Versatile Task Assignment for Heterogeneous Soft Dual-Processor Platforms

Matin Hashemi, *Student Member, IEEE,* Soheil Ghiasi, *Member, IEEE*

***Abstract*—This manuscript presents proofs of the theorems.**

Theorem 1: For a convex cut on a directed acyclic graph:

$$\underbrace{\sum_{v\in V_1}\theta_1(v)}_{\substack{\text{weight of}\\ G_1 \text{ vertices}}}+\underbrace{\sum_{e\in E_1}\theta_1(e)}_{\substack{\text{weight of}\\ G_1 \text{ edges}}}+\underbrace{\sum_{e\in C}\theta_1(e)}_{\substack{\text{weight of}\\ \text{cut edges}}}=\underbrace{\sum_{e\in C}\theta_1'(e)}_{\substack{\text{total weight}\\ \text{held by cut } C}}$$

Lemma: The transformation propagates the attribute of an arbitrary vertex (edge) along exactly one directed path, referred to as the *propagation path*, from the vertex (edge) to the unique sink vertex.

Proof of Lemma: Let a be an arbitrary vertex in the directed acyclic graph. The transformation propagates $\theta_1(a)$ along exactly one of its outgoing edges, which is selected at random. Let b be the destination vertex of the randomly-selected outgoing edge. If b is the sink vertex, then the lemma is proved. Otherwise, $\theta_1(a)$ is propagated along exactly one of the outgoing edges of b, and the same argument can be made iteratively. Since graph is acyclic, we will never visit a vertex that we have visited before. The graph has finite number of vertices and hence, the iteration will have to end by arriving at the sink vertex. Similar argument can be made for edge attributes. ■

Proof of Theorem 1: We argue that C and the propagation path of an arbitrary vertex $a \in G_1$ intersect at exactly one edge. If C and the propagation path of vertex a do not intersect, then $a \in G_1$ is connected to the sink vertex via a connected path and hence, C is not a cut. If they intersect in more than one edge, then the propagation path direction goes out of G_1 and then back into G_1, which means that C is not convex. Therefore, the two edge set intersect at exactly one edge, and hence, $\theta_1(a)$ is accurately captured in θ_1' of the edge. Similar arguments can be made for arbitrary edges in G_1 or C. ■

Theorem 2: similarly we have:

$$\underbrace{\sum_{v\in V_2}\theta_2(v)}_{\substack{\text{weight of}\\ G_2 \text{ vertices}}}+\underbrace{\sum_{e\in E_2}\theta_2(e)}_{\substack{\text{weight of}\\ G_2 \text{ edges}}}+\underbrace{\sum_{e\in C}\theta_2(e)}_{\substack{\text{weight of}\\ \text{cut edges}}}=\underbrace{\sum_{e\in C}\theta_2'(e)}_{\substack{\text{total weight}\\ \text{held by cut } C}}$$

The authors are with the Department of Electrical and Computer Engineering, University of California, Davis, CA 95616.
E-mail: {hashemi,ghiasi}@ucdavis.edu

Proof: Construct a new graph G^r from G by reversing the direction of all edges, and then, apply Theorem 1 to G^r. ■

Lemma 1: We have $\frac{\beta}{1+\delta} < \ddot{\beta} \le \beta$

Proof:

$$\left\lfloor \log_{1+\delta}^{\beta} \right\rfloor \le \log_{1+\delta}^{\beta} < \left\lfloor \log_{1+\delta}^{\beta} \right\rfloor + 1$$
$$(1+\delta)^{\left\lfloor \log_{1+\delta}^{\beta} \right\rfloor} \le (1+\delta)^{\log_{1+\delta}^{\beta}} < (1+\delta)^{\left\lfloor \log_{1+\delta}^{\beta} \right\rfloor+1}$$
$$\ddot{\beta} \le \beta < (1+\delta)\ddot{\beta}$$ ■

Theorem 3: If we set $\delta = \sqrt[F]{1+\epsilon} - 1$, where $\epsilon > 0$ and F is the number of faces in task graph G, then we have

$$\frac{\beta(P^*)}{1+\epsilon} < \ddot{\beta}(P^*) \le \beta(P^*)$$

Proof: Let P^{*j} denote a partial path consisting of the first j edges of path P^*. For example in Figure 10, for path $P_3^* = \{\text{longdash}, \text{dot}\}$, the partial path P_3^{*1} has only one edge $(s^*[0,0,0,0], r^*[0,2,2,4])$, and P_3^{*2} has two edges $(s^*[0,0,0,0], r^*[0,2,2,4])$ and $(r^*[0,2,2,4], t^*[4,2,8,4])$. Since k is the number of edges in P^*, P^{*k} is equal to P^*.

Let $\ddot{\beta}(P^{*j})$ denote the approximated β value of P^{*j} and $\beta(P^{*j})$ is its original value. In our example, $\ddot{\beta}(P_3^{*1}) = [0,2,2,4]$ and $\ddot{\beta}(P_3^{*2}) = [4,2,8,4]$ (Figure 10), and also, $\beta(P_3^{*1}) = [0,3,2,7]$ and $\beta(P_3^{*2}) = [4,3,8,8]$ (Figure 8). In addition, let $\beta(e_j)$ denote the attribute vector of the jth edge in P^*, e.g., $\beta(e_1) = [0,3,2,7]$ and $\beta(e_2) = [4,0,6,1]$. We have:

$$\begin{aligned}
&\beta(P^{*1}) = \beta(e_1) && \ddot{\beta}(P^{*1}) = f\big(\beta(e_1)\big)\\
&\beta(P^{*j}) = \beta(P^{*j-1}) + \beta(e_j) \rightarrow && \ddot{\beta}(P^{*j}) = f\big(\ddot{\beta}(P^{*j-1}) + \beta(e_j)\big)\\
&\beta(P^*) = \beta(P^{*k}) && \ddot{\beta}(P^*) = \ddot{\beta}(P^{*k})
\end{aligned}$$

We prove the theorem by induction. For $k=1$, $\beta(P^{*1}) = \beta(e_1)$ and $\ddot{\beta}(P^{*1}) = f\big(\beta(e_1)\big)$. Based on Lemma 1:

$$\frac{\beta(P^{*1})}{1+\delta} < \ddot{\beta}(P^{*1}) \le \beta(P^{*1})$$

Let us assume that the theorem holds for $k = j-1$. Our objective is to prove that it holds for $k = j$. Based on the induction assumption:

$$\frac{\beta(P^{*j-1})}{(1+\delta)^{j-1}} < \ddot{\beta}(P^{*j-1}) \le \beta(P^{*j-1})$$

$$\frac{\beta(P^{*j-1})}{(1+\delta)^{j-1}} + \beta(e_j) < \ddot{\beta}(P^{*j-1}) + \beta(e_j) \le \beta(P^{*j-1}) + \beta(e_j)$$

$$\frac{\beta(P^{*j-1}) + (1+\delta)^{j-1}\beta(e_j)}{(1+\delta)^{j-1}} < ... \le ...$$

Since $1 < (1+\delta)^{j-1}$:

$$\frac{\beta(P^{*j-1})+\beta(e_j)}{(1+\delta)^{j-1}} < \ddot{\beta}(P^{*j-1})+\beta(e_j) \leq \beta(P^{*j-1})+\beta(e_j)$$

Since $\beta(P^{*j}) = \beta(P^{*j-1}) + \beta(e_j)$:

$$\frac{\beta(P^{*j})}{(1+\delta)^{j-1}} < \ddot{\beta}(P^{*j-1})+\beta(e_j) \leq \beta(P^{*j}) \qquad \text{(I)}$$

We also know $\ddot{\beta}(P^{*j}) = f(\ddot{\beta}(P^{*j-1}) + \beta(e_j))$. Based on Lemma 1:

$$\frac{\ddot{\beta}(P^{*j-1})+\beta(e_j)}{1+\delta} < \ddot{\beta}(P^{*j}) \leq \ddot{\beta}(P^{*j-1})+\beta(e_j) \qquad \text{(II)}$$

From (I) and (II) we have

$$\frac{\beta(P^{*j})}{(1+\delta)^{j}} < \ddot{\beta}(P^{*j}) \leq \beta(P^{*j})$$

Therefore, induction is complete and $\frac{\beta(P^{*k})}{(1+\delta)^k} < \ddot{\beta}(P^{*k}) \leq \beta(P^{*k})$. Since $P^{*k} = P^*$ we have

$$\frac{\beta(P^{*})}{(1+\delta)^{k}} < \ddot{\beta}(P^{*}) \leq \beta(P^{*})$$

Since $k \leq F$, we have $(1+\delta)^k \leq (1+\delta)^F$. After replacing $\delta = \sqrt[F]{1+\epsilon} - 1$, we get $(1+\delta)^F = 1+\epsilon$, which implies, $(1+\delta)^k \leq 1+\epsilon$. Hence,

$$\frac{\beta(P^{*})}{1+\epsilon} < \ddot{\beta}(P^{*}) \leq \beta(P^{*}) \quad \blacksquare$$

Corollary 2: The original hard constraints

$$\beta_3(P^*) \leq \beta_3^{\max} \quad \text{and} \quad \beta_4(P^*) \leq \beta_4^{\max}$$

can be replaced with following constraints, which use approximated values

$$\ddot{\beta}_3(P^*) \leq \frac{\beta_3^{\max}}{1+\epsilon} \quad \text{and} \quad \ddot{\beta}_4(P^*) \leq \frac{\beta_4^{\max}}{1+\epsilon}$$

Proof: From Theorem 3, we have $\beta(P^*) < (1+\epsilon)\ddot{\beta}(P^*)$. Therefore, to guarantee the original constraint is satisfied, we need $(1+\epsilon)\ddot{\beta}_3(P^*)$ to be bounded, i.e., $(1+\epsilon)\ddot{\beta}_3(P^*) \leq \beta_3^{\max}$. The same argument holds for β_4. $\blacksquare$

Corollary 3: Let $\ddot{\beta}^{\max} = f(\beta^{\max})$. The constraints

$$\ddot{\beta}_3(P^*) \leq \ddot{\beta}_3^{\max} \quad \text{and} \quad \ddot{\beta}_4(P^*) \leq \ddot{\beta}_4^{\max}$$

guarantee that

$$\beta_3(P^*) \leq (1+\epsilon)\beta_3^{\max} \quad \text{and} \quad \beta_4(P^*) \leq (1+\epsilon)\beta_4^{\max}$$

Proof: From Theorem 3, we have $\frac{\beta_3(P^*)}{1+\epsilon} < \ddot{\beta}_3(P^*)$, and also, $\ddot{\beta}_3^{\max} \leq \beta_3^{\max}$. The same argument holds for β_4. $\blacksquare$

Theorem 4: Let $\ddot{Q}(P^*) = F\big(\ddot{\beta}_1(P^*), \ddot{\beta}_2(P^*), \beta_5(P^*)\big)$ denote the approximated value of our cost function $Q(P^*) = F\big(\beta_1(P^*), \beta_2(P^*), \beta_5(P^*)\big)$, for the path P^*. We have

$$\big(1 - \frac{\epsilon}{1+\epsilon} S(P^*)\big)\ Q(P^*) \ \leq\ \ddot{Q}(P^*) \ \leq\ Q(P^*)$$

where $S(P^*)$ is defined as

$$S(P^*) = \frac{\beta_1(P^*)}{Q(P^*)} \max \frac{\partial Q}{\partial \beta_1} + \frac{\beta_2(P^*)}{Q(P^*)} \max \frac{\partial Q}{\partial \beta_2}$$

Note that although Q is originally a discrete function, throughout the following mathematical analysis, we look at it as a continuous function. That is, we use the same formula for Q, but assume its domain is $\mathbb{R}$ instead of $\mathbb{I}$. Note that Q does not have to be differentiable. As long as Q is differentiable on several intervals and continuous (i.e., piece-wise differentiable), we are able to calculate the maximum slope. For example, $\max \frac{\partial Q}{\partial \beta_1} = \max \frac{\partial Q}{\partial \beta_1} = 1$ for $Q = \max\{W_1 + \alpha_1 N, \alpha_2 N + W_2\} = \max\{\beta_1 + \alpha_1\beta_5, \alpha_2\beta_5 + \beta_2\}$, because the slope of Q with respect to both β_1 and β_2 is either 0 or 1 on its entire domain.

Proof: Note that for better readability, we write Q instead of $Q(P^*)$, and so on. When $\ddot{\beta} = \beta$, we have $\ddot{Q} = Q$. Since Q is non-descending in β, maximum value of $Q - \ddot{Q}$ happens when $\ddot{\beta}$ is farthest away from β. This point is on the top left corner of the box shown Figure 1.A, i.e., $\ddot{\beta}^{tl} = \frac{\beta}{1+\epsilon}$. Hence, $Q - \ddot{Q} \leq Q - \ddot{Q}^{tl}$. In addition, based on properties of $\vec{\nabla} Q$, we know that $Q - \ddot{Q}^{tl} \leq (\vec{\beta} - \vec{\beta^{tl}}) \bullet \max \vec{\nabla} Q$. Therefore

$$Q - \ddot{Q} \leq (\vec{\beta} - \frac{\vec{\beta}}{1+\epsilon}) \bullet \max \vec{\nabla} Q = (\frac{\epsilon}{1+\epsilon})\vec{\beta} \bullet \max \vec{\nabla} Q$$

After dividing both sides by Q:

$$\frac{Q - \ddot{Q}}{Q} \leq (\frac{\epsilon}{1+\epsilon}) \underbrace{\frac{\vec{\beta}}{Q} \bullet \max \vec{\nabla} Q}_{S}$$

By rearranging the above equation we have:

$$(1 - \frac{\epsilon}{1+\epsilon} S)\ Q \leq \ddot{Q} \quad \blacksquare$$

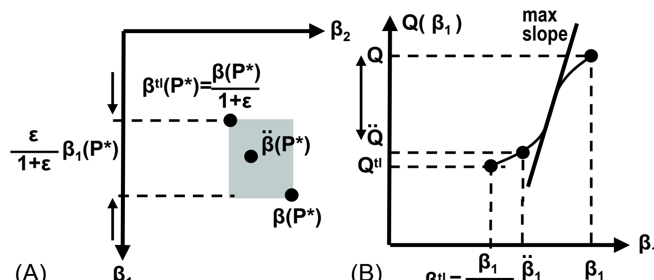

Fig. 1. A) Point $\beta(P^*)$ in 2-D space. The gray box shows the area where the approximated value $\ddot{\beta}(P^*)$ could be. Based on Theorem 3, the point farthest away from $\beta(P^*)$ is when $\ddot{\beta}(P^*) = \frac{\beta(P^*)}{1+\epsilon}$. B) Calculating the bound for function Q.

Corollary 4: Let $S^{\max}$ be the maximum possible value of S over the domain of function Q. We have

$$\forall P^* : (1 - \frac{\epsilon}{1+\epsilon} S^{\max})\ Q(P^*) \leq \ddot{Q}(P^*) \leq Q(P^*)$$

The above theorem states that the error in calculating cost function is bounded within a constant factor. The main objective of graph bi-partitioning is to find the optimal path $P^* = P^*_{opt}$ which minimizes our cost function $Q(P^*)$. Using the approximation method, however, $\ddot{Q}(P^*)$ is minimized for some near-optimum path $P^* = P^*_{near}$.

Corollary 5: Let $\xi = \dfrac{\epsilon}{1+\epsilon} S^{\max}$, and $T = \dfrac{1}{Q}$ denote the throughput. We have

$$(1-\xi)\ T_{opt} \leq T_{near} \leq T_{opt}$$

Proof: P^*_{near} minimizes $\ddot{Q}$, thus, $\ddot{Q}(P^*_{near}) \leq \ddot{Q}(P^*_{opt})$. Based on Corollary 4, $(1-\xi)\ Q(P^*_{near}) \leq \ddot{Q}(P^*_{near})$ and also $\ddot{Q}(P^*_{opt}) \leq Q(P^*_{opt})$. Therefore, $(1-\xi)\ Q(P^*_{near}) \leq Q(P^*_{opt})$. As a result, $(1-\xi)\ T(P^*_{opt}) \leq T(P^*_{near})$. ■