EE 364a: Convex Optimization I — R. Pathak

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Midterm Quiz Solutions

1. *True or false.* Write T or F if each statement is true or false. Suppose that $f, g : \mathbf{R}^n \to \mathbf{R}$ and $\phi : \mathbf{R} \to \mathbf{R}$ are given functions.

   (a) **F** If f, g are convex, then $h(x,y) = (f(x) + g(y))^2$ is convex.

   (b) **T** If f, ϕ are convex, differentiable, and $\phi' > 0$, then $\phi(f(x))$ is convex.

   (c) **T** If f, g are concave and positive, then $\sqrt{f(x)g(x)}$ is concave.

   Solution.

   (a) *False.* A simple counterexample is $g(y) = 0$ and $f(x) = x \log x$. Then $h(x,y) = (x \log x)^2$ is not convex, even though f and g are convex. If in addition f and g were guaranteed to be nonnegative, then $h(x,y)$ would be convex by the composition rules.

   (b) *True.* Since $\phi' > 0$, ϕ is increasing, so $\phi(f(x))$ is convex by the composition rules.

   (c) *True.* The function $h(x,y) = \sqrt{xy}$ is concave on $\mathbf{R}^2_{++}$ and increasing in each argument. Since f and g are concave and positive, $h(f(x), g(x))$ is concave by the composition rules.

2. *DCP rules.* The function $f(x,y) = -1/(xy)$ with $\mathbf{dom}\, f = \mathbf{R}^2_{++}$ is concave. Briefly explain how to represent it, using disciplined convex programming (DCP), limited to the atoms $1/u$, $\sqrt{uv}$, $\sqrt{v}$, u^2, u^2/v, addition, subtraction, and scalar multiplication. Justify any statement about the curvature, monotonicity, or other properties of the functions you use. Assume these atoms take their usual domains (*e.g.*, $\sqrt{u}$ has domain $u \geq 0$), and that DCP is sign-sensitive (*e.g.*, u^2/v is increasing in u when $u \geq 0$).

   Solution.

   Since $f(x,y) = -(1/\sqrt{xy})^2$, it can be seen as the composition, $f(x,y) = -g_3(g_2(g_1(x,y)))$, where $g_3(u) = u^2$, $g_2(u) = 1/u$, and $g_1(u,v) = \sqrt{uv}$. Note that g_1 is concave on $\mathbf{R}^2_{++}$ and positive, g_2 is convex and decreasing on $\mathbf{R}_{++}$, and g_3 is convex and increasing. Thus, $g_2(g_1(x,y))$ is DCP convex, and hence $g_3(g_2(g_1(x,y)))$ is also DCP convex, so f is DCP concave.

3. *Curvature of some functions.* Determine the curvature of the functions below.

   (a) the product $f(u,v) = uv$, with $\mathbf{dom}\, f = \mathbf{R}^2$

   ☐ convex ☐ concave ■ **neither**

   (b) the function $f(x,u,v) = \log(v - x^T x/u)$, with $\mathbf{dom}\, f = \{(x,u,v) \mid uv > x^T x, u > 0\}$

   ☐ convex ■ **concave** ☐ neither

(c) the 'exponential barrier' of polyhedral constraints

$$f(x) = \sum_{i=1}^{m} \exp\left(\frac{1}{b_i - a_i^T x}\right),$$

with $\mathbf{dom}\, f = \{x \mid a_i^T x < b_i, i = 1, \ldots, m\}$, and $a_i \in \mathbf{R}^n$, $b \in \mathbf{R}^m$

■ **convex** □ concave □ neither

Solution.

(a) *Neither.* The Hessian $\nabla f^2(u, v)$ has a positive and negative eigenvalue, thus this function is neither convex nor concave (though it is quasiconcave).

(b) *Concave.* The function $x^T x/u$ is jointly convex in x and u. Hence, $v - x^T x/u$ is concave, and also positive on the given domain. The result follows as log is concave and increasing.

(c) *Convex.* The function $1/u$ is convex on $\mathbf{R}_{++}$ and on the given domain, $b_i - a_i^T x > 0$, so $1/(b_i - a_i^T x)$ is convex in x. The result follows since $\exp(1/(b_i - a_i^T x))$ is the composition of a convex increasing function with a convex function and because sum of convex functions is convex.

4. *Convexity of some sets.* Determine if each set is necessarily convex.

(a) $\{P \in \mathbf{R}^{n \times n} \mid x^T P x \geq 0 \text{ for all } x \succeq 0\}$

■ **convex** □ not convex

(b) $\{(u, v) \in \mathbf{R}^2 \mid \cos(u + v) \geq \sqrt{2}/2,\ u^2 + v^2 \leq \pi^2/4\}$ (*Hint:* $\cos(\pi/4) = \sqrt{2}/2$)

■ **convex** □ not convex

(c) $\{x \in \mathbf{R}^n \mid x^T A^{-1} x \geq 0\}$, where $A \prec 0$.

■ **convex** □ not convex

Solution.

(a) *Convex.* Let $X, Y \in \{P \mid x^T P x \geq 0, \text{ for all } x \succeq 0\}$. If $0 \leq \theta \leq 1$ and $x \succeq 0$, then

$$x^T (\theta P + (1 - \theta) Q) x = \theta x^T P x + (1 - \theta) x^T Q x \geq 0,$$

and thus $\theta P + (1 - \theta) Q$ also lies in this set.

(b) *Convex.* The second condition implies $u, v \in [-\pi/2, \pi/2]$. Using the hint, $\cos(u + v) \geq \sqrt{2}/2$ if and only if $-\pi/4 \leq u + v \leq \pi/4$. As $f(u, v) = u^2 + v^2$ is convex, the given set can be written as

$$\{(u, v) \mid -\pi/4 \leq u + v \leq \pi/4\} \cap \{(u, v) \mid f(u, v) \leq \pi^2/4\}.$$

(The second set is also the ball of radius $\pi/2$ centered about the origin in $\mathbf{R}^2$.) The intersection of a slab with a sublevel set of a convex function is convex.

(c) *Convex.* The function $f(x) = x^T A^{-1} x$ is concave, since its Hessian is $2A^{-1}$ which is negative semidefinite since $A \prec 0$. This set is the 0-superlevel set of a concave function, hence convex. Another valid argument would be that if $A \prec 0$, then $A^{-1} \prec 0$, so $x^T A^{-1} x < 0$ for all nonzero $x \in \mathbf{R}^n$. In particular, this means that the set is just $\{0\}$, which is convex.

5. *DCP compliance.* Determine if each expression below is (sign-sensitive) DCP compliant, and check the applicable box.

(a) `sqrt(1 + 4 * square(x) + 16 * square(y))`

☐ DCP convex ☐ DCP concave ■ **not compliant**

(b) `min(x, log(y)) - max(y, z)`

☐ DCP convex ■ **DCP concave** ☐ not compliant

(c) `log(exp(2 * x + 3) + exp(4 * y + 5))`

☐ DCP convex ☐ DCP concave ■ **not compliant**

Solution.

(a) *Not compliant.* Although the function $\sqrt{1 + 4x^2 + 16y^2}$ is convex, the given composition violates the DCP ruleset, since $\sqrt{u}$ is concave (so any precomposition could only result in a concave function, under the DCP rules). One way to reformulate this function is `norm2(1, 2 * x, 4 * y)`, which is the composition of an affine function with the norm, thus DCP convex.

(b) *DCP concave.* The minimum of two concave functions is concave, and the maximum of two affine functions is convex, so the given function is concave and also complies with the DCP rules.

(c) *Not compliant.* Although the function $\log(\exp(2x + 3) + \exp(2y + 5))$ is convex, the given composition violates the DCP ruleset, since log is concave and increasing (in particular, any precomposition could only result in a concave function, under the DCP rules). One way to reformulate this function is `log_sum_exp(2 * x + 3, 4 * y + 5)`, which is the precomposition of the (convex) function log-sum-exp with affine functions.