$\hat{\mathbf{X}}(s) = \frac{\sqrt{V_1/V_2}}{s + \sqrt{V_1/V_2}}\,\mathbf{Y}(s), \tag{4-126}$$

where $\hat{\mathbf{X}}(s)$ and $\mathbf{Y}(s)$ are the Laplace transforms of $\hat{\xi}(t)$ and $\eta(t)$, respectively. As can be seen, the observer is a first-order filter with unity gain at zero frequency and break frequency $\sqrt{V_1/V_2}$.

Example 4.4. *Positioning system*

In Example 2.4 (Section 2.3), we considered a positioning system which is described by the state differential equation

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \kappa \end{pmatrix} \mu(t). \tag{4-127}$$

Here $x(t) = \text{col}\,[\xi_1(t), \xi_2(t)]$, where $\xi_1(t)$ denotes the angular displacement $\theta(t)$ and $\xi_2(t)$ the angular velocity $\dot{\theta}(t)$. Let us now assume, as in Example 2.4, that a disturbing torque $\tau_d(t)$ acts upon the shaft of the motor. Accordingly, the state differential equation must be modified as follows:

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \kappa \end{pmatrix} \mu(t) + \begin{pmatrix} 0 \\ \gamma \end{pmatrix} \tau_d(t), \qquad \textbf{4-128}$$

where $1/\gamma$ is the rotational moment of inertia of all the rotating parts. If the fluctuations of the disturbing torque are fast as compared to the motion of the system itself, the assumption might be justified that $\tau_d(t)$ is white noise. Let us therefore suppose that $\tau_d(t)$ is white noise, with constant, scalar intensity V_d. Let us furthermore assume that the observed variable is given by

$$\eta(t) = (1, 0)x(t) + \nu_m(t), \qquad \textbf{4-129}$$

where $\nu_m(t)$ is white noise with constant, scalar intensity V_m.

We compute the steady-state optimal observer for this system. The variance Riccati equation takes the form

$$\dot{\underline{Q}}(t) = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix} \underline{Q}(t) + \underline{Q}(t) \begin{pmatrix} 0 & 0 \\ 1 & -\alpha \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & \gamma^2 V_d \end{pmatrix} - \underline{Q}(t) \begin{pmatrix} 1 \\ 0 \end{pmatrix} \frac{1}{V_m} (1, 0) \underline{Q}(t). \qquad \textbf{4-130}$$

In terms of the entries $q_{ij}(t)$, $i, j = 1, 2$, of $\underline{Q}(t)$, we obtain the following set of differential equations (using the fact that $q_{12}(t) = q_{21}(t)$):

$$\begin{aligned} \dot{q}_{11}(t) &= 2q_{12}(t) - \frac{1}{V_m} q_{11}^2(t), \\ \dot{q}_{12}(t) &= q_{22}(t) - \alpha q_{12}(t) - \frac{1}{V_m} q_{11}(t) q_{12}(t), \\ \dot{q}_{22}(t) &= -2\alpha q_{22}(t) + \gamma^2 V_d - \frac{1}{V_m} q_{12}^2(t). \end{aligned} \qquad \textbf{4-131}$$

It can be found that the steady-state solution of the equations as $t \to \infty$ is given by

$$\bar{\underline{Q}} = V_m \begin{pmatrix} -\alpha + \sqrt{\alpha^2 + 2\beta} & \alpha^2 + \beta - \alpha\sqrt{\alpha^2 + 2\beta} \\ \alpha^2 + \beta - \alpha\sqrt{\alpha^2 + 2\beta} & -\alpha^3 - 2\alpha\beta + (\alpha^2 + \beta)\sqrt{\alpha^2 + 2\beta} \end{pmatrix}, \qquad \textbf{4-132}$$

where

$$\beta = \gamma\sqrt{V_d/V_m}. \qquad \textbf{4-133}$$

It follows that the steady-state optimal gain matrix is given by

$$\bar{K} = \begin{pmatrix} -\alpha + \sqrt{\alpha^2 + 2\beta} \\ \alpha^2 + \beta - \alpha\sqrt{\alpha^2 + 2\beta} \end{pmatrix}. \qquad \textbf{4-134}$$

The characteristic polynomial of the matrix $A - \bar{K}C$ can be found to be

$$\det(sI - A + \bar{K}C) = s^2 + s\sqrt{\alpha^2 + 2\beta} + \beta, \qquad \textbf{4-135}$$

from which it can be derived that the poles of the steady-state optimal observer are

$$\tfrac{1}{2}(-\sqrt{\alpha^2 + 2\beta} \pm \sqrt{\alpha^2 - 2\beta}). \qquad \textbf{4-136}$$

Let us adopt the following numerical values:

$$\begin{aligned} \kappa &= 0.787 \text{ rad/(V s}^2), \\ \alpha &= 4.6 \text{ s}^{-1}, \\ \gamma &= 0.1 \text{ kg}^{-1} \text{ m}^{-2}, \\ V_d &= 10 \text{ N}^2 \text{ m}^2 \text{ s}, \\ V_m &= 10^{-7} \text{ rad}^2 \text{ s}. \end{aligned} \qquad \textbf{4-137}$$

It is supposed that the value of V_d is derived from the knowledge that the disturbing torque has an rms value of $\sqrt{1000} \simeq 31.6$ N m and that its power spectral density is constant from about -50 to 50 Hz and zero outside this frequency band. Similarly, we assume that the observation noise, which has an rms value of 0.01 rad, has a flat power spectral density function from about -500 to 500 Hz and is zero outside this frequency range. We carry out the calculations as if the noises were white with intensities as indicated in **4-137** and then see if this assumption is justified.

With the numerical values as given, the steady-state gain matrix is found to be

$$\bar{K} = \begin{pmatrix} 40.36 \\ 814.3 \end{pmatrix}. \qquad \textbf{4-138}$$

The observer poles are $-22.48 \pm j22.24$. These pole locations apparently provide an optimal compromise between the speed of convergence of the reconstruction error and the immunity against observation noise.

The break frequency of the optimal observer can be determined from the pole locations. The observer characteristic polynomial is

$$s^2 + s\sqrt{\alpha^2 + 2\beta} + \beta \simeq s^2 + 45s + 1000, \qquad \textbf{4-139}$$

which represents a second-order system with undamped natural frequency $\omega_0 = 31.6$ rad/s $\simeq$ 5 Hz and relative damping of about 0.71. The undamped natural frequency is also the break frequency of the observer. Since this frequency is quite small as compared to the observation noise bandwidth of about 500 Hz and the disturbance bandwidth of about 50 Hz, we conjecture that it is safe to approximate both processes as white noise. We must compare both the disturbance bandwidth and the observation noise bandwidth to the *observer* bandwidth, since as can be seen from the error differential equation **4-82** both processes directly influence the behavior of the reconstruction error. In Example 4.5, at the end of Section 4.3.5, we compute the optimal filter without approximating the observation noise as white noise and see whether or not this approximation is justified.

The steady-state variance matrix of the reconstruction error is given by

$$\bar{Q} = \begin{pmatrix} 0.000004036 & 0.00008143 \\ 0.00008143 & 0.003661 \end{pmatrix}. \qquad \textbf{4-140}$$

By taking the square roots of the diagonal elements, it follows that the rms reconstruction error of the position is about 0.002 rad, while that of the angular velocity is about 0.06 rad/s.

We conclude this example with a discussion of the optimal observer that has been found. First, we note that the filter is completely determined by the ratio V_d/V_m, which can be seen as a sort of "signal-to-noise" ratio. The expression **4-136** shows that as this ratio increases, which means that β increases, the observer poles move further and further away. As a result, the observer becomes faster, but also more sensitive to observation noise. For $\beta = \infty$ we obtain a differentiating filter, which can be seen as follows. In transfer matrix form the observer can be represented as

$$\begin{aligned}\hat{\mathrm{X}}(s) &= (sI - A + \bar{K}C)^{-1}[\bar{K}\mathrm{Y}(s) + B\mathrm{U}(s)] \\ &= \frac{1}{s^2 + s\sqrt{\alpha^2 + 2\beta} + \beta} \\ &\quad \cdot \left\{ \begin{pmatrix} s(-\alpha + \sqrt{\alpha^2 + 2\beta}) + \beta \\ s(\alpha^2 + \beta - \alpha\sqrt{\alpha^2 + 2\beta}) \end{pmatrix} \mathrm{Y}(s) + \begin{pmatrix} \kappa \\ s - \alpha + \sqrt{\alpha^2 + 2\beta} \end{pmatrix} \mathrm{U}(s) \right\}.\end{aligned} \qquad \textbf{4-141}$$

Here $\hat{\mathrm{X}}(s)$, $\mathrm{Y}(s)$, and $\mathrm{U}(s)$ are the Laplace transforms of $\hat{x}(t)$, $y(t)$, and $u(t)$, respectively. As the observation noise becomes smaller and smaller, that is, $\beta \to \infty$, **4-141** converges to

$$\hat{\mathrm{X}}(s) = \begin{pmatrix} 1 \\ s \end{pmatrix} \mathrm{Y}(s). \qquad \textbf{4-142}$$

This means that the observed variable is taken as the reconstructed angular position and that the observed variable is differentiated to obtain the reconstructed angular velocity.

4.3.3* The Nonsingular Optimal Observer Problem with Correlated State Excitation and Observation Noises

In this section the results of the preceding section are extended to the case where the state excitation noise and the measurement noise are correlated, that is, $V_{12}(t) \neq 0$, $t \geq t_0$. To determine the optimal observer, we proceed in a fashion similar to the correlated case. Again, let $\tilde{Q}(t)$ denote the variance matrix of the reconstruction error when the observer is implemented with an arbitrary gain matrix $K(t)$, $t \geq t_0$. Using Theorem 1.52 (Section 1.11.2), we obtain the following differential equation for $\tilde{Q}(t)$, which is an extended version of **4-89**:

$$\dot{\tilde{Q}}(t) = [A(t) - K(t)C(t)]\tilde{Q}(t) + \tilde{Q}(t)[A(t) - K(t)C(t)]^T + V_1(t) - V_{12}(t)K^T(t) - K(t)V_{12}^T(t) + K(t)V_2(t)K^T(t), \qquad t \geq t_0, \tag{4-143}$$

with the initial condition

$$\tilde{Q}(t_0) = Q_0. \tag{4-144}$$

To convert the problem of finding the optimal gain matrix to a familiar problem, we reverse time in this differential equation. It then turns out that the present problem is dual to the "extended regulator problem" discussed in Problem 3.7 in which the integral criterion contains a cross-term in the state x and the input u. By using the results of Problem 3.7, it can easily be shown that the solution of the present problem is as follows (see, e.g., Wonham, 1963).

Theorem 4.6. *Consider the optimal observer problem of Definition 4.3 (Section 4.3.1). Suppose that the problem is nonsingular, that is, $V_2(t) > 0$, $t \geq t_0$. Then the solution af the optimal observer problem is achieved by choosing the gain matrix $K(t)$ of the observer* **4-73** *as*

$$K^0(t) = [Q(t)C^T(t) + V_{12}(t)]V_2^{-1}(t), \qquad t \geq t_0, \tag{4-145}$$

where $Q(t)$ is the solution of the matrix Riccati equation

$$\begin{aligned}\dot{Q}(t) = {} & [A(t) - V_{12}(t)V_2^{-1}(t)C(t)]Q(t) \\ & + Q(t)[A(t) - V_{12}(t)V_2^{-1}(t)C(t)]^T \\ & - Q(t)C^T(t)V_2^{-1}(t)C(t)Q(t) \\ & + V_1(t) - V_{12}(t)V_2^{-1}(t)V_{12}^T(t), \qquad t \geq t_0,\end{aligned} \tag{4-146}$$

with the initial condition

$$Q(t_0) = Q_0. \tag{4-147}$$

The initial condition of the observer is

$$\hat{x}(t_0) = \bar{x}_0. \tag{4-148}$$

For the choices **4-145** *and* **4-148**, *the mean square reconstruction error*

$$E\{[x(t) - \hat{x}(t)]^T W(t)[x(t) - \hat{x}(t)]\} \tag{4-149}$$

is minimized for all $t \geq t_0$. *The variance matrix of the reconstruction error is given by*

$$E\{[x(t) - \hat{x}(t)][x(t) - \hat{x}(t)]^T\} = Q(t), \tag{4-150}$$

hence

$$E\{[x(t) - \hat{x}(t)]^T W(t)[x(t) - \hat{x}(t)]\} = \text{tr}\,[W(t)Q(t)], \qquad t \geq t_0. \tag{4-151}$$

4.3.4* The Time-Invariant Singular Optimal Observer Problem

This section is devoted to the derivation of the optimal observer for the singular case, namely, the case where the matrix $V_2(t)$ is not positive-definite. To avoid the difficulties that occur when $V_2(t)$ is positive-definite during certain periods and singular during other periods, we restrict the derivation of this section to the time-invariant case, where all the matrices occurring in Definition 4.3 (Section 4.3.1) are constant. Singular observation problems arise when some of the components of the observed variable are free of observation noise, and also when the observation noise is not a white noise process, as we see in the following section. The present derivation roughly follows that of Bryson and Johansen (1965).

First, we note that when V_2 is singular the derivation of Section 4.3.2 breaks down; upon investigation it turns out that an infinite gain matrix would be required for a full-order observer as proposed. As a result, the problem formulation of Definition 4.3 is inadequate for the singular case. What we do in this section is to reduce the singular problem to a nonsingular problem (of lower dimension) and then apply the results of Sections 4.3.2 or 4.3.3.

Since V_2 is singular, we can always introduce another white noise process $w_2'(t)$, with *non*singular intensity V_2', such that

$$w_2(t) = Hw_2'(t), \tag{4-152}$$

with $\dim(w_2') < \dim(w_2)$, and where H has full rank. This means that the observed variable is given by

$$y(t) = Cx(t) + Hw_2'(t). \tag{4-153}$$

With this assumption the intensity of $w_2(t)$ is given by

$$V_2 = HV_2'H^T. \tag{4-154}$$

Since V_2 is singular, it is possible to decompose the observed variable into two parts: a part that is "completely noisy," and a part that is noise-free. We shall see how this decomposition is performed.

Since dim $(w_2') <$ dim (w_2), it is always possible to find an $l \times l$ nonsingular matrix T (l is the dimension of the observed variable y) partitioned as

$$T = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}, \tag{4-155}$$

such that

$$\begin{pmatrix} T_1 \\ T_2 \end{pmatrix} H = \begin{pmatrix} H_1 \\ 0 \end{pmatrix}. \tag{4-156}$$

Here H_1 is square and nonsingular, and the partitioning of T has been chosen corresponding to that in the right hand side of **4-156.** Multiplying the output equation

$$y(t) = Cx(t) + Hw_2'(t) \tag{4-157}$$

by T we obtain

$$y_1(t) = C_1x(t) + H_1w_2'(t), \tag{4-158a}$$

$$y_2(t) = C_2x(t), \tag{4-158b}$$

where

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} y(t), \qquad \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} C. \tag{4-159}$$

We see that **4-158** represents the decomposition of the observed variable $y(t)$ into a "completely noisy" part $y_1(t)$ (since $H_1V_2'H_1^T$ is nonsingular), and a noise-free part $y_2(t)$.

We now suppose that C_2 has full rank. If this is not the case, we can redefine $y_2(t)$ by eliminating all components that are linear combinations of other components, so that the redefined C_2 has full rank. We denote the dimension of $y_2(t)$ by k.

Equation **4-158b** will be used in two ways. First, we conclude that since $y_2(t)$ provides us with k linear equations for $x(t)$ we must reconstruct only $n - k$ (n is the dimension of x) additional linear combinations of $x(t)$. Second, since $y_2(t)$ does not contain white noise it can be differentiated in order to extract more data. Let us thus define, as we did in Section 4.2.3, an $(n - k)$-dimensional vector variable

$$p(t) = C_2'x(t), \tag{4-160}$$

where C_2' is so chosen that the $n \times n$ matrix

$$\begin{pmatrix} C_2 \\ C_2' \end{pmatrix} \tag{4-161}$$

is nonsingular. From $y_2(t)$ and $p(t)$ we can reconstruct $x(t)$ exactly by the relations

$$\begin{aligned} y_2(t) &= C_2 x(t), \\ p(t) &= C_2' x(t), \end{aligned} \tag{4-162}$$

or

$$x(t) = \begin{pmatrix} C_2 \\ C_2' \end{pmatrix}^{-1} \begin{pmatrix} y_2(t) \\ p(t) \end{pmatrix}. \tag{4-163}$$

It is convenient to introduce the notation

$$\begin{pmatrix} C_2 \\ C_2' \end{pmatrix}^{-1} = (L_1, L_2), \tag{4-164}$$

so that

$$x(t) = L_1 y_2(t) + L_2 p(t). \tag{4-165}$$

Our next step is to construct an observer for $p(t)$. The reconstructed $p(t)$ will be denoted by $\hat{p}(t)$. It follows from **4-165** that $\hat{x}(t)$, the reconstructed state, is given by

$$\hat{x}(t) = L_1 y_2(t) + L_2 \hat{p}(t). \tag{4-166}$$

The state differential equation for $p(t)$ is obtained by differentiation of **4-160**. It follows with **4-165**

$$\begin{aligned} \dot{p}(t) &= C_2' \dot{x}(t) = C_2' A x(t) + C_2' B u(t) + C_2' w_1(t) \\ &= C_2' A [L_1 y_2(t) + L_2 p(t)] + C_2' B u(t) + C_2' w_1(t), \end{aligned} \tag{4-167}$$

or

$$\dot{p}(t) = A' p(t) + B' u(t) + B'' y_2(t) + C_2' w_1(t), \tag{4-168}$$

where

$$A' = C_2' A L_2, \qquad B' = C_2' B, \qquad B'' = C_2' A L_1. \tag{4-169}$$

Note that both $u(t)$ and $y_2(t)$ are forcing variables for this equation. The observations that are available are $y_1(t)$, as well as $\dot{y}_2(t)$, for which we find

$$\begin{aligned} \dot{y}_2(t) &= C_2 \dot{x}(t) = C_2 A x(t) + C_2 B u(t) + C_2 w_1(t) \\ &= C_2 A [L_1 y_2(t) + L_2 p(t)] + C_2 B u(t) + C_2 w_1(t). \end{aligned} \tag{4-170}$$

For $y_1(t)$ we write

$$\begin{aligned} y_1(t) &= C_1 x(t) + H_1 w_2'(t) \\ &= C_1 [L_1 y_2(t) + L_2 p(t)] + H_1 w_2'(t). \end{aligned} \tag{4-171}$$

Combining $y_1(t)$ and $\dot{y}_2(t)$ we write for the observed variable of the system **4-168**

$$y'(t) = \begin{pmatrix} y_1(t) \\ \dot{y}_2(t) \end{pmatrix} = C'p(t) + D'u(t) + D''y_2(t) + H'\begin{pmatrix} w_1(t) \\ w_2'(t) \end{pmatrix}, \quad \textbf{4-172}$$

where

$$C' = \begin{pmatrix} C_1L_2 \\ C_2AL_2 \end{pmatrix}, \quad D' = \begin{pmatrix} 0 \\ C_2B \end{pmatrix}, \quad D'' = \begin{pmatrix} C_1L_1 \\ C_2AL_1 \end{pmatrix}, \quad H' = \begin{pmatrix} 0 & H_1 \\ C_2 & 0 \end{pmatrix}. \quad \textbf{4-173}$$

Note that in the state differential equation **4-168** and in the output equation **4-172** we treat both $u(t)$ and $y_2(t)$ as *given* data. To make the problem formulation complete, we must compute the *a priori* statistical data of the auxiliary variable $p(t_0)$:

$$\hat{p}(t_0) = E\{C_2'x(t_0) \mid y_2(t_0)\} \quad \textbf{4-174}$$

and

$$Q(t_0) = E\{[p(t_0) - \hat{p}(t_0)][p(t_0) - \hat{p}(t_0)]^T \mid y_2(t_0)\}. \quad \textbf{4-175}$$

It is outlined in Problem 4.4 how these quantities can be found.

The observation problem that we now have obtained, and which is defined by **4-168**, **4-172**, **4-174**, and **4-175**, is an observation problem with correlated state excitation and observation noises. It is either singular or nonsingular. If it is nonsingular it can be solved according to Section 4.3.3, and once $\hat{p}(t)$ is available we can use **4-166** for the reconstruction of the state. If the observation problem is still singular, we repeat the entire procedure by choosing a new transformation matrix T for **4-172** and continuing as outlined. This process terminates in one of two fashions:

(a) A nonsingular observation problem is obtained.

(b) Since the dimension of the quantity to be estimated is reduced at each step, eventually a stage can be reached where the matrix C_2 in **4-162** is square and nonsingular. This means that we can solve for $x(t)$ directly and no dynamic observer is required.

We conclude this section by pointing out that if **4-168** and **4-172** define a nonsingular observer problem, in the actual realization of the optimal observer it is not necessary to take the derivative of $y_2(t)$, since later this derivative is integrated by the observer. To show this consider the following observer for $p(t)$:

$$\dot{\hat{p}}(t) = A'\hat{p}(t) + B'u(t) + B''y_2(t) + K(t)[y'(t) - D'u(t) - D''y_2(t) - C'\hat{p}(t)]. \quad \textbf{4-176}$$

Partitioning

$$K(t) = [K_1(t), K_2(t)], \tag{4-177}$$

it follows for **4-176**:

$$\dot{\hat{p}}(t) = [A' - K(t)C']\hat{p}(t) + B'u(t) + B''y_2(t) + K_1(t)y_1(t) + K_2(t)\dot{y}_2(t) - K(t)[D'u(t) + D''y_2(t)]. \tag{4-178}$$

Now by defining

$$q(t) = \hat{p}(t) - K_2(t)y_2(t), \tag{4-179}$$

a state differential equation for $q(t)$ can be obtained with $y_1(t)$, $y_2(t)$, and $u(t)$, but not $\dot{y}_2(t)$, as inputs. Thus, by using **4-179**, $\hat{p}(t)$ can be found without using $\dot{y}_2(t)$.

4.3.5 The Colored Noise Observation Problem

This section is devoted to the case where the state excitation noise $w_1(t)$ and the observation noise $w_2(t)$ cannot be represented as white noise processes. In this case we assume that these processes can be modeled as follows:

$$\begin{aligned} w_1(t) &= C_1(t)x'(t) + w_1'(t), \\ w_2(t) &= C_2(t)x'(t) + w_2'(t), \end{aligned} \tag{4-180}$$

with

$$\dot{x}'(t) = A'(t)x'(t) + w_3(t). \tag{4-181}$$

Here $w_1'(t)$, $w_2'(t)$, and $w_3(t)$ are white noise processes that in general need not be uncorrelated. Combining **4-180** and **4-181** with the state differential and output equations

$$\begin{aligned} \dot{x}(t) &= A(t)x(t) + B(t)u(t) + w_1(t), \\ y(t) &= C(t)x(t) + w_2(t), \end{aligned} \tag{4-182}$$

we obtain the augmented state differential and output equations

$$\begin{aligned} \begin{pmatrix} \dot{x}(t) \\ \dot{x}'(t) \end{pmatrix} &= \begin{pmatrix} A(t) & C_1(t) \\ 0 & A'(t) \end{pmatrix} \begin{pmatrix} x(t) \\ x'(t) \end{pmatrix} + \begin{pmatrix} B(t) \\ 0 \end{pmatrix} u(t) + \begin{pmatrix} w_1'(t) \\ w_3(t) \end{pmatrix}, \\ y(t) &= [C(t), C_2(t)] \begin{pmatrix} x(t) \\ x'(t) \end{pmatrix} + w_2'(t). \end{aligned} \tag{4-183}$$

To complete the problem formulation the mean and variance matrix of the initial augmented state col $[x(t), x'(t)]$ must be given. In many cases the white noise $w_2'(t)$ is absent, which makes the observation problem singular. If the problem is time-invariant, the techniques of Section 4.3.4 can then be applied. This approach is essentially that of Bryson and Johansen (1965).

We illustrate this section by means of an example.

Example 4.5. *Positioning system with colored observation noise*

In Example 4.4 we considered the positioning system with state differential equation

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \kappa \end{pmatrix} \mu(t) + \begin{pmatrix} 0 \\ \gamma \end{pmatrix} \tau_d(t) \tag{4-184}$$

and the output equation

$$\eta(t) = (1, 0)x(t) + \nu_m(t). \tag{4-185}$$

The measurement noise $\nu_m(t)$ was approximated as white noise with intensity V_m. Let us now suppose that a better approximation is to model $\nu_m(t)$ as exponentially correlated noise (see Example 1.30, Section 1.10.2) with power spectral density function

$$\Sigma_m(\omega) = \frac{2\sigma^2\theta}{1 + \omega^2\theta^2}. \tag{4-186}$$

This means that we can write (Example 1.36, Section 1.11.4)

$$\nu_m(t) = \xi_3(t), \tag{4-187}$$

where

$$\dot{\xi}_3(t) = -\frac{1}{\theta}\xi_3(t) + \omega(t). \tag{4-188}$$

Here $\omega(t)$ is white noise with scalar intensity $2\sigma^2/\theta$. In Example 4.4 we assumed that $\tau_d(t)$ is also white noise with intensity V_d. In order not to complicate the problem too much, we stay with this hypothesis. The augmented problem is now represented by the state differential and output equations:

$$\begin{pmatrix} \dot{\xi}_1(t) \\ \dot{\xi}_2(t) \\ \dot{\xi}_3(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & -\frac{1}{\theta} \end{pmatrix} \begin{pmatrix} \xi_1(t) \\ \xi_2(t) \\ \xi_3(t) \end{pmatrix} + \begin{pmatrix} 0 \\ \kappa \\ 0 \end{pmatrix} \mu(t) + \begin{pmatrix} 0 \\ \gamma\tau_d(t) \\ \omega(t) \end{pmatrix}, \tag{4-189}$$

$$y(t) = (1, 0, 1)\begin{pmatrix} \xi_1(t) \\ \xi_2(t) \\ \xi_3(t) \end{pmatrix},$$

where col $[\xi_1(t), \xi_2(t)] = x(t)$. This is obviously a singular observation problem, because the observation noise is absent. Following the argument of Section 4.3.4, we note that the output equation is already in the form **4-158,** where C_1 and H_1 are zero matrices. It is natural to choose

$$p(t) = x(t), \qquad \textbf{4-190}$$

so that

$$C_2' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \qquad \textbf{4-191}$$

Writing

$$p(t) = \text{col}\,[\pi_1(t), \pi_2(t)], \qquad \textbf{4-192}$$

it follows by matrix inversion from

$$\begin{pmatrix} \eta(t) \\ \pi_1(t) \\ \pi_2(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \xi_1(t) \\ \xi_2(t) \\ \xi_3(t) \end{pmatrix} \qquad \textbf{4-193}$$

that

$$\begin{pmatrix} \xi_1(t) \\ \xi_2(t) \\ \xi_3(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} \eta(t) \\ \pi_1(t) \\ \pi_2(t) \end{pmatrix}. \qquad \textbf{4-194}$$

Since $p(t) = x(t)$, it immediately follows that $p(t)$ satisfies the state differential equation

$$\dot{p}(t) = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix} p(t) + \begin{pmatrix} 0 \\ \kappa \end{pmatrix} \mu(t) + \begin{pmatrix} 0 \\ \gamma \end{pmatrix} \tau_d(t). \qquad \textbf{4-195}$$

To obtain the output equation, we differentiate $\eta(t)$:

$$\dot{\eta}(t) = (1, 0)\dot{x}(t) + \dot{\xi}_3(t). \qquad \textbf{4-196}$$

Using **4-184, 4-188,** and **4-194,** it follows that we can write

$$\dot{\eta}(t) = \left(\frac{1}{\theta}, 1\right) p(t) - \frac{1}{\theta}\eta(t) + \omega(t). \qquad \textbf{4-197}$$

Together, **4-195** and **4-197** constitute an observation problem for $p(t)$ that is nonsingular and where the state excitation and observation noises happen to be uncorrelated. The optimal observer is of the form

$$\dot{\hat{p}}(t) = \begin{pmatrix} 0 & 1 \\ 0 & -\alpha \end{pmatrix} \hat{p}(t) + \begin{pmatrix} 0 \\ \kappa \end{pmatrix} \mu(t) + K^0(t)\left[\dot{\eta}(t) + \frac{1}{\theta}\eta(t) - \left(\frac{1}{\theta}, 1\right)\hat{p}(t)\right], \qquad \textbf{4-198}$$

where the optimal gain matrix $K^0(t)$ can be computed from the appropriate Riccati equation. From **4-194** we see that the optimal estimates $\hat{x}(t)$ of the state of the plant and $\hat{\xi}_3(t)$ of the observation noise are given by

$$\begin{aligned} \hat{x}(t) &= \hat{p}(t), \\ \hat{\xi}_3(t) &= \eta(t) + (-1, 0)\hat{p}(t). \end{aligned} \qquad \textbf{4-199}$$

Let us assume the following numerical values:

$$\begin{aligned} \kappa &= 0.787 \text{ rad}/(\text{Vs}^2), \\ \alpha &= 4.6 \text{ s}^{-1}, \\ \gamma &= 0.1 \text{ kg}^{-1}\text{ m}^{-2}, \\ V_d &= 10 \text{ N}^2\text{ m}^2\text{ s}, \\ \theta &= 5 \times 10^{-4}\text{ s}, \\ \sigma &= 0.01 \text{ rad}. \end{aligned} \qquad \textbf{4-200}$$

The numerical values for σ and θ imply that the observation noise has an rms value of 0.01 rad and a break frequency of $1/\theta = 2000$ rad/s $\simeq$ 320 Hz. With these values we find for the steady-state optimal gain matrix in **4-198**

$$\bar{K}^0 = \begin{pmatrix} 0.01998 \\ 0.4031 \end{pmatrix}. \qquad \textbf{4-201}$$

The variance matrix of the reconstruction error is

$$\begin{pmatrix} 0.000003955 & 0.00007981 \\ 0.00007981 & 0.003628 \end{pmatrix}. \qquad \textbf{4-202}$$

Insertion of $\bar{K}^0$ for $K^0(t)$ into **4-198** immediately gives us the optimal steady-state observer for $x(t)$. An implementation that does not require differentiation of $\eta(t)$ can easily be found.

The problem just solved differs from that of Example 4.4 by the assumption that ν_m is colored noise and not white noise. The present problem reduces to that of Example 4.4 if we approximate ν_m by white noise with an intensity V_m which equals the power spectral density of the colored noise for low frequencies, that is, we set

$$V_m = 2\sigma^2\theta. \qquad \textbf{4-203}$$

The numerical values in the present example and in Example 4.4 have been chosen consistently. We are now in a position to answer a question raised in Example 4.4: Are we justified in considering ν_m white noise because it has a large bandwidth, and in computing the optimal observer accordingly? In order to deal with this question, let us compute the reconstruction error variance matrix for the present problem by using the observer found in Example 4.4. In Example 4.4 the reconstruction error obeys the differential equation

$$\dot{e}(t) = \begin{pmatrix} -k_1 & 1 \\ -k_2 & -\alpha \end{pmatrix} e(t) + \begin{pmatrix} 0 \\ \gamma \end{pmatrix} \tau_d(t) - \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \nu_m(t), \qquad \text{4-204}$$

where we have set $\bar{K} = \text{col}\,(k_1, k_2)$. With the aid of **4-187** and **4-188**, we obtain the augmented differential equation

$$\begin{pmatrix} \dot{\varepsilon}_1(t) \\ \dot{\varepsilon}_2(t) \\ \dot{\xi}_3(t) \end{pmatrix} = \begin{pmatrix} -k_1 & 1 & -k_1 \\ -k_2 & -\alpha & -k_2 \\ 0 & 0 & -\dfrac{1}{\theta} \end{pmatrix} \begin{pmatrix} \varepsilon_1(t) \\ \varepsilon_2(t) \\ \xi_3(t) \end{pmatrix} + \begin{pmatrix} 0 \\ \gamma\tau_d(t) \\ \omega(t) \end{pmatrix}, \qquad \text{4-205}$$

where $e(t) = \text{col}\,[\varepsilon_1(t), \varepsilon_2(t)]$. It follows from Theorem 1.52 (Section 1.11.2) that the variance matrix $Q(t)$ of $\text{col}\,[\varepsilon_1(t), \varepsilon_2(t), \xi_3(t)]$ satisfies the matrix differential equation

$$\dot{Q}(t) = \begin{pmatrix} -k_1 & 1 & -k_1 \\ -k_2 & -\alpha & -k_2 \\ 0 & 0 & -\dfrac{1}{\theta} \end{pmatrix} Q(t) + Q(t) \begin{pmatrix} -k_1 & -k_2 & 0 \\ 1 & -\alpha & 0 \\ -k_1 & -k_2 & -\dfrac{1}{\theta} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \gamma^2 V_d & 0 \\ 0 & 0 & \dfrac{2\sigma^2}{\theta} \end{pmatrix}. \qquad \text{4-206}$$

Numerical solution with the numerical values **4-200** and **4-138** yields for the steady-state variance matrix of the reconstruction error $e(t)$

$$\begin{pmatrix} 0.000003995 & 0.00008062 \\ 0.00008062 & 0.003645 \end{pmatrix}. \qquad \text{4-207}$$

Comparison with **4-202** shows that the rms reconstruction errors that result from the white noise approximation of Example 4.4 are only very slightly greater than for the more accurate approach of the present example. This confirms the conjecture of Example 4.4 where we argued that for the optimal observer the observation noise $\nu_m(t)$ to a good approximation is white noise, so that a more refined filter designed on the assumption that $\nu_m(t)$ is actually exponentially correlated noise gives very little improvement.

4.3.6* Innovations

Consider the optimal observer problem of Definition 4.3 and its solution as given in Sections 4.3.2, 4.3.3, and 4.3.4. In this section we discuss an interesting property of the process

$$y(t) - C(t)\hat{x}(t), \qquad t \geq t_0, \tag{4-208}$$

where $\hat{x}(t)$ is the optimal reconstruction of the state at time t based upon data up to time t. In fact, we prove that this process, **4-208**, is *white noise* with intensity $V_2(t)$, which is precisely the intensity of the observation noise $w_2(t)$. This process is called the *innovation process* (Kailath, 1968), a term that can be traced back to Wiener. The quantity $y(t) - C(t)\hat{x}(t)$ can be thought of as carrying the new information contained in $y(t)$, since $y(t) - C(t)\hat{x}(t)$ is the extra driving variable that together with the model of the system constitutes the optimal observer. The innovations concept is useful in understanding the separation theorem of linear stochastic optimal control theroy (see Chapter 5). It also has applications in state reconstruction problems outside the scope of this book, in particular the so-called optimal smoothing problem (Kailath, 1968).

We limit ourselves to the situation where the state excitation noise w_1 and the observation noise w_2 are uncorrelated and have intensities $V_1(t)$ and $V_2(t)$, respectively, where $V_2(t) > 0$, $t \geq t_0$. In order to prove that $y(t) - C(t)\hat{x}(t)$ is a white noise process with intensity $V_2(t)$, we compute the covariance matrix of its integral and show that this covariance matrix is identical to the covariance matrix of the integral of a white noise process with intensity $V_2(t)$.

Let us denote by $s(t)$ the integral of $y(t) - C(t)\hat{x}(t)$, so that

$$\begin{aligned} \dot{s}(t) &= y(t) - C(t)\hat{x}(t), \\ s(t_0) &= 0. \end{aligned} \tag{4-209}$$

Furthermore,

$$e(t) = x(t) - \hat{x}(t) \tag{4-210}$$

is the reconstruction error. Referring back to Section 4.3.2, we obtain from **4-209** and **4-82** the following joint state differential equation for $s(t)$ and $e(t)$:

$$\begin{pmatrix} \dot{s}(t) \\ \dot{e}(t) \end{pmatrix} = \begin{pmatrix} 0 & C(t) \\ 0 & A(t) - K^0(t)C(t) \end{pmatrix} \begin{pmatrix} s(t) \\ e(t) \end{pmatrix} + \begin{pmatrix} 0 & I \\ I & -K^0(t) \end{pmatrix} \begin{pmatrix} w_1(t) \\ w_2(t) \end{pmatrix}, \tag{4-211}$$

where $K^0(t)$ is the gain of the optimal observer. Using Theorem 1.52 (Section 1.11.2), we obtain the following matrix differential equation for the variance matrix $\tilde{Q}(t)$ of col $[s(t), e(t)]$:

$$\dot{\tilde{Q}}(t) = \begin{pmatrix} 0 & C(t) \\ 0 & A(t) - K^0(t)C(t) \end{pmatrix} \tilde{Q}(t) + \tilde{Q}(t) \begin{pmatrix} 0 & 0 \\ C^T(t) & A^T(t) - C^T(t)K^{0T}(t) \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & I \\ I & -K^0(t) \end{pmatrix} \begin{pmatrix} V_1(t) & 0 \\ 0 & V_2(t) \end{pmatrix} \begin{pmatrix} 0 & I \\ I & -K^{0T}(t) \end{pmatrix}, \quad \textbf{4-212}$$

with the initial condition

$$\tilde{Q}(t_0) = \begin{pmatrix} 0 & 0 \\ 0 & Q_0 \end{pmatrix}, \quad \textbf{4-213}$$

where Q_0 is the variance matrix of $x(t_0)$. Let us partition $\tilde{Q}(t)$ as follows:

$$\tilde{Q}(t) = \begin{pmatrix} Q_{11}(t) & Q_{12}(t) \\ Q_{12}^T(t) & Q_{22}(t) \end{pmatrix}. \quad \textbf{4-214}$$

Then we can rewrite the matrix differential equation **4-212** in the form

$$\dot{Q}_{11}(t) = C(t)Q_{12}^T(t) + Q_{12}(t)C^T(t) + V_2(t), \qquad Q_{11}(t_0) = 0, \quad \textbf{4-215}$$

$$\dot{Q}_{12}(t) = C(t)Q_{22}(t) + Q_{12}(t)[A(t) - K^0(t)C(t)]^T - V_2(t)K^{0T}(t),$$
$$Q_{12}(t_0) = 0, \quad \textbf{4-216}$$

$$\dot{Q}_{22}(t) = [A(t) - K^0(t)C(t)]Q_{22}(t) + Q_{22}(t)[A(t) - K^0(t)C(t)]^T$$
$$+ V_1(t) + K^0(t)V_2(t)K^{0T}(t), \qquad Q_{22}(t_0) = Q_0. \quad \textbf{4-217}$$

As can be seen from **4-217**, and as could also have been seen beforehand, $Q_{22}(t) = Q(t)$, where $Q(t)$ is the variance matrix of the reconstruction error. It follows with **4-105** that in **4-216** we have

$$C(t)Q_{22}(t) - V_2(t)K^{0T}(t) = 0, \quad \textbf{4-218}$$

so that **4-216** reduces to

$$\dot{Q}_{12}(t) = Q_{12}(t)[A(t) - K^0(t)C(t)]^T, \qquad Q_{12}(t_0) = 0, \quad \textbf{4-219}$$

which has the solution

$$Q_{12}(t) = 0, \qquad t \geq t_0. \quad \textbf{4-220}$$

Consequently, **4-215** reduces to

$$\dot{Q}_{11}(t) = V_2(t), \qquad Q_{11}(t_0) = 0, \quad \textbf{4-221}$$

so that

$$Q_{11}(t) = \int_{t_0}^{t} V_2(\tau)\, d\tau. \quad \textbf{4-222}$$

By invoking Theorem 1.52 once again, the covariance matrix of col $[s(t), e(t)]$ can be written as

$$\tilde{R}(t_1, t_2) = \begin{cases} \tilde{Q}(t_1)\tilde{\Psi}^T(t_2, t_1) & \text{for} \quad t_2 \geq t_1, \\ \tilde{\Psi}(t_1, t_2)\tilde{Q}(t_2) & \text{for} \quad t_1 \geq t_2, \end{cases} \tag{4-223}$$

where $\tilde{\Psi}(t_1, t_0)$ is the transition matrix of the system

$$\begin{pmatrix} \dot{s}(t) \\ \dot{e}(t) \end{pmatrix} = \begin{pmatrix} 0 & C(t) \\ 0 & A(t) - K^0(t)C(t) \end{pmatrix} \begin{pmatrix} s(t) \\ e(t) \end{pmatrix}. \tag{4-224}$$

It is easily found that this transition matrix is given by

$$\tilde{\Psi}(t_1, t_0) = \begin{pmatrix} I & \int_{t_0}^{t_1} C(t)\Psi(t, t_0)\, dt \\ 0 & \Psi(t_1, t_0) \end{pmatrix}. \tag{4-225}$$

where $\Psi(t_1, t_0)$ is the transition matrix of the system

$$\dot{e}(t) = [A(t) - K^0(t)C(t)]e(t). \tag{4-226}$$

The covariance matrix of $s(t)$ is the $(1, 1)$-block of $\tilde{R}(t_1, t_2)$, which can be found to be given by

$$R_s(t_1, t_2) = \int_{t_0}^{\min(t_1, t_2)} V_2(t)\, dt. \tag{4-227}$$

This is the covariance matrix of a process with uncorrelated increments (see Example 1.29, Section 1.10.1). Since the process $y(t) - C(t)\hat{x}(t)$ is the derivative of the process $s(t)$, it is white noise with intensity $V_2(t)$ (see Example 1.33, Section 1.11.1).

We summarize as follows.

Theorem 4.7. *Consider the solution of the nonsingular optimal observer problem with uncorrelated state excitation noise and observation noise as given in Theorem 4.5. Then the innovation process*

$$y(t) - C(t)\hat{x}(t), \qquad t \geq t_0, \tag{4-228}$$

is a white noise process with intensity $V_2(t)$.

It can be proved that this theorem is also true for the singular optimal observer problem with correlated state excitation and observation noises.

4.4* THE DUALITY OF THE OPTIMAL OBSERVER AND THE OPTIMAL REGULATOR; STEADY-STATE PROPERTIES OF THE OPTIMAL OBSERVER

4.4.1* Introduction

In this section we study the steady-state and stability properties of the optimal observer. All of these results are based upon the properties of the optimal regulator obtained in Chapter 3. These results are derived through the *duality* of the optimal regulator and the optimal observer problem (Kalman and Bucy, 1961). Section 4.4.2 is devoted to setting forth this duality, while in Section 4.4.3 the steady-state properties of the optimal observer are discussed. Finally, in Section 4.4.4 we study the asymptotic behavior of the steady-state time-invariant optimal observer as the intensity of the observation noise goes to zero.

4.4.2* The Duality of the Optimal Regulator and the Optimal Observer Problem

The main result of this section is summarized in the following theorem.

Theorem 4.8. *Consider the optimal regulator problem (ORP) of Definition* 3.2 (*Section* 3.3.1) *and the nonsingular optimal observer problem (OOP) with uncorrelated state excitation and observation noises of Definition* 4.3 (*Section* 4.3.1). *In the observer problem let the matrix $V_1(t)$ be given by*

$$V_1(t) = G(t)V_3(t)G^T(t), \qquad t \geq t_0, \tag{4-229}$$

where

$$V_3(t) > 0, \qquad t \geq t_0. \tag{4-230}$$

Let the various matrices occurring in the definitions of the ORP and the OOP be related as follows:

$$\begin{array}{lll} A(t) & \text{of the ORP equals } A^T(t^* - t) & \text{of the OOP,} \\ B(t) & \text{of the ORP equals } C^T(t^* - t) & \text{of the OOP,} \\ D(t) & \text{of the ORP equals } G^T(t^* - t) & \text{of the OOP,} \\ R_3(t) & \text{of the ORP equals } V_3(t^* - t) & \text{of the OOP,} \\ R_2(t) & \text{of the ORP equals } V_2(t^* - t) & \text{of the OOP,} \\ P_1 & \text{of the ORP equals } Q_0 & \text{of the OOP,} \end{array} \tag{4-231}$$

all for $t \leq t_1$. Here

$$t^* = t_0 + t_1. \tag{4-232}$$

Under these conditions the solutions of the optimal regulator problem (Theorem

3.4, *Section* 3.3.3) *and the nonsingular optimal observer problem with uncorrelated state excitation and observation noises* (*Theorem* 4.5, *Section* 4.3.2) *are related as follows:*

(a) *$P(t)$ of the ORP equals $Q(t^* - t)$ of the OOP for $t \le t_1$;*
(b) *$F^0(t)$ of the ORP equals $K^{0T}(t^* - t)$ of the OOP for $t \le t_1$;*
(c) *The closed-loop regulator of the ORP:*

$$\dot{x}(t) = [A(t) - B(t)F^0(t)]x(t), \tag{4-233}$$

and the unforced reconstruction error equation of the OOP:

$$\dot{e}(t) = [A(t) - K^0(t)C(t)]e(t), \tag{4-234}$$

are dual with respect to t^* in the sense of Definition 1.23 (*Section* 1.8).

The proof of this theorem easily follows by comparing the regulator Riccati equation **3-130** and the observer Riccati equation **4-106**, and using time reversal (Lemma 4.1, Section 4.3.2).

In Section 4.4.3 we use the duality of the optimal regulator and the optimal observer problem to obtain the steady-state properties of the optimal observer from those of the optimal regulator. Moreover, this duality enables us to use computer programs designed for optimal regulator problems for optimal observer problems, and vice versa, by making the substitutions **4-231**.

4.4.3* Steady-State Properties of the Optimal Observer

Theorem 4.8 enables us to transfer from the regulator to the observer problem the steady-state properties (Theorem 3.5, Section 3.4.2), the steady-state stability properties (Theorem 3.6, Section 3.4.2), and various results for the time-invariant case (Theorems 3.7, Section 3.4.3, and 3.8, Section 3.4.4).

In this section we state some of the more important steady-state and stability properties. Theorem 3.5, concerning the steady-state behavior of the Riccati equation, can be rephrased as follows (Kalman and Bucy, 1961).

Theorem 4.9. *Consider the matrix Riccati equation*

$$\dot{Q}(t) = A(t)Q(t) + Q(t)A^T(t) + G(t)V_3(t)G^T(t) - Q(t)C^T(t)V_2^{-1}(t)C(t)Q(t). \tag{4-235}$$

Suppose that $A(t)$ is continuous and bounded, that $C(t)$, $G(t)$, $V_3(t)$, and $V_2(t)$ are piecewise continuous and bounded, and furthermore that

$$V_3(t) \ge \alpha I, \qquad V_2(t) \ge \beta I \qquad \text{for all } t, \tag{4-236}$$

where α and β are positive constants.

(i) *Then if the system*

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + G(t)w_3(t),\\ y(t) &= C(t)x(t),\end{aligned} \qquad \textbf{4-237}$$

is either

(a) *completely reconstructible, or*

(b) *exponentially stable,*

the solution $Q(t)$ *of the Riccati equation* **4-235** *with the initial condition* $Q(t_0) = 0$ *converges to a nonnegative-definite matrix* $\bar{Q}(t)$ *as* $t_0 \to -\infty$. $\bar{Q}(t)$ *is a solution of the Riccati equation* **4-235.**

(ii) *Moreover, if the system* **4-237** *is either*

(c) *both uniformly completely reconstructible and uniformly completely controllable, or*

(d) *exponentially stable,*

the solution $Q(t)$ *of the Riccati equation* **4-235** *with the initial condition* $Q(t_0) = Q_0$ *converges to* $\bar{Q}(t)$ *as* $t_0 \to -\infty$ *for any* $Q_0 \geq 0$.

The proof of this theorem immediately follows by applying the duality relations of Theorem 4.8 to Theorem 3.5, and recalling that if a system is completely reconstructible its dual is completely controllable (Theorem 1.41, Section 1.8), and that if a system is exponentially stable its dual is also exponentially stable (Theorem 1.42, Section 1.8).

We now state the dual of Theorem 3.6 (Section 3.4.2):

Theorem 4.10. *Consider the nonsingular optimal observer problem with uncorrelated state excitation and observation noises and let*

$$V_1(t) = G(t)V_3(t)G^T(t), \qquad \textit{for all } t, \qquad \textbf{4-238}$$

where $V_3(t) > 0$, *for all* t. *Suppose that the continuity, boundedness, and positive-definiteness conditions of Theorem* 4.9 *concerning* A, C, G, V_3, *and* V_2 *are satisfied. Then if the system* **4-237** *is either*

(a) *uniformly completely reconstructible and uniformly completely controllable, or*

(b) *exponentially stable,*

the following facts hold.

(i) *The* ***steady-state optimal observer***

$$\dot{\hat{x}}(t) = A(t)\hat{x}(t) + \bar{K}(t)[y(t) - C(t)\hat{x}(t)], \qquad \textbf{4-239}$$

where

$$\bar{K}(t) = \bar{Q}(t)C^T(t)V_2^{-1}(t), \qquad \textbf{4-240}$$

is exponentially stable. Here $\bar{Q}(t)$ *is as defined in Theorem* **4.9.**

(ii) *The steady-state optimal observer gain $\bar{K}(t)$ minimizes*

$$\lim_{t_0 \to -\infty} E\{e^T(t)W(t)e(t)\} \tag{4-241}$$

for every $Q_0 \geq 0$. The minimal value of **4-241**, *which is achieved by the steady-state optimal observer, is given by*

$$\mathrm{tr}\,[\bar{Q}(t)W(t)]. \tag{4-242}$$

We also state the counterpart of Theorem 3.7 (Section 3.4.3), which is concerned with time-invariant systems.

Theorem 4.11. *Consider the time-invariant nonsingular optimal observer problem of Definition 4.3 with uncorrelated state excitation and observation noises for the system*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Gw_3(t), \\ y(t) &= Cx(t) + w_2(t). \end{aligned} \tag{4-243}$$

Here w_3 is white noise with intensity V_3, and w_2 has intensity V_2. It is assumed that $V_3 > 0$, $V_2 > 0$, and $Q_0 \geq 0$. The associated Riccati equation is given by

$$\dot{Q}(t) = AQ(t) + Q(t)A^T + GV_3G^T - Q(t)C^TV_2^{-1}CQ(t), \tag{4-244}$$

with the initial condition

$$Q(t_0) = Q_0. \tag{4-245}$$

(a) *Assume that $Q_0 = 0$. Then as $t_0 \to -\infty$ the solution of the Riccati equation approaches a constant steady-state value $\bar{Q}$ if and only if the system* **4-243** *possesses no poles that are at the same time unstable, unreconstructible, and controllable.*

(b) *If the system* **4-243** *is both detectable and stabilizable, the solution of the Riccati equation approaches the value $\bar{Q}$ as $t_0 \to -\infty$ for every $Q_0 \geq 0$.*

(c) *If $\bar{Q}$ exists, it is a nonnegative-definite symmetric solution of the algebraic Riccati equatian*

$$0 = AQ + QA^T + GV_3G^T - QC^TV_2^{-1}CQ. \tag{4-246}$$

If the system **4-243** *is detectable and stobilizable, $\bar{Q}$ is the unique nonnegative-definite solution of the algebraic Riccati equation.*

(d) *If $\bar{Q}$ exists, it is positive-definite if and only if the system is completely controllable.*

(e) *If $\bar{Q}$ exists, the steady-state optimal observer*

$$\dot{\hat{x}}(t) = A\hat{x}(t) + \bar{K}[y(t) - C\hat{x}(t)], \tag{4-247}$$

where

$$\bar{K} = \bar{Q}C^TV_2^{-1}, \tag{4-248}$$

is asymptotically stable if and only if the system is detectable and stabilizable.

(f) *If the system is detectable and stabilizable, the steady-state optimal observer* **4-247** *minimizes*

$$\lim_{t_0 \to -\infty} E\{e^T(t)We(t)\} \tag{4-249}$$

for all $Q_0 \geq 0$. *For the steady-state optimal observer,* **4-249** *is given by*

$$\text{tr}\,[\bar{Q}W]. \tag{4-250}$$

We note that the conditions (b) and (c) are sufficient but not necessary.

4.4.4* Asymptotic Properties of Time-Invariant Steady-State Optimal Observers

In this section we consider the properties of the steady-state optimal filter for the time-invariant case, when the intensity of the observation noise approaches zero. This section is quite short since we are able to obtain our results immediately by "dualizing" the results of Section 3.8.

We first consider the case in which both the state excitation noise $w_3(t)$ (see **4-237**) and the observed variable are scalar. From Theorem 3.11 (Section 3.8.1), the following result is obtained almost immediately.

Theorem 4.12. *Consider the n-dimensional time-invariant system*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + g\omega_3(t), \\ \eta(t) &= cx(t) + \omega_2(t), \end{aligned} \tag{4-251}$$

where ω_3 *is scalor white noise with constant intensity* V_3, ω_2 *scalor white noise uncorrelated with* ω_3 *with positive constant intensity* V_2, g *a column vector, and* c *a row vectar. Suppose that* $\{A, g\}$ *is stabilizable and* $\{A, c\}$ *detectable. Let* $H(s)$ *be the scalar transfer function*

$$H(s) = c(sI - A)^{-1}g = \frac{\psi(s)}{\phi(s)} = \frac{\alpha \prod_{i=1}^{p} (s - \nu_i)}{\prod_{i=1}^{n} (s - \pi_i)}, \tag{4-252}$$

where $\phi(s)$ *is the characteristic polynomial of the system, and* π_i, $i = 1, 2, \cdots, n$, *its choracteristic values. Then the characteristic values of the steady-stote optimal observer are the left-half plane zeroes of the palynomial*

$$(-1)^n \phi(s)\phi(-s)\left[1 + \frac{V_3}{V_2} H(-s)H(s)\right]. \tag{4-253}$$

As a result, the following statements hold.

(a) *As $V_2/V_3 \to 0$, p of the n steady-state optimal observer poles approach the numbers $\hat{\nu}_i$, $i = 1, 2, \cdots, p$, where*

$$\hat{\nu}_i = \begin{cases} \nu_i & \text{if } \operatorname{Re}(\nu_i) \leq 0, \\ -\nu_i & \text{if } \operatorname{Re}(\nu_i) > 0. \end{cases} \tag{4-254}$$

(b) *As $V_2/V_3 \to 0$, the remaining $n - p$ observer poles asymptotically approach straight lines which intersect in the origin and make angles with the negative real axis of*

$$\begin{aligned} &\pm l \frac{\pi}{n-p}, \qquad l = 0, 1, \cdots, \frac{n-p-1}{2}, \qquad n-p \text{ odd}, \\ &\pm \frac{(l+\frac{1}{2})\pi}{n-p}, \qquad l = 0, 1, \cdots, \frac{n-p}{2} - 1, \qquad n-p \text{ even}. \end{aligned} \tag{4-255}$$

These faraway observer poles asymptotically are at a distance

$$\omega_0 = \left(\alpha^2 \frac{V_3}{V_2}\right)^{1/[2(n-p)]} \tag{4-256}$$

from the origin.

(c) *As $V_2/V_3 \to \infty$, the n observer poles approach the numbers $\hat{\pi}_i$, $i = 1, 2, \cdots, n$, where*

$$\hat{\pi}_i = \begin{cases} \pi_i & \text{if } \operatorname{Re}(\pi_i) \leq 0, \\ -\pi_i & \text{if } \operatorname{Re}(\pi_i) > 0. \end{cases} \tag{4-257}$$

It follows from (b) that the faraway poles approach a Butterworth configuration.

For the general case we have the following results, which follow from Theorem 3.12 (Section 3.8.1).

Theorem 4.13. *Consider the n-dimensional time-invariant system*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t) + Gw_3(t), \\ y(t) &= Cx(t) + w_2(t), \end{aligned} \tag{4-258}$$

where w_3 is white noise with constant intensity V_3 and w_2 is white noise uncorrelated with w_3 with constant intensity $V_2 > 0$. Suppose that $\{A, G\}$ is stabilizable and $\{A, C\}$ detectable. Then the poles of the steady-state optimal observer are the left-half plane zeroes of the polynomial

$$(-1)^n \phi(s)\phi(-s) \det [I + V_2^{-1} H(s) V_3 H^T(-s)], \tag{4-259}$$

where $H(s)$ is the transfer matrix

$$H(s) = C(sI - A)^{-1}G, \tag{4-260}$$

and $\phi(s)$ *is the characteristic polynomial of the system* **4-258**. *Suppose that* $\dim(w_3) = \dim(y) = k$, *so that* $H(s)$ *is a* $k \times k$ *transfer matrix. Let*

$$\det[H(s)] = \frac{\psi(s)}{\phi(s)} = \frac{\alpha \prod_{i=1}^{p}(s - \nu_i)}{\prod_{i=1}^{n}(s - \pi_i)}, \qquad \textbf{4-261}$$

and assume that $\alpha \neq 0$. *Also, suppose that*

$$V_2 = \rho N, \qquad \textbf{4-262}$$

with $N > 0$ *and* ρ *a positive scalar.*

(a) *Then as* $\rho \downarrow 0$, p *of the optimal observer poles approach the numbers* $\hat{\nu}_i$, $i = 1, 2, \cdots, p$, *where*

$$\hat{\nu}_i = \begin{cases} \nu_i & \text{if } \operatorname{Re}(\nu_i) \leq 0, \\ -\nu_i & \text{if } \operatorname{Re}(\nu_i) > 0. \end{cases} \qquad \textbf{4-263}$$

The remaining observer poles go to infinity and group into several Butterworth configurations of different orders and different radii. A rough estimate of the distance of the faraway poles to the origin is

$$\left(\alpha^2 \frac{\det(V_3)}{\rho^k \det(N)}\right)^{1/[2(n-p)]}. \qquad \textbf{4-264}$$

(b) *As* $\rho \to \infty$, *the* n *optimal observer poles approach the numbers* $\hat{\pi}_i$, $i = 1, 2, \cdots, n$, *where*

$$\hat{\pi}_i = \begin{cases} \pi_i & \text{if } \operatorname{Re}(\pi_i) \leq 0, \\ -\pi_i & \text{if } \operatorname{Re}(\pi_i) > 0. \end{cases} \qquad \textbf{4-265}$$

Some information concerning the behavior of the observer poles when $\dim(w_3) \neq \dim(y)$ follows by dualizing the results of Problem 3.14.

We finally transcribe Theorem 3.14 (Section 3.8.3) as follows.

Theorem 4.14. *Consider the time-invariant system*

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Gw_3(t), \\ y(t) &= Cx(t) + w_2(t), \end{aligned} \qquad \textbf{4-266}$$

where G *and* C *hove full rank,* w_3 *is white noise with constant intensity* V_3 *and* w_2 *is white noise uncorrelated with* w_3 *with constant nonsingular intensity* $V_2 = \rho N$, $\rho > 0$, $N > 0$. *Suppose that* $\{A, G\}$ *is stabilizable and* $\{A, C\}$ *detectable and let* $\bar{Q}$ *be the steady-state solution of the variance Riccati equation* **4-244** *associated with the optimal observer problem. Then the following facts hold.*

(a) *The limit*

$$\lim_{\rho \downarrow 0} \bar{Q} = Q_1 \qquad \textbf{4-267}$$

exists.
(b) *Let $e_s(t)$ denote the contribution of the state excitation noise to the reconstruction error $e(t) = x(t) - \hat{x}(t)$, and $e_o(t)$ the contribution of the observation noise to $e(t)$. Then for the steady-state optimal observer the following limits hold:*

$$\lim_{\rho \downarrow 0} E\{e^T(t)We(t)\} = \text{tr}\,(Q_1 W),$$

$$\lim_{\rho \downarrow 0} E\{e_s^T(t)We_s(t)\} = \text{tr}\,(Q_1 W), \qquad \textbf{4-268}$$

$$\lim_{\rho \downarrow 0} E\{e_o^T(t)We_o(t)\} = 0.$$

(c) *If* dim $(w_3) >$ dim (y), *then* $Q_1 \neq 0$.
(d) *If* dim $(w_3) =$ dim (y), *and the numerator polynomial $\psi(s)$ of the square transfer matrix*

$$C(sI - A)^{-1}G \qquad \textbf{4-269}$$

is nonzero, then $Q_1 = 0$ if and only if $\psi(s)$ has zeroes with nonpositive real parts only.
(e) *If* dim $(w_3) <$ dim (y), *then a sufficient condition for Q_1 to be the zero matrix is that there exists a rectangular matrix M such that the numerator polynomial of the square transfer matrix $MC(sI - A)^{-1}G$ is nonzero and has zeroes with nonpositive real parts only.*

This theorem shows that if no observation noise is present, completely accurate reconstruction of the state of the system is possible only if the number of components of the observed variable is at least as great as the number of components of the state excitation noise $w_1(t)$. Even if this condition is satisfied, completely faultless reconstruction is possible only if the transfer matrix from the system noise w_3 to the observed variable y possesses no right-half plane zeroes.

The following question now comes to mind. For very small values of the observation noise intensity V_2, the optimal observer has some of its poles very far away, but some other poles may remain in the neighborhood of the origin. These nearby poles cause the reconstruction error to recover relatively slowly from certain initial values. Nevertheless, Theorem 4.14 states that the reconstruction error variance matrix can be quite small. This seems to be a contradiction. The answer to this question must be that the structure of the system to be observed is so exploited that the reconstruction error cannot be driven into the subspace from which it can recover only slowly.

We conclude this section by remarking that Q_1, the limiting variance matrix for $\rho \downarrow 0$, can be computed by solving the singular optimal observer problem that results from setting $w_2(t) \equiv 0$. As it turns out, occasionally the reduced-order observation problem thus obtained involves a nondetectable system, which causes the appropriate algebraic Riccati equation to possess more than one nonnegative-definite solution. In such a case one of course has to select that solution that makes the reduced-order observer stable (asymptotically or in the sense of Lyapunov), since the full-order observer that approaches the reduced-order observer as $V_2 \to 0$ is always asymptotically stable.

The problem that is dual to computing Q_1, that is, the problem of computing

$$P_0 = \lim_{R_2 \to 0} \bar{P} \qquad \textbf{4-270}$$

for the optimal deterministic regulator problem (Section 3.8.3), can be solved by formulating the dual observer problem and attacking the resulting singular optimal observer problem as outlined above. Butman (1968) gives a direct approach to the "control-free costs" linear regulator problem.

Example 4.6. *Positioning system*

In Example 4.4 (Section 4.3.2), we found that for the positioning system under consideration the steady-state solution of the error variance matrix is given by

$$\bar{Q} = V_m \begin{pmatrix} -\alpha + \sqrt{\alpha^2 + 2\beta} & \alpha^2 + \beta - \alpha\sqrt{\alpha^2 + 2\beta} \\ \alpha^2 + \beta - \alpha\sqrt{\alpha^2 + 2\beta} & -\alpha^3 - 2\alpha\beta + (\alpha^2 + \beta)\sqrt{\alpha^2 + 2\beta} \end{pmatrix}, \qquad \textbf{4-271}$$

where

$$\beta = \gamma\sqrt{V_d/V_m}. \qquad \textbf{4-272}$$

As $V_m \downarrow 0$, the variance matrix behaves as

$$\bar{Q} \simeq \begin{pmatrix} 2^{1/2}\gamma^{1/2}V_d^{1/4}V_m^{3/4} & \gamma V_d^{1/2}V_m^{1/2} \\ \gamma V_d^{1/2}V_m^{1/2} & 2^{1/2}\gamma^{3/2}V_d^{3/4}V_m^{1/4} \end{pmatrix}. \qquad \textbf{4-273}$$

Obviously, $\bar{Q}$ approaches the zero matrix as $V_m \downarrow 0$. In Example 4.4 we found that the optimal observer poles are

$$\tfrac{1}{2}(-\sqrt{\alpha^2 + 2\beta} \pm \sqrt{\alpha^2 - 2\beta}). \qquad \textbf{4-274}$$

Asymptotically, these poles behave as

$$\tfrac{1}{2}\sqrt{2}\left(\frac{V_d}{V_m}\right)^{1/4} \gamma^{1/2}(-1 \pm j), \qquad \textbf{4-275}$$

which represents a second-order Butterworth configuration. All these facts accord with what we might suppose, since the system transfer function is given by

$$H(s) = c(sI - A)^{-1}g = \frac{1}{s(s + \alpha)}, \qquad \text{4-276}$$

which possesses no zeroes. As we have seen in Example 4.4, for $V_m \downarrow 0$ the optimal filter approaches the differentiating reduced-order filter

$$\begin{aligned} \hat{\xi}_1(t) &= \eta(t), \\ \hat{\xi}_2(t) &= \dot{\eta}(t). \end{aligned} \qquad \text{4-277}$$

If no observation noise is present, this differentiating filter reconstructs the state completely accurately, no matter how large the state excitation noise.

4.5 CONCLUSIONS

In this chapter we have solved the problem of reconstructing the state of a linear differential system from incomplete and inaccurate measurements. Several versions of this problem have been discussed. The steady-state and asymptotic properties of optimal observers have been reviewed. It has been seen that some of the results of this chapter are reminiscent of those obtained in Chapter 3, and in fact we have derived several of the properties of optimal observers from the corresponding properties of optimal regulators as obtained in Chapter 3.

With the results of this chapter, we are in a position to extend the results of Chapter 3 where we considered linear state feedback control systems. We can now remove the usually unacceptable assumption that all the components of the state can always be accurately measured. This is done in Chapter 5, where we show how *output feedback control systems* can be designed by connecting the state feedback laws of Chapter 3 to the observers of the present chapter.

4.6 PROBLEMS

4.1. *An observer for the inverted pendulum positioning system*

Consider the inverted pendulum positioning system described in Example

1.1 (Section 1.2.3). The state differential equation of this system is given by

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -\frac{F}{M} & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{g}{L'} & 0 & \frac{g}{L'} & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ \frac{1}{M} \\ 0 \\ 0 \end{pmatrix} \mu(t). \qquad \textbf{4-278}$$

Suppose we choose as the observed variable the angle $\phi(t)$ that the pendulum makes with the vertical, that is, we let

$$\eta_1(t) = \left(-\frac{1}{L'}, 0, \frac{1}{L'}, 0\right) x(t). \qquad \textbf{4-279}$$

Consider the problem of finding a time-invariant observer for this system.

(a) Show that it is impossible to find an asymptotically stable observer. Explain this physically.

(b) Show that if in addition to the angle $\phi(t)$ the displacement $s(t)$ of the carriage is also measured, that is, we add a component

$$\eta_2(t) = (1, 0, 0, 0)x(t) \qquad \textbf{4-280}$$

to the observed variable, an asymptotically stable time-invariant observer can be found.

4.2. *Reconstruction of the angular velocity*

Consider the angular velocity control system of Example 3.3 (Section 3.3.1), which is described by the state differential equation

$$\dot{\xi}(t) = -\alpha\xi(t) + \kappa\mu(t), \qquad \textbf{4-281}$$

where $\xi(t)$ is the angular velocity and $\mu(t)$ the driving voltage. Suppose that the system is disturbed by a stochastically varying torque operating on the shaft, so that we write

$$\dot{\xi}(t) = -\alpha\xi(t) + \kappa\mu(t) + \omega_1(t), \qquad \textbf{4-282}$$

where $\omega_1(t)$ is exponentially correlated noise with rms value σ_1 and time constant θ_1. The observed variable is given by

$$\eta(t) = \xi(t) + \omega_2(t), \qquad \textbf{4-283}$$

where ω_2 is exponentially correlated noise with rms value σ_2 and time constant θ_2. The processes ω_1 and ω_2 are uncorrelated.

The following numerical values are assumed:

$$\begin{aligned} \alpha &= 0.5\ \text{s}^{-1}, \\ \kappa &= 150\ \text{rad/(V s}^2\text{)}, \\ \sigma_1 &= 54.78\ \text{rad/s}^2, \\ \theta_1 &= 0.1\ \text{s}, \\ \sigma_2 &= 5\ \text{rad/s}, \\ \theta_2 &= 0.01\ \text{s}. \end{aligned} \quad \text{4-284}$$

(a) Since the state excitation noise and the observation noise have quite large bandwidths as compared to the system bandwidth, we first attempt to find an optimal observer for the angular velocity by approximating both the state excitation noise and the observation noise as white noise processes, with intensities equal to the power spectral densities of ω_1 and ω_2 at zero frequency. Compute the steady-state optimal observer that results from this approach.

(b) To verify whether or not it is justified to represent ω_1 and ω_2 as white noise processes, model ω_1 and ω_2 as exponentially correlated noise processes, and find the augmented state differential equation that describes the angular velocity control system. Using the observer differential equation obtained under (a), obtain a three-dimensional augmented state differential equation for the reconstruction error $\varepsilon(t) = \xi(t) - \hat{\xi}(t)$ and the state variables of the processes ω_1 and ω_2. Next compute the steady-state variance of tbe reconstruction error and compare this number to the value that has been predicted under (a). Comment on the difference and the reason that it exists.

(c) Attempt to reach a better agreement between the predicted and the actual results by reformulating the observation problem as follows. The state excitation noise is modeled as exponentially correlated noise, but the approximation of the observation noise by white noise is maintained, since the observation noise bandwidth is very large. Compute the steady-state optimal observer for this situation and compare its predicted steady-state mean square reconstruction error with the actual value (taking into account that the observation noise is exponentially correlated noise). Comment on the results.

(d)* Determine the completely accurate solution of the optimal observer problem by modeling the observation noise as exponentially correlated noise also. Compare the performance of the resulting steady-state optimal observer to that of the observer obtained under (c) and comment.

4.3. *Solution of the observer Riccati equation*

Consider the matrix Riccati equation

$$\dot{Q}(t) = A(t)Q(t) + Q(t)A^T(t) + V_1(t) - Q(t)C^T(t)V_2^{-1}(t)C(t)Q(t) \quad \text{4-285}$$

with the initial condition

$$Q(t_0) = Q_0. \quad \text{4-286}$$

Define $\Psi(t, t_0)$ as the $(2n \times 2n)$-dimensional $[Q(t)$ is $n \times n]$ solution of

$$\frac{d}{dt}\Psi(t, t_0) = \begin{pmatrix} -A^T(t) & C^T(t)V_2^{-1}(t)C(t) \\ V_1(t) & A(t) \end{pmatrix}\Psi(t, t_0), \qquad \textbf{4-287}$$

$$\Psi(t_0, t_0) = I.$$

Partition $\Psi(t, t_0)$ corresponding to the partitioning occurring in **4-287** as follows.

$$\Psi(t, t_0) = \begin{pmatrix} \Psi_{11}(t, t_0) & \Psi_{12}(t, t_0) \\ \Psi_{21}(t, t_0) & \Psi_{22}(t, t_0) \end{pmatrix}. \qquad \textbf{4-288}$$

Show that the solution of the Riccati equation can be written as

$$Q(t) = [\Psi_{21}(t, t_0) + \Psi_{22}(t, t_0)Q_0][\Psi_{11}(t, t_0) + \Psi_{12}(t, t_0)Q_0]^{-1}. \qquad \textbf{4-289}$$

4.4.* *Determination of a priori data for the singular optimal observer*

When computing an optimal observer for the singular observation problem as described in Section 4.3.4, we must determine the a priori data

$$\hat{p}(t_0) = E\{C_2' x(t_0) \mid y_2(t_0)\} \qquad \textbf{4-290}$$

and

$$Q(t_0) = E\{[p(t_0) - \hat{p}(t_0)][p(t_0) - \hat{p}(t_0)]^T \mid y_2(t_0)\}, \qquad \textbf{4-291}$$

where

$$y_2(t_0) = C_2 x(t_0). \qquad \textbf{4-292}$$

We assume that

$$E\{x(t_0)\} = \bar{x}_0 \qquad \textbf{4-293}$$

and

$$E\{[x(t_0) - \bar{x}_0][x(t_0) - \bar{x}_0]^T\} = Q_0 \qquad \textbf{4-294}$$

are given. Prove that if $x(t_0)$ is Gaussian then

$$E\{x(t_0) \mid y_2(t_0)\} = \hat{x}(t_0) = \bar{x}_0 + Q_0 C_2^T (C_2 Q_0 C_2^T)^{-1}[y_2(t_0) - C_2\bar{x}_0] \qquad \textbf{4-295}$$

and

$$E\{[x(t_0) - \hat{x}(t_0)][x(t_0) - \hat{x}(t_0)]^T \mid y_2(t_0)\} = Q_0 - Q_0 C_2^T (C_2 Q_0 C_2^T)^{-1} C_2 Q_0. \qquad \textbf{4-296}$$

Determine from these results expressions for **4-290** and **4-291**. *Hint:* Use the vector formula for a multidimensional Gaussian density function (compare **1-434**) and the expression for the inverse of a partitioned matrix as given by Noble (1969, Exercise 1.59, p. 25).